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Geometric Representation of Fundamental Particles' Masses

L.Schachter[1] and J. Spencer[2]

[1] *Technion-Israel Institute of Technology, Haifa 32000, Israel*

[2] *SLAC National Accelerator Laboratory, Menlo Park, CA 94025-7015, USA*

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A geometric representation of the lowest lying inertial masses of the known particles (N=312) was introduced by employing a Riemann Sphere facilitating the interpretation of the N masses in terms of a single, hypothetical particle we call the Masson (M). Geometrically, its mass is the radius of the Sphere. Dynamically, its derived mass is near the mass of the only stable hadron regardless of whether it is determined from all N particles or only the hadrons (294), the mesons (160) or the baryons (134) separately. Ignoring all other properties of these particles, it is shown that the eigenvalues, the polar representation θ_ν of the masses on the Sphere, satisfy the symmetry $\theta_\nu + \theta_{N+1-\nu} = \pi$ within less than 1% relative error for the fundamental particles. These pair correlations form at least 6 distinct clusters for all N particles. In the limit of no quark-quark interactions the Masson becomes a quark and the meson assumes a mass of 2M.

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Spanning from zero to more than 100GeV, we introduce a geometric representation of the masses of the known particles allowing us to posit a generating particle - the Masson (pronounced as one does the Muon). Associated with it, there is a generating function whose zeros are the normalized masses of the N particles[1, 2]. These masses can then be projected onto a 2D Riemann Sphere[3] of radius equal to the mass of the Masson that is determined by imposing the equivalent of a minimum action criterion; throughout this study whenever we refer to mass, our intention is to the *inertial mass*.

We do *not* consider antiparticles here because there has never been a fermion discovered that did not lead to the discovery of its corresponding antiparticle of identical mass as first implied by Dirac[4]. The only particle we fully understand is the photon with zero mass that must move at the speed of light because there is no rest frame to measure the mass explicitly based on $m/\sqrt{1-\beta^2}$. Thus, while we know how to determine the extreme, in general, we do not know the fundamentals underlying the other values beyond the interesting work of Brodsky and collaborators[5] based on QCD.

In contrast, we consider the mass as a direct mechanical observable. We do know, according to Sommerfeld[6], that it is not associated with the charge alone. He pointed out that given a macroscopic charge of finite radius and mass, the energy associated with the two is different. His approach was simple: denoting by $E_{\rm EM}^{(\rm rest)}$ the electrostatic energy of the charged particle when at rest and subtracting this energy from the electric and magnetic energy when the particle is in motion $E_{\rm EM}^{(\rm motion)}$, it was shown that the difference does not equal the kinetic energy of the particle.

Here we introduce a geometric (polar θ_ν) representation of the N masses on a Riemann Sphere. This allows us to interpret them in terms of a single particle, the Masson, that may be in one of the N states and whose mass M we take as the radius of the Sphere as shown in Figure 1. Ignoring the other properties of these particles, it is shown that these values satisfy the symmetry $\theta_\nu + \theta_{N+1-\nu} = \pi$ within less than 1% relative error. These eigenvalues form at least 6 clusters suggestive of a "Periodic" Chart of the Fundamental Particles. This mapping is not unique but was chosen for its simplicity whereas others might reveal further relationships.

Specifically, the masses are organized in ascending order along the horizontal axis "x". A circle of radius M has its center at x=0, z=M and the intersection of the straight-line, connecting the top of the circle with z=0, x=m_ν defines a unique angle θ_ν on the sphere for each particle m_ν given by

$$\theta_\nu = 2\arctan\left(\frac{2\mathrm{M}}{m_\nu}\right). \tag{1}$$

This transformation represents the projection of any one of the masses on the sphere whose radius we attribute to the mass of the Masson. It preserves the original order of the masses. Next we establish M based on the experimental data and a minimal action criterion. We note that the masses were organized in ascending order and

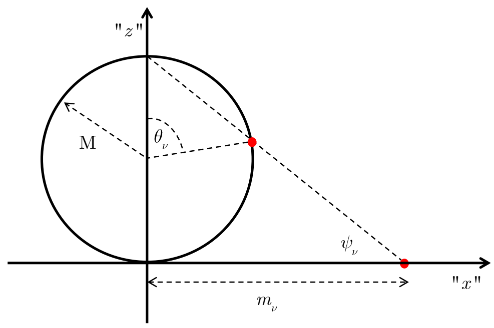

FIG. 1: The mass of a particle is marked on the axis (red-dot). Projection of the mass of the particle on the Riemann Sphere, whose radius represents the mass of the Masson M, is uniquely determined by the polar angle θ_ν.

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that we defined the interval-spread of any two adjacent angles as

$$\mathcal{E}\,(\mathrm{M}) = \frac{1}{\pi}\sqrt{\frac{1}{N+1}\sum_{\nu=0}^{N}\left(\theta_{\nu+1}-\theta_{\nu}\right)^2}\ . \tag{2}$$

M is the value that *minimizes* this functional; $\theta_{\nu=0} = 0$ and $\theta_{\nu=N+1} = \pi$ represent the upper and lower limits of the masses in this polar representation.

For the trivial case of a *single* particle, its mass is represented by an angle θ and there are two intervals: $\theta - 0$ and $\pi - \theta$ so the intervals spread is proportional to $\theta^2 + (\pi-\theta)^2$ with a minimum at $\theta = \pi/2$ implying that the radius of the sphere is half the mass of the particle i.e. M$= m/2$ or, equivalently, the particle's mass is twice the mass of the Masson: m = 2M consistent with the structure of mesons as being composed of two weakly interacting quarks of mass M.

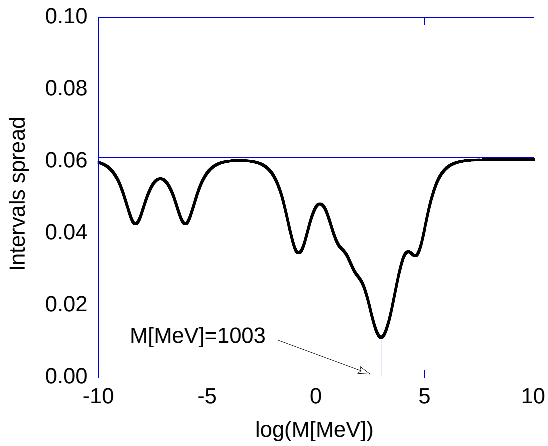

FIG. 2: Spread of intervals for the N=312 particles as a function of M. The dominant minimum is calculated numerically and occurs at M[MeV] = 1003 near the lowest lying baryon mass which is the only stable hadron mass.

When considering the entire ensemble of particles (ν=1,2,...N) their spread intervals in Figure 2 clearly shows resonance-like behavior. The absolute minimum, occurring at 1003 MeV, we take to be the mass of the Masson. The 2D Riemann Sphere is illustrated in Figure 3 for this value. Two facts are evident – first, as anticipated, most of the particles are located in the $\theta \sim \pi/2$ region and, second, close to zero and π there are voids although these are not symmetrically disposed nor correlated in any obvious way.

Repeating the same procedure but only for hadrons, baryons or mesons separately, leads only to relatively small deviations from the above value for the Masson's mass. To begin, consider only the hadrons ($N = 294$). If we were to establish the Masson based on the hadrons alone, its mass would be only slightly reduced to M$^{(\mathrm{H})}$=962.2 MeV. Moreover, if we attribute a separate

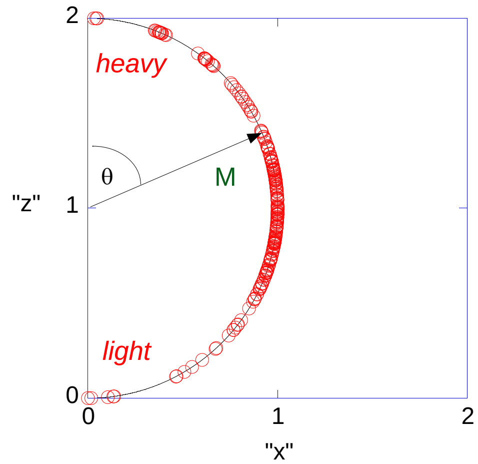

FIG. 3: Projection of the masses of all 312 particles where the mass of the Masson is determined from the requirement that the spread of the intervals in Figure 2 is minimal. Light particles ($\theta \sim \pi$) are the gamma, gluon and neutrinos. The heavy ones ($\theta \sim 0$) the gauge-particles, Higgs and top quark.

Masson to baryons ($N = 134$) and to mesons ($N = 160$) the corresponding masses would be M$^{(\mathrm{B})}$= 1094 MeV and M$^{(\mathrm{M})}$=964 MeV. All of these and especially M$^{(\mathrm{H})}$ and M$^{(\mathrm{M})}$ are close to both M as well as to the only stable hadron mass, the nucleon N(940). Also, there are more mesons than baryons even though their confined quarks(2) are fewer than for the baryons(3). Their corresponding "intervals spread", similar to Figure 2 for all particles, gave a *single* comparable minimum for M.

Another perspective on the polar representation of the masses can be obtained by ordering the $\{\theta_\nu\}$ in ascending order and plotting them as a function of the normalized index ν (quantum number) as the red squares in Figure 4. For comparison, the N zeros of the Legendre polynomial of order $N = 312$ are organized in ascending order and given by the black diamonds $[P_N(\cos\zeta_\nu) = 0;\ \nu = 1, 2, ...N]$. While the latter is virtually linear, the former has a more complex structure with distinct "band-gaps" in the range $\nu < 0.2N$ and $\nu > 0.9N$.

At least four observations can now be made: (i) the Legendre function zeroes are *linear* on the index (ν/N). (ii) If the absolute value of the argument of the Legendre polynomial is larger than unity the behavior is hyperbolic and the function has no zeros in this range which is consistent with the existence of band-gaps. (iii) Having in mind that the argument of the Legendre polynomials ($\cos\theta$) varies between -1 and 1, we consider another function which is defined in this range (tanh) and we calculate the zeros of $P_N\left[\tanh\left(3.46\left(\pi/2-\theta_\nu\right)\right)\right] = 0$ which are represented by the green squares in Figure 4. (iv) In the range $0.2 < \nu/N < 0.9$ the dependence of θ_ν on the index is *linear* with an accuracy of 0.07% being defined as $100 \times \left\langle [1 - \theta_\nu^{(\mathrm{Linear})}/\theta_\nu]^2 \right\rangle_\nu$.

Further, these results indicate that the $\{\theta_\nu\}$ might be regarded as the eigenvalues of a characteristic polynomial

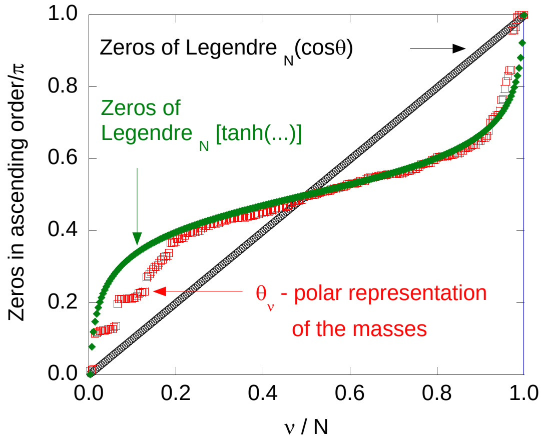

FIG. 4: Red squares represent the masses (θ_ν) in ascending order and the black diamonds the zeros of the modified Legendre function of order $N = 312$ whose argument is the hyperbolic tangent function. The green squares are discussed in the text below. The index ν is normalized by N.

of the Legendre type. These comparisons to the zeros of the Legendre polynomials suggests the possible relevance of the $\{\theta_\nu\}$ to the dynamics of the Masson which is explored next. In this our approach was inspired by the work of Liboff and Wong[7] in connection with their study of the prime numbers and the zeta function.

One of the main results of our approach relies on a property of the Legendre polynomials that the sum of *two zeros* of complementary order ($\nu + \nu' = N + 1$) equals π, or explicitly $\zeta_\nu + \zeta_{N+1-\nu} = \pi$. We have examined to what extent this rule applies to the polar representation of the masses θ_ν and found that $\theta_\nu + \theta_{N+1-\nu} = \pi\chi$ with $\chi = 0.958$ within 0.13% relative error defined as

$$\text{Error}[\%] = 100 \frac{1}{2N} \sum_{\nu=1}^{N} \left[\frac{\theta_\nu + \theta_{N+1-\nu} - \pi\chi}{\theta_\nu + \theta_{N+1-\nu}} \right]^2 . \quad (3)$$

The factor of 2 in Eq.(3) corrects the fact that each pair of masses is counted twice. According to the present spectrum of masses [1], this relation implies that the mass of the Higgs and that of the Axion (if verified) would be related $\theta_{\text{Axion}} + \theta_{\text{Higgs}} \simeq \pi$, that the mass of the electrons neutrino is related to that of the Z-gauge boson $\theta_{\nu_e} + \theta_Z \simeq \pi$ and that the $W^\pm$ gauge bosons are related similarly to the other neutrinos. However, it should be emphasized that the present estimate of the error is dominated by the light particles with $\theta \sim \pi$ and that it is larger if the deviation is compared to the smallest angle between the two. In fact, due to uncertainty associated with the measurements of many of those masses and especially the neutrinos, comparing to the calculated deviation of χ from unity, one can hypothesize that $\chi \equiv 1$ or explicitly

$$\theta_\nu + \theta_{N+1-\nu} = \pi \ . \quad (4)$$

For further insight into this result, we plot in Figure 5 the normalized symmetry-pairs $(\theta_\nu + \theta_{N+1-\nu})/\pi$ as a function of the normalized masses (θ_ν/π). Several interesting aspects are reflected in this plot: (i) the pairs linked by Eq.(4) form (at least) six clusters. (ii) The error or deviation from unity is dominated by light particles ($\theta \sim \pi$). When both particles have similar mass, the deviation is negligible as shown by the right cluster. (iii) Further splitting is expected when including additional quantum numbers that are expected to produce a Riemann hypersphere. (iv) Subject to the condition $\chi \equiv 1$, the error defined above for hadrons is 0.47%, for baryons 0.07% and for mesons it is 0.63%.

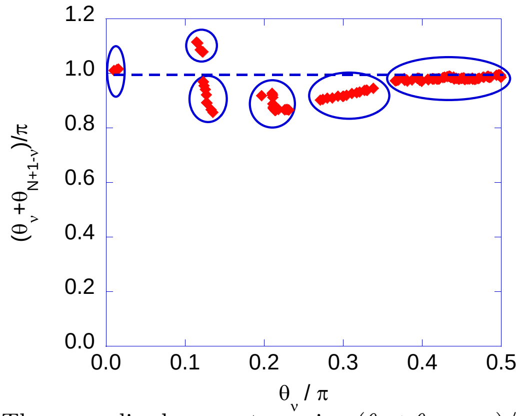

FIG. 5: The normalized symmetry-pairs, $(\theta_\nu + \theta_{N+1-\nu})/\pi$, as a function of the normalized geometric representation of the masses (θ_ν/π). These pairs form at least six clusters suggestive of a "Periodic Table" for the fundamental particles.

Before proceeding it is important to assess whether such small errors are not the result of pure coincidence. For this purpose let us postulate that the Masson has a fixed inertial mass of M_0=1003 MeV and between the two extremes the various particles (312) are randomly distributed. We represent the inertial masses in terms of a random variable $m_\nu\,[\text{MeV}] = 10^{p_\nu}$ wherein p_ν is uniformly distributed $-8 \leq p_\nu \leq 5 + \log(1.26) = 5.104$. As in the case of the real particles, we employ the transformation in Eq. 1. It is tacitly assumed that the mass of the Masson is not dependent on the specific distribution. Once the θ_ν are established, the error is calculated based on Eq. 3 with $\chi \equiv 1$.

Figure 6 illustrates the errors, $7.5 \leq \text{Error}[\%] \leq 10.5$, associated with the polar representation of a random distribution of masses for 1000 different seeds based on using Eq. 3 with $\chi = 1$. For comparison, in the case of using all of the actual particles, its value is 0.225% indicating that the roughly two orders of magnitude (8%) difference is not a result of coincidence.

So far we have focused our attention on the kinematics of the Masson. Based on the observations discussed above, we can make a few assessments regarding the dynamics. Our starting point is our first observation of the *linear* dependence of the zeros of the Legendre polynomial of order N namely, $\xi_\nu \sim \pi\nu/N$ wherein

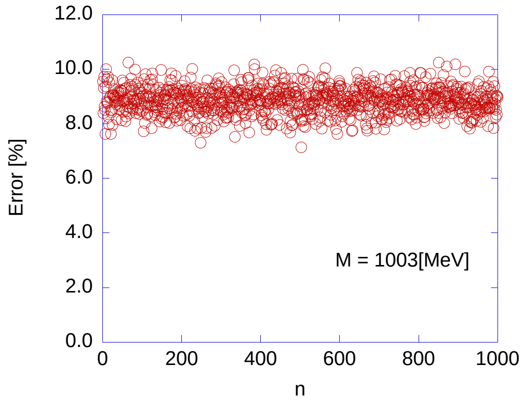

FIG. 6: The errors associated with the polar representation of a random distribution of masses for 1000 different seeds based on using Eq. [3]. For comparison, in the case of using all of the actual particles, its value is 0.225%.

$P_N(\cos\xi_\nu) \equiv 0$. The latter, $\psi_N^{(\mathrm{Leg})}(\theta) = P_N(\cos\theta)$, is a solution of Legendre's equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + N(N+1)\right]\psi_N^{(\mathrm{Leg})}(\theta) = 0. \qquad (5)$$

By analogy, the polar representation of the masses θ_ν are the N zeros of $\psi_N^{(\mathrm{Masson})}(\theta) \simeq \Pi_{\nu=1}^{N}(1-\theta/\theta_\nu)$ which is a solution of the "Legendre-like" differential operator

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + V_N(\theta)\right]\psi_N^{(\mathrm{Masson})}(\theta) = 0 \qquad (6)$$

with $V_N(\theta)$ representing the normalized potential that is responsible for the difference between ξ_ν and θ_ν. In Supplement A we discuss the Table of Masses and partition it into twenty natural subgroups. In Supplement B we show that since the zeroes of the solution $[\psi_N^{(\mathrm{Masson})}(\theta)]$ determine up to a constant its polynomial representation and employing the orthogonality of the Legendre polynomials then the normalized potential $V_N(\theta) = \sum_{n=0}^{\infty}\alpha_n P_n(\cos\theta)$ is given by

$$\vec{\alpha} = \mathbf{D^{-1}}\vec{b} \qquad (7)$$

with $b_n = a_n 2n(n+1)/(2n+1)$ and the matrix $D_{n,m}$ as

$$\sum_{n'=0}^{\infty} a_{n'} \int_0^{\pi} d\theta \sin\theta P_n(\cos\theta)P_{n'}(\cos\theta)P_m(\cos\theta) \ . \qquad (8)$$

However, it is important to appreciate the more general significance of this potential i.e. in a fully coupled-channel representation of the fundamental particles and their interactions.

In conclusion, a geometric representation of the inertial masses of the reasonably established, lowest lying hadrons was introduced by employing a Riemann Sphere. It allowed us to interpret their masses in terms of a single entity, the Masson, that might be in one of the N eigenstates but whose mass, in the limit of weakly interacting quarks, was the quark mass itself.

Geometrically, the mass of the Masson was the radius of the Riemann Sphere while its numerical value was closest to the mass of the proton, the only stable hadron, regardless of whether it was computed from all of the particles (312), the hadrons (294), or just the mesons (160) or baryons (134) separately clearly reflecting the strong interactions between confined quarks.

Ignoring the other properties of these particles, it was shown that the eigenvalues, the polar representation of the masses θ_ν on the sphere, satisfied the symmetry $\theta_\nu + \theta_{N+1-\nu} = \pi$ within less than 1% relative error. A function was established (to be discussed in a subsequent paper) whose zeros were, to good approximation, the polar representation of the masses θ_ν.

Although we did not include antiparticles, they are important for cosmology where the lack of any apparent antimatter in the universe is an ongoing scientific question[8]. We did not consider gravity notwithstanding a new result on the mass of the graviton $m_g < 7.7{\times}10^{-17}\mathrm{MeV/c^2}$[9]. Because the only stable hadron is the relatively heavy proton, presumably because it contains no antiquarks, one sees the weakness of using only classical concepts to understand the microscopic particle world. Finally, we note that different mappings than the one used here could well reveal additional relations comparable to Eq. 4 as given in the previous paragraph.

Finally, we acknowledge one of the great scientific achievements of the last two centuries on this approximate anniversary of Mendeleev's Periodic Table of the Chemical Elements. Also, we thank Stan Brodsky, Achim Weidemann and Kent Wootton for their comments on our manuscript. This work was supported in part by Department of Energy contract DE-AC02-76SF00515.

[1] Rev. Part. Prop., Phys. Lett. B, Vol. **667/1** (1-1340) (2008); Rev. Part. Phys., J. Phys. G, Vol. **37**, 7A(1-1422) (2010) and Chin. Phys. C, **40**, 100001 (2016). The Table of Masses can be accessed from the link in Supplement A. In Supplement B we show that the zeroes of the solution $[\psi_N^{(\mathrm{Masson})}(\theta)]$ determine up to a constant its polynomial representation.

[2] We take the "rest mass" as simply the "mass" – a relativistic invariant with neither "rest" nor subscript "0" appended. The observed masses [1] are understood to be greater than the bare masses due to self interaction contributions. The Axion was included but not the graviton even though neither of these has yet been observed.

[3] Our definition is from *The Encyclopedia of Mathematics*.

[4] P. A. M. Dirac, "The Quantum Theory of the Electron", Proc. Roy. Soc.**A:** Math., Phys. & Eng. Sci.**117,** 610 (1928).

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[6] Sommerfeld A. , *Electrodynamics* in *Lectures on Theoretical Physics*, (Academic Press, New York, 1952), pp. 278.

[7] Richard L. Liboff and Michael Wong, "Quasi-Chaotic Property of Prime-Number Sequence", Int. J. Theo. Phys. **37,** 3109-3117 (1998).

[8] Large (1 km across) isolated clouds of positrons may have been observed recently but with short lifetimes (0.2 s); J.R.Dwyer et al., J. Plasma Physics **81**, 475810405 (2015).

[9] B.P. Abbott et al., GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett., PRL **118** 221101 (2017).

SUPPLEMENT A

The Table of Masses used here was developed from Ref. [1]. It shows 3 groups of 6 particles each comprising the leptons, gauge bosons, and quarks as well as the lowest lying 294 composite hadrons such as the light, unflavored mesons with B=C=S=0, the strange mesons with S=±1 and B=C=0 and so on together with the individual quantum numbers for each particle. In this way, it is partitioned into 20 natural subgroups. The Table is available from the authors or as a SLAC Report: SLAC-R-1081.

SUPPLEMENT B

Our goal here is to demonstrate that given the solution of Eq.(6) $[\psi_N^{(\mathrm{Masson})}(\theta)]$ in its polynomial representation (in terms of its zeroes), we can determine the effective normalized potential responsible for the Masson's dynamics. Since the Legendre Polynomials form a complete orthonormal set

$$\int_0^\pi d\theta \sin\theta P_n(\cos\theta) P_{n'}(\cos\theta) = \frac{2}{2n+1}\delta_{n,n'},$$

we may expand both the solution function $\psi_N^{(\mathrm{Masson})}(\theta)$ and the normalized potential $V_N(\theta)$ in terms of the Legendre functions, namely

$$\psi_N^{(\mathrm{Masson})}(\theta) \simeq \Pi_{\nu=1}^N (1-\theta/\theta_\nu) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta)$$

and

$$V_N(\theta) = \sum_{n=0}^{\infty} \alpha_n P_n(\cos\theta).$$

Explicitly, our goal is to establish α_n in terms of a_n. Substituting $\sum_{n=0}^{\infty} a_n P_n(\cos\theta)$ in Eq.6 we get

$$\sum_{n=0}^{\infty} a_n \left[-n(n+1) + V_N(\theta)\right] P_n(\cos\theta) = 0.$$

Next we substitute the expression for $V_N(\theta)$

$$\sum_{n=0}^{\infty} a_n \left[-n(n+1) + \sum_{m=0}^{\infty} \alpha_m P_m(\cos\theta)\right] P_n(\cos\theta) = 0.$$

Note that the normalization coefficient of the solution can not affect the normalized potential (α_n). Our last step is to multiply the last equation by $P_{n'}(\cos\theta)$ and employ the orthogonality condition

$$\sum_{n,m=0}^{\infty} \alpha_m \int_0^\pi d\theta \sin\theta P_m(\cos\theta) P_n(\cos\theta) P_{n'}(\cos\theta) = a_{n'} \frac{2n'(n'+}{2n'+1}$$

In the text [Eq.(7)] this expression is given in matrix and vector form with the corresponding definitions.