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On Energy Transfer in the Plasma Wake Field Accelerator*

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ABSTRACT

The transfer of energy from the driving beam to the trailing beam in the plasma wake field accelerator is studied in computer simulations. We show that with an appropriate asymmetric current distribution in the driving bunch, the trailing particles can gain energies up to $\sqrt{1+k_p^2 Z^2}\Delta\gamma mc^2$, where Z is the bunch length, and $\Delta\gamma mc^2$ is the average energy loss of driving electrons. Due to the relative phase slippage and the two stream instability, the process of energy gain degrades before the driving beam loses all of its energy. However, even for initial $\gamma_i = 150$, we already see that $\Delta\gamma/\gamma_i \gtrsim 70\%$ and an energy gain $\lesssim 1$ GeV.

The idea of using one bunch of relativistic electrons to accelerate another to higher energy through the wake plasma wave set up by the leading bunch, was first suggested by Chen et.al.[1] in a paper employing the electrostatic approximation; later[2] the treatment was generalized to a fully electromagnetic one. Although the exact values depend on many parameters, accelerating fields of 1 GeV/m seemed reasonable. On the other hand, Ruth et al.[3] recognized the analogy between this scheme and collinear wake field acceleration in conventional metallic accelerating structures (MWFA), and called the former the plasma wake field accelerator (PWFA). Once this is seen, existing studies of MWFA can be directly connected to the PWFA.

One of the outstanding questions concerning wake field acceleration has been the limitation on the energy that can be transferred from the driving beam to the trailing beam. A useful parameter which describes this energy transfer is the transformer ratio R, defined as the ratio of the maximum accelerating electric field behind the driving bunch E_m^+, to the maximum retarding electric field within the bunch E_m^-. If a monoenergetic driving bunch excites a wake field, and if within distance L the particle in the bunch that experiences the maximum retarding field E_m^-, which would stop the earliest, loses energy $\Delta\gamma mc^2 = eLE_m^-$, then the maximum possible energy gain for a test charge behind the bunch will be $R\Delta\gamma mc^2$ in the same distance.

It can be proven that,[4] for any finite length bunch with a symmetric longitudinal charge distribution traversing an EM cavity supporting only a single mode, the transformer ratio cannot be larger than two; this is a generalized version of the fundamental theorem of beam loading.[5] Since the plasma in the PWFA is assumed to be cold, one expects that only a single mode, i.e. the oscillation at the plasma frequency ω_p will be excited by the driving beam. Thus the theorem should hold in the PWFA as well. Indeed, this limitation has been observed in computer simulations.[6] It was found that, for a driving beam with charge density profile $\rho \sim 1 + \sin(kz - \omega t)$, the driven beam gains energy only up to $\Delta U \lesssim 2\gamma_i mc^2$. But is this really the upper limit of energy gain using the collinear wake field acceleration scheme?

Recently, Bane, Chen and Wilson[7] have shown that this limitation can be overcome in the MWFA by introducing asymmetric current distributions in the driving bunch. Again, it is expected that these ideas also apply to the PWFA. In this letter we report on the results of our computer simulations and some theoretical studies of the PWFA. A one-and-two-halves dimensional (v_x, v_y, v_z, z) relativistic, electromagnetic particle code is used to simulate the beam-plasma system. Our results show that the transformer ratios in various cases agree very well with theoretical predictions. However, there are aspects that limit the ultimate energy gain of trailing particles. These will also be discussed.

Consider a one dimensional plasma in which the driving beam is an infinitely thin disk with uniform surface charge density $e\sigma$ and moves with speed $v_b \lesssim c$ in the positive z direction. Let us define the variable $\varsigma \equiv v_b t - z$, which measures the distance behind the driving beam. It can be shown [3] that the electric field $E(\varsigma)$ in the system is $4\pi e\sigma \cos k_p\varsigma$ for $\varsigma > 0$, $2\pi e\sigma$ at $\varsigma = 0$, and 0 for $\varsigma < 0$, where $k_p \equiv \omega_p / v_b$. Notice that the transformer ratio is 2 in this case, i.e. $E(0^+) = 2E(0)$.

We now consider a bunch of finite thickness with charge density $\rho(\varsigma)$ extending from $\varsigma = 0$ to $\varsigma = Z$. The electric field due to this bunch is then a convolution integral

$$E(\varsigma) = 4\pi \int_0^{\varsigma} \rho(\varsigma') \cos k_p(\varsigma - \varsigma') d\varsigma' \ . \tag{1}$$

For a linear ramp (a "triangular" bunch) with charge distribution $\rho(\varsigma) = \rho_0 k_p \varsigma$ for $0 \leq k_p\varsigma \leq 2\pi N$, and zero otherwise, it can be shown[7] via Eq. (1) that inside the bunch, $E^-(\varsigma) = 4\pi\rho_0 k_p^{-1}(1 - \cos k_p\varsigma)$, and behind the bunch, $E^+(\varsigma) = -8\pi^2 N\rho_0 k_p^{-1} \sin k_p\varsigma$. Identifying the extrema of $E^{\pm}$, we see that $R = \pi N$, which is larger than 2 for any $N \geq 1$. This calculation has been checked by computer simulations.[6,7] It is found for $k_p Z = 2\pi$ that $R \simeq 3.14$, in very good agreement with the theoretical prediction of $R = \pi$.

Consider next the "doorstep" charge distribution where $\rho(\varsigma) = \rho_0 = \text{const.}$ for $0 \leq k_p\varsigma \leq \pi/2$, and $\rho(\varsigma) = (2/\pi)\rho_0 k_p\varsigma$ for $\pi/2 \leq k_p\varsigma \leq k_p Z$. In this case $E(\varsigma) \propto -\sin k_p\varsigma$ for the first quarter wavelength and stays constant for the remaining bunch length. The transformer ratio for this case was calculated[7] to be $R = \sqrt{1 + (1 - \pi/2 + k_p Z)^2}$. For $k_p Z = 2\pi N, R \approx 2\pi N$, which is about twice that of a triangular bunch with the same length. A computer simulation (see Fig. 1(a)) was performed. The beam-plasma system was set up in such a way that the system was charge neutral both globally and locally at $t = 0$, meaning that the bunch charge was initially extracted from the background plasma at the far left of the system. It then traveled to the right, with $\gamma_i = 7.09$ in this case. We see that the E^+ oscillation near the left hand boundary is artifically larger due to the initial condition. The figure shows that the $E^-(\varsigma)$ across the triangular component of the bunch is not entirely flat as expected. This may be due to the particular way the system was initialized. Nevertheless, for $k_p Z = 2.5\pi$ we observe $R \simeq 6.12$ at $\omega_p t = 24$, which is reasonably close to the predicted value of $R = 7.35$. In this case the improvement in R is due to the fact that all particles in the triangular component of the bunch experience the same E_m^- and therefore contribute equally. This observation leads to the following provable assertion:[7]

"The maximum possible transformer ratio for a bunch with given length and total charge corresponds to that charge distribution which causes all particles in the bunch to see the same retarding field."

From the E field due to a thin disk we see that it is not possible to have a constant retarding field starting exactly at the head of the bunch for regular charge distributions. Therefore let us parametrize the optimal retarding field as[7]

$$E^-(\varsigma) = (1 - e^{-\alpha\varsigma})E_0 \ , \qquad 0 \leq \varsigma \leq Z \ , \tag{2}$$

which approaches the constant E_0 when $\alpha \to \infty$.

By the use of the Laplace transform Eq. (1) can be inverted[6] to give the charge distribution that produces a given $E^-(\varsigma)$ and $E^+(\varsigma)$. Applying this method to Eq. (2) gives

$$\rho(\varsigma) = -\frac{E_0}{4\pi\alpha}\left[\left(\alpha^2 + k_p^2\right) e^{-\alpha\varsigma} + k_p^2 \left(\alpha\varsigma - 1\right)\right] \ , \tag{3}$$

for $0 \leq \varsigma \leq Z$. This charge distribution is a superposition of two components: one a decaying exponential, the other a linearly rising ramp. In the asymptotic limit ($\alpha \to \infty$) the decaying exponential becomes a $\delta-$function and the ramp starts from $\varsigma = 0^+$. In this limit we get the maximum possible transformer ratio

$$R_m \equiv \lim_{\alpha\to\infty} R(\alpha) = \sqrt{1 + (k_p Z)^2} \ . \tag{4}$$

Since all particles except for those in the δ-function component experience the same retarding field and slow down at the same rate, the efficiency of energy extraction is $(1 + k_p^2 Z^2)/(2 + k_p^2 Z^2)$, which approaches 100% when $k_p Z \gg 1$. It is thus comforting to see that the optimal charge distribution provides not only the maximum transformer ratio but also the best efficiency.

Note that to achieve the constant retarding field the ratio of the charges in the two components of the charge distribution is not arbitrary, and must be equal to $2/(k_p Z)^2$. Computer simulations were performed for this arrangement. The resulting retarding field is very close to a constant (see Fig. 1(b)), and the transformer ratio is indeed better than the corresponding doorstep bunch. For $k_p Z = 2.5\pi$, $R \simeq 7.1$ at $\omega_p t = 24$, where the predicted value is 7.92. Again the initial condition probably contributes to the difference.

A physical picture is helpful in understanding how these asymmetric bunches can give large transformer ratios. The ideal driving bunch has two components. The leading component, such as the $\lambda_p/4$ rectangular pulse in a doorstep bunch or the δ-function pulse in an optimal distribution, serves as a precursor. The precursor gives background electrons an impulse so that they flow out of the local region at a rate that builds up in time. It is designed such that when the end of the precursor enters the region, the depletion rate of plasma electrons is just balanced by the replacement rate of electrons in the driving bunch. By this time the system becomes locally neutral and Gauss's law implies that $E^-(\varsigma)$ reaches a zero slope. The long ramp component then follows during which charge neutrality is sustained. At the time the driving bunch leaves the region, the plasma suddenly becomes nonneutral. The displaced electrons are then strongly attracted back to the ions and large amplitude plasma oscillations begin.

Although the transformer ratio increases linearly as the bunch length increases, there are practical limitations. In the PWFA a natural limitation is the wave-breaking limit of the plasma oscillations. The charge neutrality assumption implies that the beam density at the tail of the bunch should be the same as n_{p1}. Thus the peak charge density of the bunch is limited to n_{p0}. Given the total charge of a bunch there is then a trade-off between having a longer bunch but a smaller rate of increase in the charge density and a shorter bunch but a charge density which increases more rapidly. The former has a larger transformer ratio but smaller acceleration gradient whereas the latter is the opposite.

To study the energy gain of the trailing particles, we put a test charge into a plasma driven by the optimal distribution presented above in a computer simulation. The length of the driving bunch was chosen to be $4.25\lambda_p$ and the ratio of the peak driving beam density n_b to n_{p0} was 0.23. The transformer ratio is expected to be ~ 26.7 from Eq. (4). In the first trial we put the test charge at the first peak of the $E^+(\varsigma)$ oscillations. The γ of the test charge increases from $\gamma_{2i} = 8.0$ to its first peak value of 34.8 at $\omega_p t \simeq 157$ (see Fig. 2(a)), at which time the mean energy of the driving beam drops from $\bar{\gamma}_{1i} = 7.6$ to $\bar{\gamma}_{1f} \simeq 5.5$ (see Fig. 2(b)) with standard deviation $\simeq 0.61$. Ideally, we may expect the test charge to reach $\gamma_{2f} = \gamma_{2i} + R(\bar{\gamma}_{1i} - \bar{\gamma}_{1f}) \simeq 64$. As we shall explain below, by the time $\omega_p t \simeq 157$ the test charge has slipped by $\pi/2$ in phase in the wake field. As a result, during this time the test charge sees not a constant field E_m^+, but rather $E_m^+ \cos \omega_p t$. Thus we expect there to be a smaller energy gain $(2/\pi) R(\bar{\gamma}_{1i} - \bar{\gamma}_{1f})$. With this correction $\gamma_{2f} \simeq 40.7$, which is not too far from the observed value of 34.8.

After $\omega_p t \simeq 157$, γ_2 starts to fluctuate in this example. Although it eventually climbs to higher values, the process is rather random. Thus we consider the energy gain to be degraded by $\omega_p t \simeq 157$. Diagnostics indicate that there are two effects that caused the degradation in energy gain: the two stream instability in the driving bunch and the relative phase slippage between the driving and trailing beams. One of the results due to the two stream instability is that the oscillations in the driving bunch generate side-bands in k around c/ω_p, causing a modulation in the wake field amplitude. Secondly, when the instability becomes sizable some beam particles start to gain energy while others lose energy more rapidly. The driving beam thus acquires a large energy spread. At this point the wake field becomes turbulent, and the driving mechanism is largely degraded.

The relative phase shift between two relativistic particles is given in general by[3]

$$\delta \simeq \frac{\pi L}{\lambda_p}\left[\frac{1}{\gamma_{1i}\gamma_{1f}} - \frac{1}{\gamma_{2i}\gamma_{2f}}\right], \tag{5}$$

where L is the distance of travel, and i, f stand for initial and final values for both particles, respectively. The degradation in energy gain with a given δ is minimized by phasing the test charge initially at a phase $-\delta/2$ behind the crest and letting it slip over the crest to a phase $+\delta/2$. The energy gain is then $eE_m^+ L(\sin\delta/2)/(\delta/2)$.

In the specific case discussed above, the two stream instability becomes sizable when $\omega_p t \simeq 200$ (see Fig. 2(b)). On the other hand, we observe that the particles near the tail of the driving bunch have energies around one standard deviation below the mean $\bar{\gamma}_1$ at $\omega_p t \simeq 157$. To look for the relative phase slippage between the tail particles of the driving beam and the test charge, we put $\gamma_{1i} = 7.6$, $\gamma_{2i} = 8.0$ and $\gamma_{1f} \simeq 4.9$, $\gamma_{2f} \simeq 34$ into Eq. (5) at $\omega_p t \simeq 157$ and find that $\delta \simeq \pi/2$. Thus the degradation in our case is at first caused by the relative phase slippage. But the decrease of γ_2 after $\omega_p t \simeq 157$ is sustained only until the two stream instability becomes sizable, after which γ_2 begins to fluctuate. To confirm this, two other simulations were performed with test charge placed at angles 15° and 45° behind the first maximum of E^+. This arrangement allows for larger phase slippage and thus a longer driving time. Indeed, for $\phi = -15°$ γ_2 reaches ~ 39.0 at $\omega_p t \simeq 167$, and for $\phi = -45°$, we find $\gamma_2 \simeq 44.5$ at $\omega_p t \simeq 180$.

Redoing the simulation with $\gamma_{1i} = \gamma_{2i} = 150$ shows that the test charge (initially at $\phi = 0°$) reaches $\gamma_2 f \simeq 1840$ ($\lesssim 1$ GeV) before it slips into the deceleration phase at $\omega_p t \simeq 6000$. At this time $\bar{\gamma}_1$ has dropped to ~ 45, and $\Delta\bar{\gamma}_1/\gamma_{1i} \simeq 70\%$. We see therefore that the energy extraction efficiency improves as the initial energy increases

Again in this case the two stream instability only becomes sizable after the relative phase has slipped by $\delta = \pi/2$, at $\omega_p t \simeq 7000$. Notice, however, that with a reasonable energy extraction rate the phase slippage scales as γ_{1i}^{-2} whereas the growth rate of the two stream instability (with fixed n_b/n_{p0}) scales as γ_{1i}^{-1}.[8] Thus above a certain energy, the degradation of the energy gain will be influenced by the two stream instability first. One possible method to control the two stream instability is to introduce a spread in particle energy within the driving bunch. Preliminary results indicate that this technique can lead to an increase in beam stability.

So far we have reported result from one-and-two-halves dimensional simulations. The transformer ratios agree very well with theoretical predictions, and the energy gain of a test charge is much larger than the previous upper limit of $\Delta U \lesssim 2\gamma_i mc^2$ for symmetric driving bunches. In particular, from various trials $(\gamma_{1i} - \gamma_{1f})/\gamma_{1i} \gtrsim 70\%$ seem possible for the optimal current distributions. One may wonder whether the finite transverse size of a driving beam alters the physical picture presented here. As far as the transformer ratio is concerned, studies in two dimensional simulation show no fundamental changes.[9]

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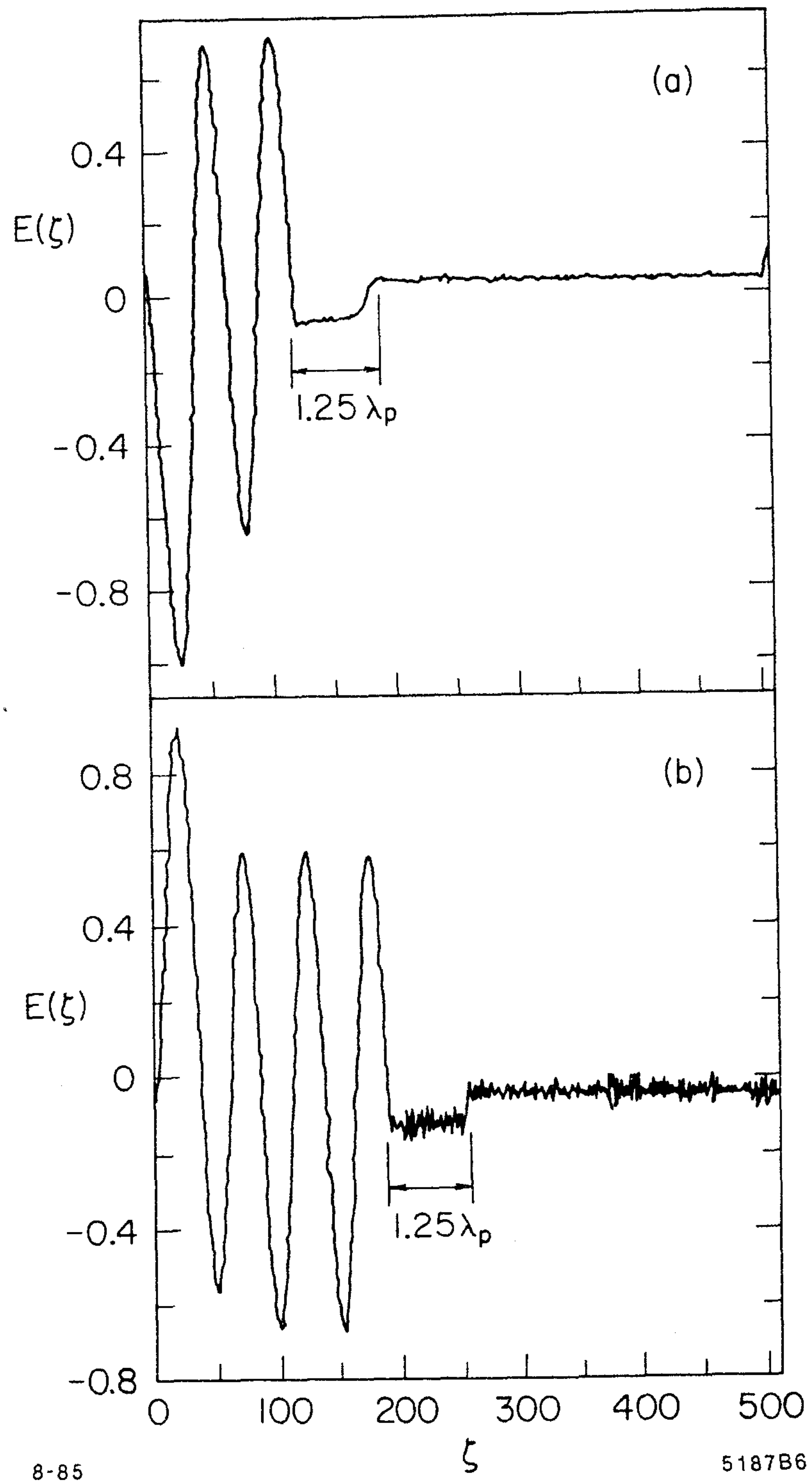
FIGURE CAPTIONS

Fig. 1. Plots of the longitudinal E at $\omega_p t = 24$. (a) A "doorstep" charge distribution with bunch length $Z = 1.25\lambda_p$. Note that $R \simeq 6.12$. (b) An optimal distribution with the same length, $R \simeq 7.1$.

Fig. 2. (a) Energy of the test charge as a function of time. The charge is initially on a crest of the wake field. At $\omega_p t \simeq 157$ where $\delta = \pi/2$ (see the arrow), it slips into the deceleration phase. (b) Mean energy of the driving bunch with $Z = 4.25\lambda_p$. The two stream instability becomes sizable at $\omega_p t \simeq 200$ (see the arrow).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0.4
E(ζ)
0
-0.4
-0.8
1.25 λp
(b)
0.8
0.4
E(ζ)
0
-0.4
-0.8
1.25 λp
0
100
200
300
400
500
ζ
8-85
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Fig. 1

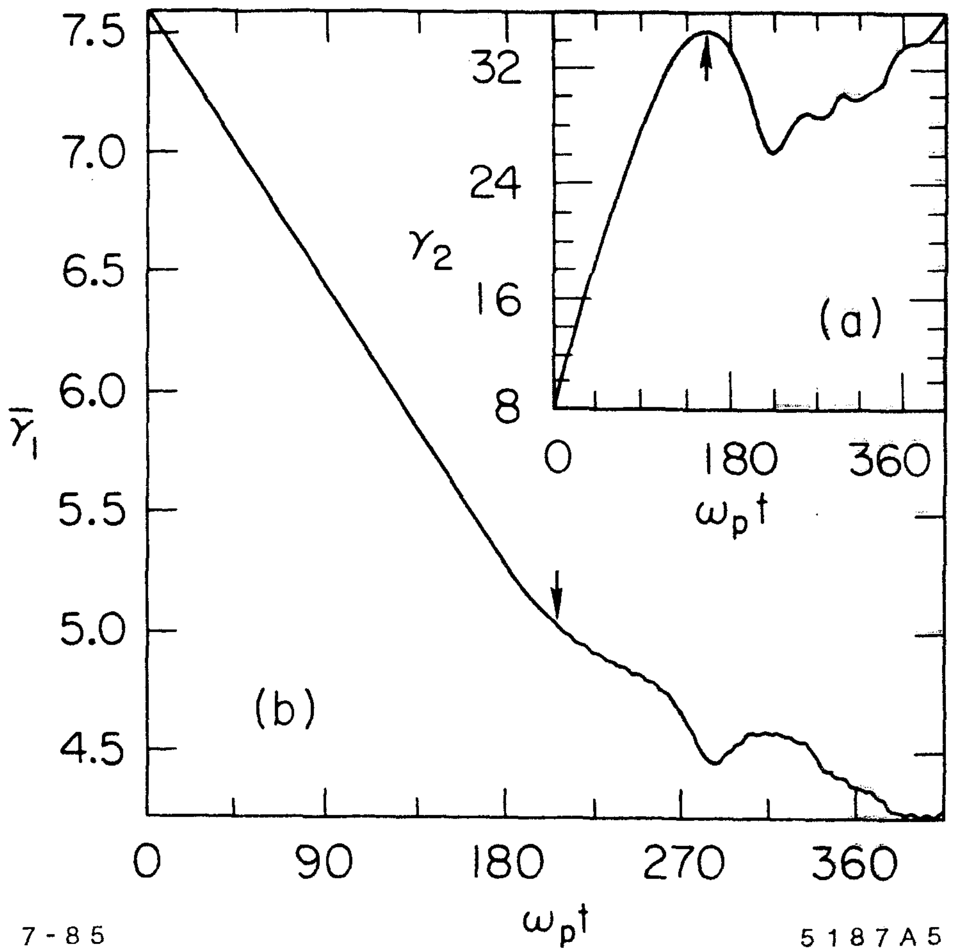
7.5
7.0
6.5
6.0
5.5
5.0
4.5
$\bar{\gamma}_1$
(b)
0
90
180
270
360
$\omega_p t$
32
24
16
8
γ_2
(a)
0
180
360
$\omega_p t$
7-85
5187A5

Fig. 2