Implications of the Principle of Maximum Conformality for the QCD Strong Coupling

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Abstract

The Principle of Maximum Conformality (PMC) provides scale-fixed perturbative QCD predictions which are independent of the choice of the renormalization scheme, as well as the choice of the initial renormalization scale. In this article, we will test the PMC by comparing its predictions for the strong coupling $\alpha_{g_1}^s(Q)$, defined from the Bjorken sum rule, with predictions using conventional pQCD scale-setting. The two results are found to be compatible with each other and with the available experimental data. However, the PMC provides a significantly more precise determination, although its domain of applicability ($Q \gtrsim 1.5$ GeV) does not extend to as small values of momentum transfer as that of a conventional pQCD analysis ($Q \gtrsim 1$ GeV). We suggest that the PMC range of applicability could be improved by a modified intermediate scheme choice or using a single effective PMC scale.

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1

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I. INTRODUCTION

The gauge theory of the strong interactions, Quantum Chromodynamics (QCD) is defined to provide objective predictions for physical observables; its predictions should not depend on arbitrary theory conventions, such as the choice of the gauge or the choice of renormalization scheme (RS). However, conventional calculations are typically carried out using a perturbative formalism where the truncated high-order predictions are RS-dependent. Furthermore, the n! growth of the n^{th} order coefficient of the resulting series –the renormalon problem [1]– makes the convergence of the series problematic, even at high momentum transfer where the QCD coupling α_s becomes small. A methodology to solve these problems has been developed, starting with the BLM procedure [2], extended by Commensurate Scale Relations [3], and culminating with the Principle of Maximum Conformality (PMC) [4–8].

The PMC provides a systematic method to eliminate the renormalization scheme and scale dependences of conventional pQCD predictions for high-momentum transfer processes. It reduces in the Abelian limit $(N_c \rightarrow 0)$ [9] to the QED Gell-Mann-Low scale-setting method [10], and it provides the underlying principle for the BLM procedure, extending it unambiguously to all orders consistent with renormalization group methods. The PMC has a solid theoretical foundation, satisfying renormalization group invariance [11, 12] and all other self-consistency conditions, such as reflexivity, symmetry, and transitivity derived from the renormalization group [13].

The PMC scales in the pQCD series are determined by shifting the arguments of the strong coupling $\alpha_s(Q^2)$ at each order n to eliminate all occurrences of the non-conformal $\{\beta_i\}$ -terms. The terms involving $\{\beta_i\}$ are identified at each order using the recursive pattern dictated by the renormalization group equation (RGE) [7, 8]. This unambiguous procedure determines the scales Q_n of the strong coupling at each specific order. As in QED, the PMC scales have a physical meaning in the sense that they are proportional to the virtuality of the gluon propagators at each given order, as well as setting the effective number n_f of active quark flavors. After applying the PMC, the divergent renormalizing the coupling to experiment at a single scale, the PMC predictions become scheme-independent. The PMC has been successfully applied to many high-energy processes; see, e.g., Ref. [14].

In this paper, we shall test the applicability of the PMC by comparing its prediction for

the evolution of the QCD strong coupling $\alpha_s(Q)$ to the corresponding prediction based on conventional scale-setting, where the renormalization scale at each order is estimated as a typical momentum transfer of the process and where arbitrary range and systematic error are assigned to estimate the uncertainty of the fixed-order pQCD predictions.

The PMC will be applied in this paper in order to determine the behavior of the running coupling $\alpha_{g_1}(Q)$, using the $\overline{\text{MS}}$ -scheme as an auxiliary RS. The coupling $\alpha_{g_1}(Q)$ is an "effective charge" [15] – i.e., an observable – defined from the Bjorken sum rule [16, 17]. It involves the spin-dependent g_1 structure function; hence, its name. The PMC prediction for $\alpha_{g_1}(Q)$ is RS-independent, whereas the conventional pQCD calculation of $\alpha_{g_1}(Q)$ retains RS-dependence, typically chosen as the $\overline{\text{MS}}$ scheme.

This article is organized as follow: In Sec. II, we recall the formalism which defines the $\alpha_{\overline{\text{MS}}}(Q)$ renormalization scheme and the pQCD expansion for the effective charge $\alpha_{g_1}(Q)$ using conventional pQCD scale-setting. In Sec. III, we provide the formulae which allow the computation of $\alpha_{g_1}(Q)$ using the PMC. In Sec. IV, we compare the two calculations. In Sec. V, we discuss the possibility of using the PMC in a procedure that employs α_s to relate the fundamental QCD parameter $\Lambda_{\overline{\text{MS}}}$ to hadron masses or, equivalently, to the confinement scale κ emerging from the Light-Front Holographic QCD approach to nonperturbative QCD [18]. We summarize the results in the final section.

II. PQCD COMPUTATION OF THE EFFECTIVE CHARGE α_{g_1} IN THE $\overline{\text{MS}}$ SCHEME

In the $\overline{\text{MS}}$ -scheme, the effective charge α_{g_1} has the leading-twist perturbative expansion [19]:

$$\frac{\alpha_{g_1}(Q)}{\pi} = \sum_{i \ge 1} a_i \left(\frac{\alpha_{\overline{\mathrm{MS}}}(Q)}{\pi}\right)^i.$$
(1)

The perturbative coefficients a_i are known up to four loops [20, 21]. (The values are given explicitly in Section III, Eq. 10.) The definition of α_{g_1} stems from the Bjorken sum rule [16, 17]. At leading-twist:

$$\int_{0}^{1^{-}} g_{1}^{p-n}(x_{Bj}, Q) \ dx_{Bj} = \frac{g_{a}}{6} \left[1 - \sum_{i \ge 1} a_{i} \left(\frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \right)^{i} \right] \equiv \frac{g_{a}}{6} \left[1 - \frac{\alpha_{g_{1}}(Q)}{\pi} \right], \tag{2}$$

where the integration runs over the Bjorken scaling variable x_{Bj} . The nucleon axial charge is g_a and the label *p*-*n* indicates the isovector part of the spin structure function g_1 . The Bjorken integral is well measured, including the transition region between perturbative to nonperturbative QCD [22]. The Q^2 -evolution of the strong coupling $\alpha_{\overline{\text{MS}}}(Q^2)$ in the $\overline{\text{MS}}$ scheme is governed by the RGE:

$$Q^{2} \frac{\partial}{\partial Q^{2}} \left(\frac{\alpha_{s}}{4\pi}\right) = \beta\left(\alpha_{s}\right) = -\sum_{n \ge 0} \left(\frac{\alpha_{s}}{4\pi}\right)^{n+2} \beta_{n}, \tag{3}$$

which is known up to 5-loops:

$$\begin{aligned} \beta_0 &= 11 - \frac{2}{3}n_f, \\ \beta_1 &= 102 - \frac{38}{3}n_f, \\ \beta_2 &= \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2, \\ \beta_3 &= \frac{149753}{6} + 3564\xi_3 - \left(\frac{1078361}{162} + \frac{6508}{27}\xi_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\xi_3\right)n_f^2 + \frac{1093}{729}n_f^3, \\ \beta_4 &= \frac{8157455}{16} + \frac{621885}{2}\xi_3 - \frac{88209}{2}\xi_4 - 288090\xi_5 + \left(-\frac{336460813}{1944} - \frac{4811164}{81}\xi_3 + \frac{33935}{6}\xi_4 + \frac{1358995}{27}\xi_5\right)n_f + \left(\frac{25960913}{1944} + \frac{698531}{81}\xi_3 - \frac{10526}{9}\xi_4 - \frac{381760}{81}\xi_5\right)n_f^2 + \\ &\quad \left(-\frac{630559}{5832} - \frac{48722}{243}\xi_3 + \frac{1618}{27}\xi_4 + \frac{460}{9}\xi_5\right)n_f^3 + \left(\frac{1205}{2916} - \frac{152}{81}\xi_3\right)n_f^4, \end{aligned}$$

where ξ_n is the Riemann zeta function [23, 24]. The coefficients β_i are expressed utilizing the $\overline{\text{MS}}$ -scheme except for β_0 and β_1 which are scheme independent.

Solving Eq. (3) iteratively yields the approximate five-loop expression of $\alpha_{\overline{\text{MS}}}^{\text{pQCD}}$ [25],

$$\begin{aligned} \alpha_{\overline{\text{MS}}}^{\text{pQCD}}(Q) &= \frac{4\pi}{\beta_0 t} \left[1 - \frac{\beta_1}{\beta_0^2} \frac{\ln(t)}{t} + \frac{\beta_1^2}{\beta_0^4 t^2} \left(\ln^2(t) - \ln(t) - 1 + \frac{\beta_2 \beta_0}{\beta_1^2} \right) + \frac{\beta_1^3}{\beta_0^6 t^3} \left(-\ln^3(t) + \frac{5}{2} \ln^2(t) + 2\ln(t) - \frac{1}{2} - 3\frac{\beta_2 \beta_0}{\beta_1^2} \ln(t) + \frac{\beta_3 \beta_0^2}{2\beta_1^3} \right) + \frac{\beta_1^4}{\beta_0^8 t^4} \left(\ln^4(t) - \frac{13}{3} \ln^3(t) - \frac{3}{2} \ln^2(t) + 4\ln(t) + \frac{7}{6} + \frac{3\beta_2 \beta_0}{\beta_1^2} \left(2\ln^2(t) - \ln(t) - 1 \right) - \frac{\beta_3 \beta_0^2}{\beta_1^3} \left(2\ln(t) + \frac{1}{6} \right) + \frac{5\beta_2^2 \beta_0^2}{3\beta_1^4} + \frac{\beta_4 \beta_0^3}{3\beta_0^4} \right) \right] \\ &+ \cdots, \end{aligned}$$

$$(4)$$

where $t = \ln (Q^2/\Lambda_s^2)$ and Λ_s is the asymptotic scale. Eqs. (1) to (4) allow us to compute $\alpha_{g_1}(Q)$ in the pQCD domain. Although α_{g_1} is an observable, the $\overline{\text{MS}}$ RS-dependence remains in its pQCD approximant due to the truncations of Eqs. (1) to (4).

III. PMC SCALE-SETTING FOR $\alpha_{g_1}(Q^2)$

Following the basic PMC procedure, we first identify the conformal and nonconformal pQCD contributions for α_{g_1} . The corresponding expression (1) is then reorganized as [8, 26]

$$\frac{\alpha_{g_1}(Q)}{\pi} = r_{1,0} \frac{\alpha_{\overline{\mathrm{MS}}}(Q)}{\pi} + (r_{2,0} + \beta_0 r_{2,1}) \left(\frac{\alpha_{\overline{\mathrm{MS}}}(Q)}{\pi}\right)^2 + (r_{3,0} + \beta_1 r_{2,1} + 2\beta_0 r_{3,1} + \beta_0^2 r_{3,2}) \left(\frac{\alpha_{\overline{\mathrm{MS}}}(Q)}{\pi}\right)^3 + (r_{4,0} + \beta_2 r_{2,1} + 2\beta_1 r_{3,1} + \frac{5}{2}\beta_0 \beta_1 r_{3,2} + 3\beta_0 r_{4,1} + 3\beta_0^2 r_{4,2} + \beta_0^3 r_{4,3}) \left(\frac{\alpha_{\overline{\mathrm{MS}}}(Q)}{\pi}\right)^4 + \cdots$$
(5)

where the coefficients $r_{i,0}$ for i > 0 are the conformal coefficients of pQCD for $\beta = 0$, and $r_{i,j}$ for i > 0, j > 0 are the non-conformal coefficients of the $\{\beta_i\}$ -terms.

Here as for Eq. (1), we have implicitly set the initial renormalization scale μ as Q, although as a basic property of PMC scale-setting, the determined scales of the coupling Q_i at each order turn out to be minimally dependent on the initial choice of scale. Any residual initial scale dependence at finite order in pQCD is highly suppressed, especially at the presently considered four-loop order. (One can test the initial scale dependence by recomputing the PMC predictions for $\mu \neq Q$; this can be conveniently done by applying the RGE.)

The conformal coefficients $r_{i,0}$ are:

$$\begin{aligned} r_{1,0} &= \frac{3}{4} \gamma_1^{\rm ns}, \\ r_{2,0} &= \frac{3}{4} \gamma_2^{\rm ns} - \frac{9}{16} (\gamma_1^{\rm ns})^2, \\ r_{3,0} &= \frac{3}{4} \gamma_3^{\rm ns} - \frac{9}{8} \gamma_2^{\rm ns} \gamma_1^{\rm ns} + \frac{27}{64} (\gamma_1^{\rm ns})^3, \\ r_{4,0} &= \frac{3}{4} \gamma_4^{\rm ns} - \frac{9}{8} \gamma_3^{\rm ns} \gamma_1^{\rm ns} - \frac{9}{16} (\gamma_2^{\rm ns})^2 + \frac{81}{64} \gamma_2^{\rm ns} (\gamma_1^{\rm ns})^2 - \frac{81}{256} (\gamma_1^{\rm ns})^4, \end{aligned}$$

and the non-conformal coefficients $r_{i,j}$ read:

$$\begin{aligned} r_{2,1} &= \frac{3}{4}\Pi_1^{\rm ns} + K_1^{\rm ns}, \\ r_{3,1} &= \frac{3}{4}\Pi_2^{\rm ns} + \frac{1}{2}K_2^{\rm ns} - \frac{\gamma_1^{\rm ns}}{4} \left(\frac{3}{2}K_1^{\rm ns} + \frac{9}{4}\Pi_1^{\rm ns}\right), \quad r_{3,2} = 0, \\ r_{4,1} &= \frac{3}{4}\Pi_3^{\rm ns} + \frac{1}{3}K_3^{\rm ns} - \frac{1}{4}\gamma_1^{\rm ns} \left(K_2^{\rm ns} + 3\Pi_2^{\rm ns}\right) - \frac{\gamma_2^{\rm ns}}{4} \left(K_1^{\rm ns} + \frac{3}{2}\Pi_1^{\rm ns}\right) + \frac{\left(\gamma_1^{\rm ns}\right)^2}{16} \left(3K_1^{\rm ns} + \frac{27}{4}\Pi_1^{\rm ns}\right), \\ r_{4,2} &= -\frac{3}{16} \left(\Pi_1^{\rm ns}\right)^2 - \frac{1}{4}K_1^{\rm ns}\Pi_1^{\rm ns}, \quad r_{4,3} = 0, \end{aligned}$$

where the expressions for γ_i^{ns} , Π_i^{ns} and K_i^{ns} are given explicitly in Refs. [20, 21].

As indicated by Eq. (5), because the running of $\alpha_{\overline{\text{MS}}}$ at each order has its own $\{\beta_i\}$ -series as governed by the RGE, the β -pattern for the pQCD series at each order is a superposition of all of the $\{\beta_i\}$ -terms which govern the evolution of the lower-order α_s contributions at this particular order. All known $\{\beta_i\}$ -terms should be absorbed into $\alpha_{\overline{\text{MS}}}$ at each order according to the RGE [7, 8], thus determining its correct running behavior at each order. Hence, after applying PMC scale-setting, only the conformal coefficients remain. The result is:

$$\frac{\alpha_{g_1}(Q)}{\pi} = \sum_{i \ge 1} r_{i,0} \left(\frac{\alpha_{\overline{\mathrm{MS}}}(Q_i)}{\pi}\right)^i.$$
(6)

The elimination of the divergent renormalon terms naturally leads to a pQCD series more convergent than the original one in Eq. (5). The PMC scales Q_i are functions of Q and read:

$$\ln \frac{Q_1^2}{Q^2} = -\frac{r_{2,1}}{r_{1,0}} - \frac{\beta_0 \left(r_{1,0}r_{3,2} - r_{2,1}^2\right)}{4r_{1,0}^2} \frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi}$$

$$(7)$$

$$\left[\beta_1^2 \left(-\frac{r_{3,1}^3}{4r_{1,0}^2} - \frac{2r_{3,2}r_{3,1}}{4r_{1,0}^2} - \frac{r_{4,3}}{4r_{1,0}^2} - \frac{r_{4,3}}}{4r_{1,0}^2} - \frac{r_{4,3}}{4r_{1,0}^2}$$

$$+ \left[\frac{\beta_{0}^{2}}{16} \left(-\frac{r_{2,1}^{2}}{r_{1,0}^{3}} + \frac{2r_{3,2}r_{2,1}}{r_{1,0}^{2}} - \frac{r_{4,3}}{r_{1,0}}\right) + \frac{\beta_{1}}{16} \left(\frac{3r_{2,1}^{2}}{2r_{1,0}^{2}} - \frac{3r_{3,2}}{2r_{1,0}}\right)\right] \left(\frac{\alpha_{\overline{\mathrm{MS}}}(Q)}{\pi}\right)^{2} + \mathcal{O}\left(\left(\frac{\alpha_{\overline{\mathrm{MS}}}}{\pi}\right)^{3}\right),$$

$$Q_{2}^{2} = r_{3,1} = 3\beta_{0} \left(r_{2,0}r_{4,2} - r_{3,1}^{2}\right) \alpha_{\overline{\mathrm{MS}}}(Q) + \mathcal{O}\left(\left(\frac{\alpha_{\overline{\mathrm{MS}}}}{8}\right)^{2}\right)$$
(8)

$$\ln \frac{Q_2^2}{Q^2} = -\frac{r_{3,1}}{r_{2,0}} - \frac{3\beta_0 \left(r_{2,0}r_{4,2} - r_{3,1}^2\right)}{8r_{2,0}^2} \frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} + \mathcal{O}\left(\left(\frac{\alpha_{\overline{\text{MS}}}}{\pi}\right)^2\right),\tag{8}$$

$$\ln \frac{Q_3^2}{Q^2} = -\frac{r_{4,1}}{r_{3,0}} + \mathcal{O}\left(\frac{\alpha_{\overline{\text{MS}}}}{\pi}\right).$$
(9)

These expressions show that the PMC scales Q_i are given as a perturbative series; any residual scale dependences in Q_i is due to unknown higher-order terms. This is the first kind of residual scale dependence; the contributions from unknown high-order terms are exponentially suppressed and are thus generally small.

A number of PMC applications have been summarized in the review [27]; in each case the PMC works successfully and leads to improved agreement with experiment. Furthermore, this multi-scale PMC approach corresponds to the fact that separate renormalization scales and effective numbers of quark flavors appear for each skeleton graph. The coefficients of the resulting pQCD series match the coefficients of the corresponding conformal theory with $\beta = 0$, ensuring the scheme-independence of the PMC predictions at any fixed order.

For convenience, we provide the conformal coefficients $r_{i,0}$ and PMC scales Q_i after substitution of the γ_i^{ns} , Π_i^{ns} and K_i^{ns} into Eq. (5). They are, up to four-loop order:

$$r_{1,0} = 1,$$

$$r_{2,0} = 1.6042 - 0.1528n_f,$$

$$r_{3,0} = 5.5335 - 1.7370n_f - 0.01980n_f^2,$$

$$r_{4,0} = 21.5613 - 8.4884n_f + 0.5050n_f^2 + 0.004503n_f^3,$$

$$\begin{aligned} \ln \frac{Q_1^2}{Q^2} &= -1.08333 + (3.2274 - 0.1956n_f) \frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \\ &+ (1.6076 - 0.2282n_f - 0.03532n_f^2) \left(\frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi}\right)^2 + \mathcal{O}\left(\left(\frac{\alpha_{\overline{\text{MS}}}}{\pi}\right)^3\right), \\ \ln \frac{Q_2^2}{Q^2} &= -\frac{5.2728 - 0.5918n_f}{1.6042 - 0.1528n_f} - \frac{0.08756n_f^3 - 3.00195n_f^2 + 32.6111n_f - 114.146}{0.02334n_f^2 - 0.4902n_f + 2.5734} \frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \\ &+ \mathcal{O}\left(\left(\frac{\alpha_{\overline{\text{MS}}}}{\pi}\right)^2\right), \\ \ln \frac{Q_3^2}{Q^2} &= -\frac{44.1983 - 8.7216n_f + 0.2165n_f^2}{5.5335 - 1.7371n_f - 0.01980n_f^2} + \mathcal{O}\left(\frac{\alpha_{\overline{\text{MS}}}}{\pi}\right). \end{aligned}$$

The PMC scale of the last known order, Q_4 , remains undetermined because the five-loop and higher order $\{\beta_i\}$ -terms are unknown. As a test, we can set $Q_4 = Q_3$ or $Q_4 = Q$, which leads to the second kind of residual scale dependence. This scale dependence, however, generates negligible uncertainty. For example, we have computed $\alpha_{g_1}(Q)$ using both prescriptions, and the results are nearly identical because of the fast convergence of the PMC series.

We note that the small values of Q (around 1 GeV), with $n_f = 3$ lead to an almost zero Q_3 ; this reflects the fact that in the soft Q-region, the intermediate gluons are effectively nonperturbative, and thus information on the behavior of α_s at low momentum is required.

We shall adopt a natural extension of the perturbative α_s -running behavior as determined from the high Q-region. Then, to avoid having Q_3 enter the nonperturbative region, we will use as the alternative scale $Q_3 = 40 \times Q$ [28]. Although we have also performed the calculations for values of n_f determined by the PMC scale Q_i , we will use $n_f = 3$ for the results in the next sections in order to compare meaningfully with the results reported in Refs. [29–31].

The results in this article use $\alpha_{g_1}(Q)$ computed with the scales Q_i calculated up to nextto-next leading order. However, for reference, we also provide here their values for $n_f = 3$ and at leading order:

$$Q_1 = 0.581Q,$$

 $Q_2 = 0.217Q,$
 $Q_3 = 40Q.$

It is informative to compare the coefficients a_i obtained from the conventional pQCD series, Eq. (1), to the PMC coefficients $r_{i,0}$. The a_i values for $n_f = 3$ are [20, 32]:

$$a_1 = 1,$$

 $a_2 = 3.58,$
 $a_3 = 20.21,$
 $a_4 = 175.7,$
 $a_5 \sim 893.38,$ (10)

which can be compared with the $r_{i,0}$ for $n_f = 3$:

$$r_{1,0} = 1,$$

 $r_{2,0} = 1.14583,$
 $r_{3,0} = 0.144097,$
 $r_{4,0} = 0.762723,$ (11)

The a_i values become very large at high orders, a manifestation of the factorial renormalon growth $(\alpha_s/\pi)^n \beta_0^n n!$ of pQCD series using conventional scale setting. In contrast, the conformal coefficients $r_{i,0}$ have reasonable values of order 1, as expected from the PMC procedure. This much-improved convergence allows for more precise predictions.

IV. COMPARISONS OF THE PMC AND CONVENTIONAL PREDICTIONS FOR THE BJORKEN SUM RULE

The PMC approach can be tested by comparing $\alpha_{g_1}(Q)$ computed using the PMC prediction (6) versus the conventional pQCD calculation (1). In each case, the prediction will be estimated up to fourth order and with $n_f = 3$. For these computations, we will evaluate



FIG. 1: (Color online) the PMC and conventional $\overline{\text{MS}}$ predictions for $\alpha_{g_1}(Q)/\pi$ computed for $n_f = 3$, with $\Lambda_{s,\overline{\text{MS}}}^{(n_f=3)} = 0.332(17)$ [33]. The various symbols represent the experimental determinations of $\alpha_{g_1}(Q^2)$ or $\alpha_{F_3}(Q^2)$.

 $\alpha_{\overline{\text{MS}}}$ up to five loops assuming $\Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 0.332(17)$ GeV [33], which is the current world average from various experimental and lattice QCD data using χ^2 minimization.

In Fig. 1, we display $\alpha_{g_1}(Q)/\pi$ calculated using the RS-independent PMC prediction versus the conventional pQCD in the $\overline{\text{MS}}$ -scheme, together with the available experimental data [19]. We also show the experimental data for $\alpha_{F_3}(Q)$, since the two effective charges α_{F_3} and α_{g_1} are in practice nearly identical [19]. We compute $\alpha_{g_1}(Q)$ for values of the argument of $\alpha_{\overline{\text{MS}}}(\mu)$ greater than 1 GeV, $\mu > 1$ GeV. In the conventional pQCD prediction of $\alpha_{g_1}(Q)$ the renormalization scale is directly set to Q and $\alpha_{g_1}(Q)$ is computed for Q > 1 GeV. For the PMC scale-setting calculation, $\mu > 1$ GeV implies that $\alpha_{g_1}(Q)$ is computed for Q > 1.48GeV, the reason for which will be discussed in the next subsection.

The total uncertainties of the two predictions stem from several sources:

- The uncertainty of the perturbative approximant for $\alpha_{\overline{\text{MS}}}$, which we estimate by taking the difference between the expressions of $\alpha_{\overline{\text{MS}}}$ at order β_3 and at order β_4 .
- The 17 MeV uncertainty on the value of $\Lambda_{\overline{\text{MS}}}^{(n_f=3)}$ [33];

• The truncation uncertainty in the PMC series (6) or in the conventional $\overline{\text{MS}}$ series (1). For the PMC series, it is estimated by taking the difference between the fourth order and third order terms: $(\alpha_{\overline{\text{MS}}}/\pi)^3 (r_{4,0}\alpha_{\overline{\text{MS}}}/\pi - r_{3,0})$. For the conventional $\overline{\text{MS}}$ pQCD series, it is taken as the difference between the estimated fifth order term and the calculated fourth order term: $(\alpha_{\overline{\text{MS}}}/\pi)^4 (a_5\alpha_{\overline{\text{MS}}}/\pi - a_4)$.

Fig. 1 shows that the four-loop PMC and the conventional pQCD calculations of $\alpha_{g_1}(Q)$ are consistent with each other, although only marginally for Q below a few GeV.

We have also performed the same calculations by computing the value of the quark flavor variable n_f , according to the quark mass threshold as determined by the value of Q, in the case of the conventional pQCD calculation of $\alpha_{g_1}(Q)$, or the values of the Q_i PMC scales for the PMC calculation. The results are similar to that shown in Fig. 1.

A notable feature in Fig. 1 is that the theoretical uncertainty of the PMC prediction is significantly smaller than that of the conventional pQCD prediction. As seen from Eqs. (10) and (11), this is due to the fact that the pQCD series using PMC scale-setting converges much faster than the conventional pQCD series.

V. MATCHING TO THE NONPERTURBATIVE DOMAIN

In Refs. [29, 30], a method has been proposed to relate the perturbative QCD asymptotic scale Λ_s to the hadron mass scale such as the proton mass. The scale Q_0 which signifies the transition between the perturbative and nonperturbative domains of QCD is also determined by this method. Both Λ_s and Q_0 are obtained in any renormalization scheme in the pQCD domain. This method uses the analytic form of α_{g_1} [34] predicted in the nonperturbative domain by Light Front Holographic QCD (LFHQCD) [18]:

$$\alpha_{g_1}(Q) = \pi \exp\left(-\frac{Q^2}{4\kappa^2}\right),\tag{12}$$

where κ is a universal nonperturbative scale derived from hadron masses, for example, $\kappa = M_{\rho}/\sqrt{2} = 0.548$ GeV, where M_{ρ} is the mass of the ρ -meson. Alternatively, κ can be obtained from fits to hadron form-factors, the Regge slopes, or the Bjorken sum rule Eq. (2). Although the value of κ is universal, in practice, the approximations used in LFHQCD induce a $\simeq 10\%$ variation. The latest determination gives $\kappa = 0.523(24)$ GeV [35]. The Gaussian form

Eq. (12) is in excellent agreement with data and the various nonperturbative calculations of $\alpha_s(Q)$ [30, 36], including the recent result based on Schwinger-Dyson Equations [37].

The basis for the matching procedure to determine Λ_s is the overlap of the domains of applicability of LFHQCD ($Q \leq 1.3 \text{ GeV}$) with the pQCD ($Q \geq 1.0 \text{ GeV}$) [38]. Continuity of $\alpha_{g_1}(Q)$ and its first derivative implies that Eq. (1) and Eq. (12), as well as their corresponding β -functions, can be equated in the overlap region. The simultaneous solution to these two equations provides an analytical relation between Λ_s and κ , as well as the transition scale Q_0 . This leads to a determination of $\Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 0.339(19) \text{ GeV}$ [31], with a precision on par with that of the averaged world data of 0.332(17) GeV [33].

Since the PMC provides a more precise determination of $\alpha_{g_1}(Q)$ than conventional renormalization scale-setting, it is interesting to investigate if the procedure is also applicable using Eq. (6) rather than Eq. (1) to improve the determination of Λ_s .



FIG. 2: (Color online) Matching procedure applied to PMC calculation (blue line). It is matched to the LFHQCD results (magenta line) by requiring the continuity of both α_{g_1} and its β -function. The blue band is the PMC prediction evaluated down to Q = 1.5 GeV, and the red band shows the conventional $\overline{\text{MS}}$ prediction from Ref. [31]. We use $\kappa = 0.523$ GeV for LFHQCD.

Following the same matching procedure, we have computed the PMC prediction using $\kappa = 0.523$ GeV. To reach the matching point Q_0 , it is necessary to extrapolate the PMC

prediction down to Q = 1 GeV, which implies that the $\alpha_{\overline{\text{MS}}}(\mu)$ must be extrapolated down to $\mu = 0.68$ GeV. The result is shown in Fig. 2. As a comparison, we also show the conventional $\overline{\text{MS}}$ prediction [31] in the figure.

The matching of the PMC prediction to LFHQCD yields a large value for $\Lambda_{s,\overline{\text{MS}}}^{(n_f=3)} = 0.406(17)$ GeV. This explains why, compared to Fig. 1, a better agreement between the matched PMC curve (blue line) and the conventional $\overline{\text{MS}}$ pQCD calculations (red band) is observed in Fig. 2.

The determined transition scale, $Q_0 = 1.14$ GeV, is below the scale at which the present PMC calculation is applicable ($Q \approx 1.48$ GeV). The failure of this self-consistency check indicates that the matching procedure cannot be used with the PMC calculation, at least when $\overline{\text{MS}}$ is used as an auxiliary RS. This explains why the matching procedure yields $\Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 0.406(17)$ GeV, which is somewhat larger than the world data. This is reflected in Fig. 2 by the fact that the blue line does not lie within the blue band.

In the case of conventional scale-setting, the renormalization scale μ is fixed at its initial value Q. In contrast, as shown by Eqs. (8) and (9), the determined PMC scale Q_i for each order is a function of Q which can result in Q_i scales that are larger or smaller than Q. This has consequences for the matching procedure proposed in Ref. [29], which requires that the transition between nonperturbative and perturbative QCD occurs at a point Q_0 rather than over a non-zero Q range.

In the case of conventional scale-setting, the meaning of the inflection point Q_0 is unambiguous: $\alpha_s(Q)$ has perturbative behavior for $Q > Q_0$ and nonperturbative behavior for $Q < Q_0$. These are determined by pQCD and LFHQCD, respectively. On the other hand, in the case of the PMC scale-setting, some PMC scales are smaller than the determined Q_0 , thus leading to an apparent incompatibility; i.e., if the determined PMC scale Q_i is less than Q_0 , the meaning of Q_0 is questionable since Q_i is now within the nonperturbative region. This is indeed the case for the present procedure. Thus, due to the fast convergence of PMC series, we have $\alpha_{g_1}(Q) \sim \alpha_{\overline{\text{MS}}}(Q_1)$, where the PMC scale $Q_1 = 0.45$ GeV is significantly smaller than the transition scale $Q_0 = 1.14$ GeV. This conflict could be due to the fact that some of nonperturbative effects which are not accounted for in the (perturbative) derivation of the PMC scales Q_i , such as those from the high-twist terms [39], may have already come into the higher-order calculations. For example, the renormalization scale for the heavy-quark loop which appears in the three-gluon coupling depends nontrivially on the virtualities of the three gluons entering the three-gluon vertex [40].



FIG. 3: (Color online) The strong coupling $\alpha_s(Q^2)$ for various renormalization schemes [30]. The lines in the perturbative region are the perturbative calculations done at order β_3 . The dashed curves are their matched LFHQCD continuations into the nonperturbative domain.

This problem may be solved by transforming to a different MS-like scheme; e.g., the R_{δ} scheme [7, 8] where the subtraction $\ln 4\pi - \Gamma_E - \delta$ is used within the minimal subtraction procedure. (The conventional MS-scheme is the R_{δ} -scheme corresponding to $\delta = 0$.) The scheme transformation between different R_{δ} -schemes corresponds simply to a displacement of their corresponding scales; $\mu_{\delta}^2 = \mu_{\overline{MS}}^2 \exp(\delta)$; thus a proper choice of δ may avoid the small scale problem found for the $\overline{\mathrm{MS}}$ -scheme. This problem may also be solved by using a different auxiliary RS, such as the MOM scheme with $\xi = 0$ (Landau gauge) [41], or the V scheme [42]. This possibility is motivated by comparing the running behaviors of α_s for different schemes; examples are presented in Fig. 3. It shows that to ensure the schemeindependence of the couplings, e.g. $\alpha_{\overline{MS}}(\mu_{\overline{MS}}) = \alpha_V(\mu_V) = \alpha_{MOM}(\mu_{MOM})$, we must have $\mu_{\overline{\text{MS}}} < \mu_{\text{V}} < \mu_{\text{MOM}}$. This fact has been observed by the LO commensurate scale relations among different effective couplings [3]. Thus a larger PMC scale can be achieved when the V-scheme or MOM-scheme is adopted as the auxiliary RS. For example, in the case of the V-scheme, the PMC prediction is applicable down to Q = 1 GeV if the perturbative behavior of $\alpha_{\overline{\text{MS}}}(\mu)$ can be extrapolated down to $\mu = 0.85 \text{ GeV} [3]$, which is larger than the corresponding value of $\mu = 0.68$ GeV required for the MS-scheme.

Another avenue to address the problem could be to use the single-scale approach for the PMC [43], where a single effective scale replaces the individual PMC scales in the sense of a mean value theorem; this can avoid the small scale problem which can appear at specific orders in the multi-scale PMC approach. These investigations will be reported in a future publication.

VI. SUMMARY AND CONCLUSION

In this paper, we have tested the PMC scale-setting procedure by comparing its prediction in the $\overline{\text{MS}}$ scheme for the effective charge $\alpha_{g_1}(Q)$ defined from the Bjorken sum rule with the prediction obtained using conventional renormalization scale-setting. To this end, we have calculated the necessary PMC coefficients and renormalization scales. We have verified that the PMC series converges much faster than the conventional $\overline{\text{MS}}$ pQCD series, which results in a significantly smaller uncertainty for the PMC pQCD prediction. Thus the central objective of the PMC is realized: it provides a determination of $\alpha_{g_1}(Q)$ compatible with the data and the conventional pQCD calculation, but without scheme-dependence and with significantly improved precision.

As an important application, we have investigated the possibility of determining $\Lambda_{\overline{\text{MS}}}$ from hadronic scales by matching the PMC calculation for pQCD to the nonperturbative lightfront holographic QCD prediction for $\alpha_{g_1}(Q)$. This had been done previously using the conventional scale-setting pQCD prediction; this worked well, giving $\Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 0.339(19)$ GeV. However, we have found that the domain of applicability of the nonperturbative LFHQCD and the domain of applicability of the perturbative PMC predictions do not overlap if the $\overline{\text{MS}}$ -scheme is used as the auxiliary scheme, causing the matching procedure to fail. This problem arises from the fact that the PMC scales at certain orders in the $\overline{\text{MS}}$ -scheme are, in some cases, smaller than the transition scale Q_0 . A detailed investigation for solving this problem, using alternative renormalization schemes and/or the single-scale PMC procedure for pQCD, is in preparation.

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1.0 GeV for the lower limit of applicability of conventional pQCD in the $\overline{\text{MS}}$ RS is determined as the value where α_{g_1} from conventional pQCD is 10% larger than the LFHQCD prediction. This agrees with the typical prescription that pQCD is applicable for Q > 1 GeV.

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