Validity and Application of the Extended Lagrangian Perturbation Theory

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Basing our discussion on the Lagrangian description of hydrodynamics, we studied the evolution of density fluctuation for nonlinear cosmological dynamics. We solved hydrodynamic equations for a self-gravitating fluid with pressure, given by a polytropic equation of state, using a perturbation method. In this conference, we discuss validity of these perturbative solutions and application for structure formation in the Universe. Especially we will notice the correspondence between past modified Lagrangian perturbation models and our model.

1. Introduction

The structure formation scenario based on gravitational instability has been studied for a long time. Zel'dovich [1] proposed a linear Lagrangian approximation for dust fluid. This approximation is called the Zel'dovich approximation (ZA) [1–3]. ZA describes the evolution of density fluctuation better than the Eulerian approximation [4–6]. After that, the second- and the third-order perturbative solution for dust fluid were derived [7, 8].

Although ZA gives an accurate description until a quasi-linear regime develops, ZA cannot describe the model after the formation of caustics. In order to proceed with a hydrodynamical description in which caustics do not form, the 'adhesion approximation' [9] (AA) was proposed based on the model equation of nonlinear diffusion (Burgers' equation). In AA, an artificial viscosity term is added to ZA. Because of the viscosity term, we can avoid caustics formation. From the standpoint of AA, the problem of structure formation has been discussed [10–12]. However, the origin of the viscosity has not yet been clarified.

Buchert and Domínguez [13] discussed the effect of velocity dispersion using the collisionless Boltzmann equation. They showed that when the velocity dispersion is regarded as small and isotropic it produces effective 'pressure' or viscosity terms. Furthermore, they posited the relation between mass density ρ and pressure P, i.e., an 'equation of state'. Buchert et al. [14] showed how the viscosity term or the effective pressure of a fluid is generated, assuming that the peculiar acceleration is parallel to the peculiar velocity.

With respect to the relation between the viscosity term and effective pressure, and the extension of the Lagrangian description to various matter, the Lagrangian perturbation theory of pressure has been considered. Actually, Adler and Buchert [15] have formulated the Lagrangian perturbation theory for a barotropic fluid. Morita and Tatekawa [16] and Tatekawa et al. [17] solved the Lagrangian perturbation equations for a polytropic fluid up to the second order. Recently, we have derived third-order perturbative solution for simple case [19]. Hereafter, we call this model the 'pressure model'.

Here we consider the correspondence between the pressure model and the modified Lagrangian models. Then we discuss the validity of linear perturbation in the pressure model. In summary, we propose an application of our model.

2. The Lagrangian description for the cosmological fluid

2.1. Basic equations

In this subsection, we briefly mention the Lagrangian description for the cosmological fluid. In the comoving coordinates, the basic equations for cosmological fluid are described as

$$\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla_x \cdot \{ \mathbf{v} (1+\delta) \} &= 0 \,, & (1) \\ \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a} (\mathbf{v} \cdot \nabla_x) \mathbf{v} + \frac{\dot{a}}{a} \mathbf{v} &= \frac{1}{a} \tilde{\mathbf{g}} - \frac{1}{a \rho} \nabla_x P \,, & (2) \\ \nabla_x \times \tilde{\mathbf{g}} = \mathbf{0} \,, \ \ \nabla_x \cdot \tilde{\mathbf{g}} &= -4 \pi G \rho_b a \delta \,, & (3) \\ \delta &\equiv \frac{\rho - \rho_b}{\rho_b} \,. & (4) \end{aligned}$$

In the Eulerian perturbation theory, the density fluctuation δ is regarded as a perturbative quantity. On the other hand, in the Lagrangian perturbation theory, the displacement from homogeneous distribution is considered.

$$\mathbf{x} = \mathbf{q} + \mathbf{s}(\mathbf{q}, t) \,, \qquad (5)$$

where $\mathbf{x}$ and $\mathbf{q}$ are the comoving Eulerian coordinates and the Lagrangian coordinates, respectively. $\mathbf{s}$ is the displacement vector that is regarded as a perturbative quantity.

The peculiar velocity is given by

$$\mathbf{v} = a \dot{\mathbf{s}} \,. \qquad (6)$$

Then we introduce the Lagrangian time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \frac{1}{a} \mathbf{v} \cdot \nabla_x \,. \qquad (7)$$

Taking divergence and rotation of Eq. (2), we obtain the evolution equations for the Lagrangian displacement [15, 16]:

$$\begin{aligned}\nabla_x \cdot \left(\ddot{\mathbf{s}} + 2\frac{\dot{a}}{a}\dot{\mathbf{s}} - \frac{\kappa\gamma\rho_b^{\gamma-1}}{a^2} J^{-\gamma}\nabla_x J \right) & \\ = -4\pi G\rho_b (J^{-1} - 1)\,, & \qquad (8)\end{aligned}$$

$$\nabla_x \times \left(\ddot{\mathbf{s}} + 2\frac{\dot{a}}{a}\dot{\mathbf{s}} \right) = 0\,, \qquad (9)$$

where $\dot{()}$ means the Lagrangian time derivative (Eq. (7)). To solve the perturbative equations, we decompose the Lagrangian perturbation to the longitudinal and transverse modes:

$$\begin{aligned}\mathbf{s} &= \nabla S + \mathbf{s}^T\,, & (10)\\ \nabla \cdot \mathbf{s}^T &= 0\,, & (11)\end{aligned}$$

where ∇ means the Lagrangian spacial derivative.

Because Eqs. (8) and (9) include the Eulerian spacial derivative, we change to the Lagrangian spacial derivative.

2.2. The Lagrangian perturbation for dust fluid

Zel'dovich derived a first-order solution of the longitudinal mode for dust fluid [1]. For the E-dS model, the solutions are written as follows:

$$S^{(1)}(\mathbf{q},t) = t^{2/3} S_+(\mathbf{q}) + t^{-1} S_-(\mathbf{q})\,. \qquad (12)$$

This first-order approximation is called the Zel'dovich approximation (ZA). Especially when we consider the plane-symmetric case, ZA gives exact solutions.

To improve approximation, higher-order perturbative solutions of Lagrangian displacement were derived. Irrotational second-order solutions (PZA) were derived by Bouchet et al. [20] and Buchert and Ehlers [21], and third-order solutions (PPZA) were obtained by Buchert [22], Bouchet et al. [7], and Catelan [8]. The second-order and third-order solutions for the E-dS model are written as follows:

$$\begin{aligned}S^{(2)}_{i,i} &= \frac{3}{14}\left(S^{(1)}_{i,j} S^{(1)}_{j,i} - S^{(1)}_{i,i} S^{(1)}_{j,j} \right)\,, & (13)\\ S^{(3)}_{i,i} &= \frac{5}{9}\left(S^{(1)}_{i,j} S^{(2)}_{j,i} - S^{(1)}_{i,i} S^{(2)}_{j,j} \right) - \frac{1}{3}\det\left(S^{(1)}_{i,j} \right)\,, & (14)\end{aligned}$$

where the superscript $S^{(n)}$ means n-th order solutions.

2.3. Adhesion approximation

Adhesion approximation (AA) [9] was proposed from a consideration based on Burgers' equation. This model is derived by the addition of an artificial viscous term to ZA. AA with small viscosity deals with the skeleton of the structure, which at an arbitrary time is found directly without a long numerical calculation.

We briefly describe the adhesion model. In ZA, the equation for 'peculiar velocity' in the E-dS model is written as follows:

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = 0\,, \quad \mathbf{u} \equiv \frac{\partial \mathbf{x}}{\partial a} = \frac{\dot{\mathbf{x}}}{\dot{a}}\,, \qquad (15)$$

where $a(\propto t^{2/3})$ means scale factor. To go beyond ZA, we add the artificial viscosity term to the right side of the equation.

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = \nu\nabla_x^2\mathbf{u}\,. \qquad (16)$$

We consider the case when the viscosity coefficient $\nu \to +0$ $(\nu \neq 0)$. In this case, the viscosity term especially affects the high-density region. Within the limits of a small ν, the analytic solution of Eq.(16) is given by

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \sum_\alpha \left(\frac{\mathbf{x}-\mathbf{q}_\alpha}{a}\right) j_\alpha \exp\left(-\frac{I_\alpha}{2\nu}\right) \\ &\quad \Big/ \sum_\alpha j_\alpha \exp\left(-\frac{I_\alpha}{2\nu}\right)\,, & (17)\end{aligned}$$

where $\mathbf{q}_\alpha$ means the Lagrangian points that minimize the action

$$\begin{aligned}I_\alpha &\equiv I(\mathbf{x}, a; \mathbf{q}_\alpha) \\ &= S_0(\mathbf{q}_\alpha) + \frac{(\mathbf{x}-\mathbf{q}_\alpha)^2}{2a} = \text{min.}\,, & (18)\\ j_\alpha &\equiv \left[\det\left(\delta_{ij} + \frac{\partial^2 S_0}{\partial q_i \partial q_j}\right)\right]^{-1/2}\Bigg|_{\mathbf{q}=\mathbf{q}_\alpha}\,, & (19)\\ S_0 &= S(\mathbf{q}, t_0)\,, & (20)\end{aligned}$$

considered as a function of $\mathbf{q}$ for fixed $\mathbf{x}$ [12]. In AA, because of the viscosity term, the caustic does not appear and a stable nonlinear structure can exist.

2.4. Pressure model

Buchert and Domínguez [13] argued that the effect of velocity dispersion becomes important beyond the caustics. They showed that when the velocity dispersion is still small and can be considered isotropic, it gives effective 'pressure' or viscosity terms. Buchert et al. [14] showed how the viscosity term is generated by the effective pressure of a fluid under the assumption that the peculiar acceleration is parallel to the peculiar velocity.

Adler and Buchert [15] have formulated the Lagrangian perturbation theory for a barotropic fluid.

Morita and Tatekawa [16] and Tatekawa et al. [17] solved the Lagrangian perturbation equations for a polytropic fluid in the Friedmann Universe. Hereafter, we call this model the 'pressure model'.

When we consider the polytropic equation of state $P = \kappa\rho^{\gamma}$, the first-order solutions for the longitudinal mode are written as follows. For $\gamma \neq 4/3$,

$$\widehat{S}(\mathbf{K}, a) \propto a^{-1/4}\, \mathcal{J}_{\pm 5/(8-6\gamma)} \left(\sqrt{\frac{2C_2}{C_1}} \frac{|\mathbf{K}|}{|4-3\gamma|}\, a^{(4-3\gamma)/2} \right), \tag{21}$$

where $\mathcal{J}_\nu$ denotes the Bessel function of order ν, and for $\gamma = 4/3$,

$$\widehat{S}(\mathbf{K}, a) \propto a^{-1/4 \pm \sqrt{25/16 - C_2 |\mathbf{K}|^2/2C_1}}, \tag{22}$$

$$C_1 \equiv 4\pi G \rho_{\rm b}(a_{\rm in})\, a_{\rm in}^{3}/3, \tag{23}$$

$$C_2 \equiv \kappa\gamma\rho_{\rm b}(a_{\rm in})^{\gamma-1}\, a_{\rm in}^{3(\gamma-1)}. \tag{24}$$

ρ_b and $\mathbf{K}$ mean background mass density and Lagrangian wavenumber, respectively. $a_{\rm in}$ means scale factor when an initial condition is given. When we take the limit $\kappa \to 0$, these solutions agree with Eq. (12).

In this model, the behavior of the solutions strongly depends on the relation between the scale of fluctuation and the Jeans scale. Here we define the Jeans wavenumber as

$$K_{\rm J} \equiv \left(\frac{4\pi G \rho_{\rm b} a^2}{{\rm d}P/{\rm d}\rho(\rho_{\rm b})} \right)^{1/2}.$$

The Jeans wavenumber, which gives a criterion for whether a density perturbation with a wavenumber will grow or decay with oscillation, depends on time in general. If the polytropic index γ is smaller than 4/3, all modes become decaying modes and the fluctuation will disappear. On the other hand, if $\gamma > 4/3$, all density perturbations will grow to collapse. In the case where $\gamma = 4/3$, the growing and decaying modes coexist at all times.

We rewrite the first-order solution Eq. (21) with the Jeans wavenumber:

$$\widehat{S}(\mathbf{K}, a) \propto a^{-1/4}\, \mathcal{J}_{\pm 5/(8-6\gamma)} \left(\frac{\sqrt{6}}{|4-3\gamma|} \frac{|\mathbf{K}|}{K_{\rm J}} \right). \tag{25}$$

For the pressure model, the Lagrangian perturbatve solutions are derived up to the third order [16, 17, 19].

In this paper, we analyze the first-order perturbation and the full-order solution. In general, it is very difficult to solve Eq. (8) for such reasons as the coordinate transformation or non-locality. Here, we imposed symmetry and avoided these difficulties.

3. Comparison between Lagrangian models

We analyzed the plane-symmetric and the spherical-symmetric cases [23]. Here we show only the sperical-symmetric case. For this case, dust collapse and void evolution have been analyzed [4–6]. Here we consider the evolution with ZA, PZA, PPZA, the exact solution for dust fluid, AA, and the pressure models. To avoid a discontinuity of the pressure gradient, we adopt the Mexican-hat type model (Fig. 1):

$$\delta(r) = \varepsilon(3 - r^2) e^{-r^2/2}. \tag{26}$$

The initial peculiar velocity is made equal with that of the growing mode in ZA. For this model, from Eq. (12)-(14), the solutions of ZA, PZA, and PPZA are given as follows:

$$S^{(1)} = -\varepsilon e^{-r^2/2}, \tag{27}$$

$$S^{(2)} = -\frac{3}{7}\varepsilon^2 e^{-r^2}, \tag{28}$$

$$S^{(3)} = -\frac{46}{189}\varepsilon^3 e^{-3r^2/2}. \tag{29}$$

In our analysis, we set the value of ε as follows:

$$\varepsilon = \pm\frac{1}{60}. \tag{30}$$

Under this condition, the initial density fluctuation at $r = 0$ becomes $\delta = \pm 0.05$. Then the scale factor is set as $a = 0.0167 (= 1/60)$ at the initial condition. In the case where $\varepsilon > 0$, the caustics appear at $a = 1$ in ZA. The initial peculiar velocity is equal with that given by the growing mode in ZA.

In past analyses [4, 5], homogeneous spherical collapse and void evolution have been analyzed. Here we consider spherical but inhomogeneous density fluctuation. We investigate time evolution in the dust model first because it may produce a result that differs from that of past analyses.

Fig. 2 shows the time evolution of the Mexican-hat type density fluctuation in the dust model. For spherical collapse, as well as in the past analyses, when we considered higher order perturbation, the occurrence time of the caustics becomes fast [4–6]. The caustic appears with an exact solution at $a \simeq 0.55$. On the other hand, the growth of the fluctuation becomes gentle, and the caustic does not appear in AA. For void evolution, the evolution of the density fluctuation stops gradually with PZA, and it starts to proceed in reverse. When we consider long-time evolution, PPZA deviates from an exact solution greatly more than ZA does. These results correspond to past analyses considering homogeneous spherical distribution.

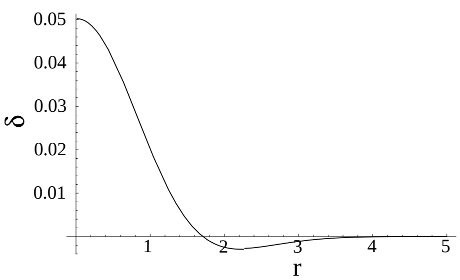

Figure 1: Mexican-hat type model. The average of density fluctuation over the whole space becomes zero.

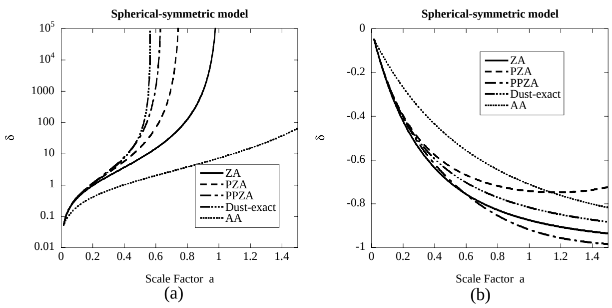

Figure 2: The evolution of the spherical-symmetric (the Mexican-hat type) density fluctuation at $r = 0$ in dust models. (a) The evolution of a density fluctuation in ZA, PZA, PPZA, the exact model, and AA in the case where $\varepsilon > 0$. The viscosity parameter in AA is set as $\nu = (1/512)^2$. The approximation is improved by higher order perturbation. In the exact model, the caustic appears at $a \simeq 0.55$. On the other hand, the density fluctuation evolves gently in AA. (b) The same as (a) but the case where $\varepsilon < 0$. In PZA, the fluctuation becomes positive during evolution. Later ($a > 0.6$), PPZA deviates from an exact solution greatly more than ZA does.

When we consider the effect of pressure, (Eq. (8)), how does the behavior of the density fluctuation change? These results are shown in Fig. 3. The pressure suppress the evolution of the density fluctuation. In the spherical-symmetric case, not only the pressure term but also the gravity term contains non-linear terms. Therefore, when we consider a full-order calculation, the contribution of not only the pressure but also the gravity becomes strong. Then if we choose a small value for K_J, the fluctuation sometimes grows earlier than in the case of linear approximation (Fig. 3 (a), (c), and (e)). In either case, the general tendency of the evolution of the fluctuation does not differ very much. According to Fig. 3 (a), (c), and (e), linear approximation of the pressure model seems to good until the quasi-nonlinear regime develops. The final state is unchanged, though a few differences are seen in the growth of the fluctuation, oscillatory amplitude, and period. In other words, even if the full-order calculation is considered, it is very difficult to explain the origin of the viscosity term in AA by the pressure model.

Next, we mention the evolution of a void (the case where $\varepsilon < 0$). When we consider a full-order calculation, unrealistic behavior, such as linear approximation in the case where $\gamma = 5/3$ (Fig. 3 (b)), does not appear. Furthermore, the oscillation of the fluctuation is suppressed, and the growth of the fluctuation comes to look like that of AA. In the evolution of the void, the linear approximation of the pressure model seems good until $\delta \simeq -0.5$. When we do not introduce linear approximation, the oscillation of the density fluctuation is almost imperceptible (Fig. 3 (b), (d), and (f)).

Although we can realize a void evolution in AA with the pressure model, we cannot reproduce the existence of a stable nonlinear structure. In other words, it is very difficult to find the origin of artificial viscosity in AA with the isotropic velocity dispersion.

According to our calculation in linear approximation, the amplitude and the period of the oscillation of the fluctuation in the intermediate state obviously depends on γ. Although the tendency of the evolution of the fluctuation in the case of $\gamma = 4/3$ looks like AA, the snapshot of the density field will be different from that in AA.

As for the validity of the linear approximation in the pressure model, the approximation is rather good until a quasi-nonlinear regime develops.

4. The effect of the higher-order approximation

We have derived perturbative solutions for the pressure model up to the third order [19]. For detail, we show our paper [19]. Here we briefly mention the result. Even if we restrict ourselves to the simplest case, the case of $\gamma = 4/3$ in the E-dS model, the third-order perturbative solutions become very complicated.

We have analyzed a planar model. The effect of the higher-order approximation suppresses the evolution of the density fluctuation. In this model, the difference between the Lagrangian approximations still be very small even just before shell-crossing.

In the one-dimensional model, the pressure only affects the nonlinear effect. However, in the three-dimensional realistic model, the gravitational force also affects the nonlinear effect, and the difference between first-, second-, and third-order approximation obviously appears. In fact, according to the comparison between the first-order approximation and full-order numerical calculation, the difference becomes large in the strongly nonlinear region [18].

5. Summary

We analyzed the corresponding relation with the viscosity term in AA and the velocity dispersion using spherical-symmetric case. In long-duration evolution, even if we consider full-order effects, the caustics will be formed. Though behavior similar to that of AA can be seen with the pressure model, more consideration is necessary for establishing the existence of stable nonlinear structure.

Next, we consider another question. When we analyze structure formation in the fluid with pressure, can we learn whether the Lagrangian linear perturbation is valid or not? From our analyses, until a quasi-nonlinear regime develops, the linear approximation of the pressure model seems rather good from the comparison with a full-order numerical calculation. Therefore, for example, if the interaction in some kind of dark matter can be described by the effective pressure, we can examine the behavior of the density fluctuation in a quasi-nonlinear stage. Furthermore, when we compare the observations and the structure that is formed by using the pressure model, we can give a limitation to the nature of the dark matter.

In our analysis for a planar model, the effect of the third-order perturbation seems very small even at the nonlinear stage. However, our result does not show that we can ignore the third-order perturbation easily, because the nonlinear term of the gravitational force disappears in the planar model. When we consider the effect of nonlinear pressure and gravitational force, the third-order perturbation is expected as a powerful tool to treat high-density regions.

Acknowledgments

This work was supported by a Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science, and Technology (Young Scientists (B) 16740152).

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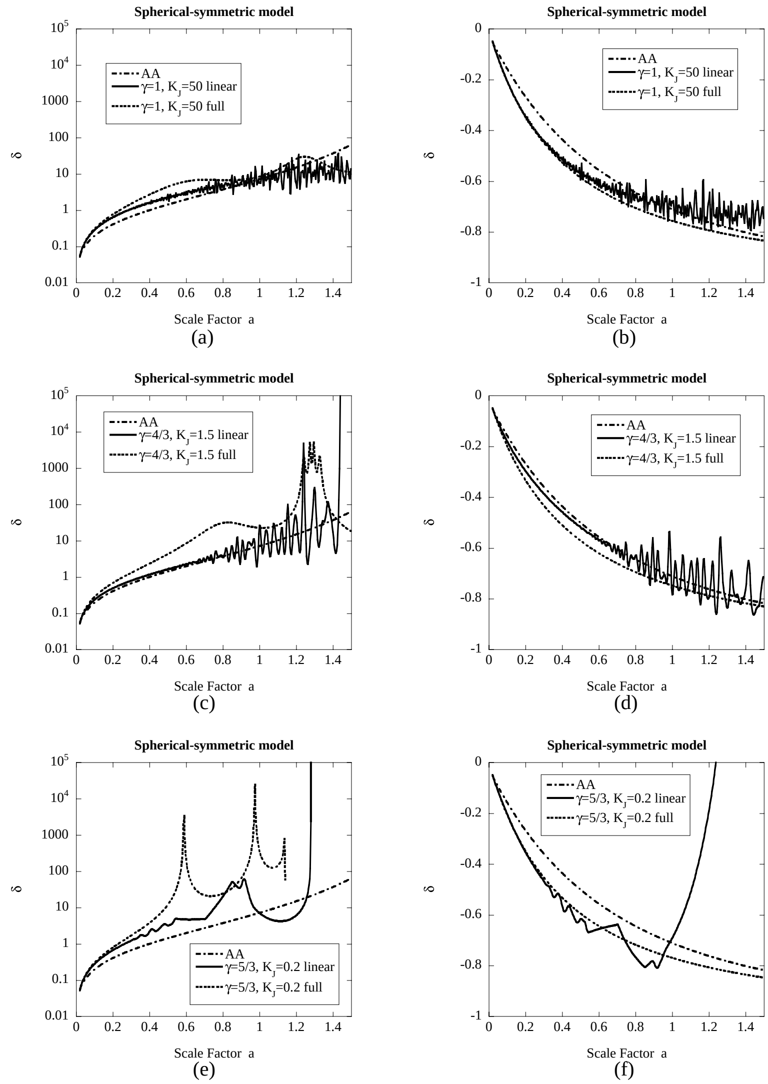

Figure 3: The evolution of a density fluctuation at $r = 0$ in the spherical-symmetric case. These figures show the evolution in AA and the pressure models (linear approximation and the full-order calculation). These figures show the case where $\varepsilon > 0$. (a) In the case where $\varepsilon > 0$ and $\gamma = 1$. In the pressure model, linear approximation seems valid until $\delta \simeq 1$. After that, in linear approximation, the fluctuation oscillates violently. In a strongly nonlinear region ($\delta > 10$), even if we consider full-order calculation in the pressure model, the evolution of a fluctuation similar to AA cannot be reproduced. (b) The same as (a), but here $\varepsilon < 0$. As in the case where $\varepsilon > 0$, when the fluctuation evolve fully ($\delta < -0.5$), the fluctuation begins to oscillate. Finally the fluctuation decays and approaches to 0. (c) The same as (a), but here $\gamma = 4/3$. In the pressure model with linear approximation, the fluctuation oscillates, and the caustic appears at $a \simeq 1.44$. When we consider a full-order equation, although we can delay the density divergence, we cannot avoid the formation of the caustic. (d) The same as (b), but here $\gamma = 4/3$. In the linear approximation in the pressure model, although the fluctuation oscillates, the density asymptotically decreases. In the pressure model, when we consider a full-order calculation, we can realize the evolution of a fluctuation similar to that of AA. (e) The same as (a), but here $\gamma = 5/3$. In the pressure model with the linear approximation, the oscillation of the fluctuation in the intermediate state grows very large. Then the caustic appears at $a \simeq 1.28$. When we consider a full-order equation, the density diverges a little to the outside at $a \simeq 1.14$, and the model fails. (f) The same as (b), but here $\gamma = 5/3$. In the linear approximation in the pressure model, the density fluctuation becomes positive during evolution because the oscillation of the fluctuation grows very large. On the other hand, when we consider a full-order equation, it is different from linear approximation, the density fluctuation always remaining negative.