The Principle of Maximal Transcendentality and the Four-Loop Collinear Anomalous Dimension

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ABSTRACT: We use the principle of maximal transcendentality and the universal nature of subleading infrared poles to extract the analytic value of the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ super-Yang-Mills theory from recent QCD results, obtaining $\hat{\mathcal{G}}_0^{(4)} = -300\zeta_7 - 256\zeta_2\zeta_5 - 384\zeta_3\zeta_4$. This value agrees with a previous numerical result to within 0.2%. It also provides the Regge trajectory, threshold soft anomalous dimension and rapidity anomalous dimension through four loops.

Dedicated to the memory of Lev Lipatov

Contents

1 Introduction

In the modern approach to scattering amplitudes, $\mathcal{N} = 4$ super Yang-Mills (SYM) theory [1, 2] has played a key conceptual role, especially in the planar limit of a large number of colors where the theory becomes integrable [3, 4, 5, 6, 7, 8] and its amplitudes become dual to Wilson loops [9, 10, 11, 12, 13, 14, 15]. So much is known about the analytic structure of scattering amplitudes in planar $\mathcal{N} = 4$ SYM that the first amplitude with non-trivial kinematic dependence, the six-point amplitude, can be bootstrapped to at least five loops [16, 17, 18, 19, 20, 21, 22].

Ironically, the finite parts of the six-point amplitude (the remainder function and ratio function), which are polylogarithmic functions of three variables, are now known to higher loop orders than is the dimensionally-regulated infrared-divergent prefactor — the BDS ansatz [23] — even though the latter depends only on four constants per loop order. One of these constants, the (light-like) cusp anomalous dimension [24, 25] is known to all orders in planar $\mathcal{N} = 4$ SYM, thanks to integrability [5]. The cusp anomalous dimension controls the double pole in ϵ in the logarithm of the dimensionally regularized BDS ansatz. The single pole is controlled by the "collinear" anomalous dimension. In planar $\mathcal{N} = 4$ SYM, it is known analytically through three loops [23, 26], and it was computed numerically at four loops a decade ago [27]. The nonplanar contribution to the four-loop collinear anomalous dimension was computed numerically very recently [28]. The collinear anomalous dimension also enters the Regge trajectory for forward scattering [10, 29, 30]. An eikonal (Wilson line) version of it enters both the threshold soft anomalous dimension for threshold resummation [31, 32, 33, 34] and the rapidity anomalous dimension for transverse momentum resummation [34, 35, 36, 37].

The BDS ansatz also contains two finite constants at each loop order, one for the four-point amplitude and one for the five-point amplitude. One of these constants is known analytically at three loops [23], but the other is only known numerically at this loop order [38].

The purpose of this paper is to provide an analytical value for one of the four constants in question at four loops, namely the collinear anomalous dimension in planar $\mathcal{N} = 4$ SYM. We do so by leveraging two recent four-loop computations in QCD in the large N_c limit [39, 40], as well as the principle of maximal transcendentality [41, 42, 43, 44]. This principle states that

for suitable quantities, such as the BFKL and DGLAP kernels, the result in $\mathcal{N} = 4$ SYM can be obtained from that in QCD by converting the fermion representation from the fundamental (quarks) to the adjoint (gluinos) and then keeping only the functions that have the highest transcendental weight. In momentum space (x-space) these functions are typically iterated integrals, and the weight is the number of iterations; in Mellin-moment space, it corresponds to the number of sums in the nested sums [45]. Here we will only need the notion of weight for ordinary Riemann zeta values, $\zeta_n \equiv \zeta(n)$, for which the weight is n. Also, the weight is additive for products, and rational numbers have weight zero.

The complete set of observables to which this principle can be applied is still unclear. Besides anomalous dimensions, it has also been successfully applied to form factors, matrix elements of gauge-invariant operators with two or three external partons [46, 47, 48, 49, 50], and to certain configurations of semi-infinite Wilson lines [33, 34]. However, it does not hold for scattering amplitudes with four or five external gluons, even at one loop [51], in the sense that there are maximally transcendental parts of the QCD one-loop amplitudes which have different rational prefactors from the corresponding $\mathcal{N} = 4$ SYM amplitudes.

Here we will apply the maximal transcendentality principle to the collinear anomalous dimension. This quantity depends on the method of regularization. We will compute its value in dimensional regularization — in fact, in a supersymmetric version of dimensional regularization such as dimensional reduction. The collinear anomalous dimension has also been computed using the so-called massive, or Higgs, regularization [52, 53, 54, 55, 56]. The Higgs-regulated result begins to differ from the dimensionally-regularized value starting at three loops [56], the last value for which it is known. A dual conformal regulator for infrared divergences has also been defined [57]; however, the multi-loop values of the collinear anomalous dimension in this scheme are still under investigation [58].

One might think that the collinear anomalous dimension in planar $\mathcal{N} = 4$ SYM could simply be read off from the leading transcendental terms in the large-N_c quark collinear anomalous dimension [39]. However, the full-color expression for this quantity is a polynomial in the adjoint and fundamental quadratic Casimirs, C_A and C_F. In the large-N_c limit, $C_A \to N_c$ while $C_F \to N_c/2$. In order to apply the principle of maximal transcendentality at large N_c, we should first set $C_F \to C_A$; that is, $C_A \to N_c$ and $C_F \to C_A \to N_c$, not $N_c/2$. There is not enough information left in the large-N_c limit to make the correct replacement.

However, if we can first convert the collinear anomalous dimension to an appropriate eikonal (Wilson line) quantity, then we can make the correct replacement. In a conformal theory, this "eikonal bypass" only requires [59] knowledge of the virtual anomalous dimension, the coefficient of $\delta(1 - x)$ in the DGLAP kernel. The virtual anomalous dimension in large-N_c QCD was computed recently at four loops [40], and we can make use of its leading transcendental part to do the conversion. Once we have the eikonal quantity, we use its non-abelian exponentiation property [60, 61], which means that it is "maximally non-abelian". That is, at any loop order, it can contain only one quadratic Casimir for the representation of the Wilson line; the remaining group theory factors must all be C_A. (A subtlety that arises when quadratic Casimir scaling does not hold is addressed in section 3.) This information

suffices to allow us to apply the principle of maximal transcendentality and extract the eikonal quantity in planar $\mathcal{N} = 4$ SYM. Then we use the virtual anomalous dimension in planar $\mathcal{N} = 4$ SYM, which has been computed to all orders using integrability [62, 63, 64], to convert back to the non-eikonal collinear anomalous dimension.

This paper is organized as follows. In section 2 we briefly review the infrared structure of scattering amplitudes, form factors and Wilson loops in planar $\mathcal{N} = 4$ SYM. In section 3 we carry out the computation and then conclude.

2 Review of Infrared Structure of Planar $\mathcal{N} = 4$ SYM

In this section we give a very brief review of the infrared structure of scattering amplitudes, form factors and Wilson loops in planar $\mathcal{N} = 4$ SYM. In general, multi-loop n-point amplitudes can be factorized into soft, collinear and hard virtual contributions, where soft gluon exchange can connect any of the n hard external legs [65, 66]. This factorization has consequences for the infrared poles in ϵ of dimensionally-regularized multi-loop amplitudes [67, 68].

In the planar limit, the soft structure simplifies enormously, because only color-adjacent lines can exchange soft gluons, and the infrared structure of the amplitude for n external adjoint particles becomes the product of n "wedges", each equivalent to the square root of a Sudakov form factor for producing two adjoint particles [23]. The infrared behavior of the Sudakov form factor was studied using factorization and renormalization group evolution, beginning in the 1970s [69, 70, 71, 72, 73, 74]. Besides the β function (which of course vanishes in $\mathcal{N} = 4$ SYM), the only quantities that enter are the (light-like) cusp anomalous dimension γ_K [24, 25] and an integration constant for a function $\mathcal{G}(q^2)$, which we will refer to as the collinear anomalous dimension and denote by $\mathcal{G}_0$ [68, 74].

We consider gauge group $SU(N_c)$ and adopt the "integrability" notation for the large-N_c coupling constant,

$$g^2 \equiv N_c \frac{g_{\mathrm{YM}}^2}{(4\pi)^2} = C_A \frac{\alpha_s}{4\pi} = \frac{\lambda}{(4\pi)^2} = \frac{a}{2}, \tag{2.1}$$

where $\alpha_s = g_{\mathrm{YM}}^2/(4\pi)$, $\lambda = N_c g_{\mathrm{YM}}^2$ is the 't Hooft coupling, and a was used e.g. in ref. [23]. The quadratic Casimir in the adjoint representation is $C_A = N_c$, while in the fundamental representation it is $C_F = (N_c^2 - 1)/(2N_c)$.

We expand the cusp and collinear anomalous dimensions in terms of g^2:

$$\gamma_K(g) = \sum_{L=1}^{\infty} g^{2L} \hat{\gamma}_K^{(L)}, \tag{2.2}$$

$$\mathcal{G}_0(g) = \sum_{L=1}^{\infty} g^{2L} \hat{\mathcal{G}}_0^{(L)}. \tag{2.3}$$

(Note that another normalization is often used for the cusp anomalous dimension, $\gamma_K = 2\Gamma_{\mathrm{cusp}}$.) The cusp anomalous dimension is known to all orders, thanks to integrability [5].

The first four terms in its perturbative expansion are:

$$\gamma_K^{\text{planar } \mathcal{N}=4} \;=\; 8\,g^2 - 16\,\zeta_2\,g^4 + 176\,\zeta_4\,g^6 - \Big(1752\,\zeta_6 + 64\,(\zeta_3)^2\Big)g^8\,. \tag{2.4}$$

We give the previously-known three-loop result for $\mathcal{G}_0(g)$ below, in eq. (3.5).

In a non-conformal theory, when the differential equation for the Sudakov form factor is integrated up, infrared poles are obtained that involve integrals over functions of the running coupling in $D = 4 - 2\epsilon$ dimensions. Because planar $\mathcal{N} = 4$ SYM is conformally invariant, the integrals can be performed analytically. One obtains for the color-ordered n-point scattering amplitude A_n [23]:

$$\ln\left(\frac{A_n}{A_n^{\text{tree}}}\right) = -\frac{1}{8}\sum_{L=1}^{\infty}\frac{g^{2L}}{L^2\,\epsilon^2}\Big(\hat{\gamma}_K^{(L)} + 2L\,\epsilon\,\hat{\mathcal{G}}_0^{(L)}\Big)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{L\epsilon} + \text{ finite}, \tag{2.5}$$

where $s_{i,i+1} = (k_i + k_{i+1})^2$.

The form factor $F(Q^2)$ for producing two adjoint particles corresponds to setting $n = 2$ in this formula, in which case the two wedges have the same kinematics,

$$\ln F(Q^2) = -\frac{1}{4}\sum_{L=1}^{\infty}\frac{g^{2L}}{L^2\,\epsilon^2}\Big(\hat{\gamma}_K^{(L)} + 2L\,\epsilon\,\hat{\mathcal{G}}_0^{(L)}\Big)\left(\frac{\mu^2}{-Q^2}\right)^{L\epsilon} + \text{ finite}, \tag{2.6}$$

Wilson loops for light-like n-gons C_n contain ultraviolet poles rather than infrared ones. These poles have a very similar form (with $\epsilon \to -\epsilon$ due to their ultraviolet nature) [10]:

$$\ln W_{C_n} = -\frac{1}{8}\sum_{L=1}^{\infty}\frac{g^{2L}}{L^2\,\epsilon^2}\Big(\hat{\gamma}_K^{(L)} - 2L\,\epsilon\,\hat{\mathcal{G}}_{0,\,\text{eik}}^{(L)}\Big)\sum_{i=1}^{n}\left(\frac{\mu_{\text{UV}}^2}{-x_{i,i+2}^2}\right)^{-L\epsilon} + \text{ finite}, \tag{2.7}$$

where $x_{i,i+2}^2 = (x_i - x_{i+2})^2$ are invariant distances between the corners of the polygons x_i^μ. The amplitude-Wilson loop duality makes the identification $(x_i - x_{i+2})^2 = (k_i + k_{i+1})^2$. While the leading double poles in Wilson loops are governed by the same quantity as in amplitudes, namely γ_K, a different quantity appears in the subleading poles, $\mathcal{G}_{0,\,\text{eik}}$, whose expansion is defined by

$$\mathcal{G}_{0,\,\text{eik}}(g) = \sum_{L=1}^{\infty}g^{2L}\hat{\mathcal{G}}_{0,\,\text{eik}}^{(L)}\,. \tag{2.8}$$

instead of $\mathcal{G}_0$.

The relation between $\mathcal{G}_0$ and $\mathcal{G}_{0,\,\text{eik}}$ was explored in ref. [59], where it was shown that for a conformal theory, they obey a particularly simple relation,

$$\mathcal{G}_0 = \mathcal{G}_{0,\,\text{eik}} + 2\,B\,. \tag{2.9}$$

(Empirical evidence for this kind of relation was given in refs. [31, 75].) Here B, sometimes called B_δ or the virtual anomalous dimension, is the coefficient of the first subleading term in the limit as $x \to 1$ of the DGLAP kernel for parton i to split to parton i:

$$P_{ii}(x) \;=\; \frac{\gamma_K}{2(1-x)_+} + B_i\,\delta(1-x) + \dots\,. \tag{2.10}$$

In a general theory, $B = B_i$ depends on the type of parton i (also the leading, cusp, term in eq. (2.10) depends on the color representation of parton i), but in $\mathcal{N} = 4$ SYM B is the same for all partons, by supersymmetry.

In planar $\mathcal{N} = 4$ SYM, thanks to dual conformal symmetry, the gluon Regge trajectory governing the forward limit of the four-point amplitude can be computed from the cusp and collinear anomalous dimensions [10, 29, 30]. The result is [10]

$$\frac{\partial}{\partial \ln s} \ln A_4(s,t)\Big|_{s \gg t} = \omega_R(-t), \tag{2.11}$$

where

$$\omega_R(-t) = \frac{1}{4}\gamma_K \ln\left(\frac{\mu^2}{-t}\right) + \frac{1}{4\epsilon}\int_0^{g^2} \frac{dg'^2}{g'^2}\gamma_K(g') + \frac{1}{2}\mathcal{G}_0 + O(\epsilon). \tag{2.12}$$

Hence our four-loop result for $\mathcal{G}_0$ will also provide $\omega_R(-t)$ to the same order.

3 The Computation

In ref. [39], the quark form factor was computed to four loops in the large N_c limit of QCD, and the cusp and collinear anomalous dimensions for large-N_c QCD were determined from it. In ref. [43] it was proposed that the $\mathcal{N} = 4$ super-Yang-Mills results for the twist-two anomalous dimensions (which includes the cusp anomalous dimension, but not the collinear anomalous dimension) could be extracted from the leading transcendental terms in the QCD result by setting $C_F \to C_A$. Through three loops, where full-color QCD results are known, the same extraction procedure also works for the collinear anomalous dimension.

Unfortunately, as mentioned in the introduction, the large N_c limit corresponds to

$$C_F = \frac{N_c^2 - 1}{2N_c} \to \frac{N_c}{2} = \frac{C_A}{2}. \tag{3.1}$$

The factor of $1/2$ means that the $C_F \to C_A$ replacement can't be deduced in general from the large N_c limit. However, there is a workaround, the eikonal bypass discussed in the introduction, which involves converting the non-eikonal quark collinear anomalous dimension to an eikonal (Wilson line) quantity [59], with the help of the recent four-loop result for the DGLAP kernels in the large N_c limit of QCD [40]. In particular, we need the coefficient of $\delta(1-x)$ in this result, the virtual anomalous dimension. We will see that th $C_F \to C_A$ replacement can be performed for the eikonal quantity we have constructed. Afterwards, one can use the virtual anomalous dimension for planar $\mathcal{N} = 4$ SYM [62, 63, 64] to convert back to the non-eikonal collinear anomalous dimension. We will find an analytic expression that is quite close to the numerical result [27].

Through four loops, the leading transcendental part of the leading-color quark collinear anomalous dimension is [39, 76, 77]

$$\gamma_q|_{\text{L.C.L.T.}} = 7\zeta_3\, g^4 - \left(68\,\zeta_5 + \frac{44}{3}\,\zeta_2\zeta_3\right)g^6 + \left(705\,\zeta_7 + 144\,\zeta_2\,\zeta_5 + 164\,\zeta_3\,\zeta_4\right)g^8. \tag{3.2}$$

Through three loops, we also know the full group-theoretical decomposition [76]:

$$
\begin{aligned}
\gamma_q|_{\text{L.T.}} = C_F \bigg\{ & (26\, C_A - 24\, C_F)\zeta_3 \left(\frac{\alpha_s}{4\pi}\right)^2 \\
& - \left[\left(136C_A^2 + 120C_F C_A - 240C_F^2\right)\zeta_5 + \left(\frac{88}{3}C_A^2 + 16C_F C_A - 32C_F^2\right)\zeta_2\zeta_3\right]\left(\frac{\alpha_s}{4\pi}\right)^3 \\
& + \dots \bigg\}.
\end{aligned}
\tag{3.3}
$$

Letting $C_F \to C_A$, the $\mathcal{N} = 4$ SYM result, for a gluon or gluino in the adjoint representation is [43, 78]:

$$
\gamma^{\mathcal{N}=4} = 2\,\zeta_3 \left(\frac{C_A\,\alpha_s}{4\pi}\right)^2 - \left(16\,\zeta_5 + \frac{40}{3}\,\zeta_2\zeta_3\right)\left(\frac{C_A\,\alpha_s}{4\pi}\right)^3 + \dots .
\tag{3.4}
$$

In refs. [23, 68], the collinear anomalous dimension $\mathcal{G}_0$ was evaluated to two and three loops in planar $\mathcal{N} = 4$ SYM, although to this order there are no subleading color terms. The result is

$$
\mathcal{G}_0^{\mathcal{N}=4} = -4\,\zeta_3 \left(\frac{C_A\,\alpha_s}{4\pi}\right)^2 + \left(32\,\zeta_5 + \frac{80}{3}\,\zeta_2\zeta_3\right)\left(\frac{C_A\,\alpha_s}{4\pi}\right)^3 - \dots .
\tag{3.5}
$$

Comparing with eq. (3.4), there is a difference in normalization convention by a factor of two: $\mathcal{G}_0 = -2\gamma$.

In ref. [40], the twist-two anomalous dimensions or DGLAP kernels were computed in the large N_c limit of QCD to four loops. In the limit that $x \to 1$, as in eq. (2.10), the coefficient of the leading $1/(1-x)_+$ term is the cusp anomalous dimension [25]. The next-to-leading term as $x \to 1$ is the coefficient of $\delta(1-x)$, sometimes called the virtual anomalous dimension, or B_δ, or just B. The large-N_c, leading transcendentality terms in B for quarks are given by [40, 79]:

$$
B_q|_{\text{L.C.L.T.}} = 20\,\zeta_5\, g^6 - \left(280\,\zeta_7 + 40\,\zeta_2\,\zeta_5 - 16\,\zeta_3\,\zeta_4\right)g^8 .
\tag{3.6}
$$

Through three loops, we also know the full group-theoretical decomposition [79]:

$$
\begin{aligned}
B_q|_{\text{L.T.}} = C_F \bigg\{ & -12(C_A - 2C_F)\zeta_3 \left(\frac{\alpha_s}{4\pi}\right)^2 \\
& + \left[\left(40\, C_A^2 + 120\, C_F C_A - 240C_F^2\right)\zeta_5 + 16C_F(C_A - 2C_F)\zeta_2\zeta_3\right]\left(\frac{\alpha_s}{4\pi}\right)^3 \\
& + \dots \bigg\}.
\end{aligned}
\tag{3.7}
$$

Letting $C_F \to C_A$, the $\mathcal{N} = 4$ SYM result, for a gluon or gluino, is [43]:

$$
B^{\mathcal{N}=4} = 12\,\zeta_3 \left(\frac{C_A\,\alpha_s}{4\pi}\right)^2 - \left(80\,\zeta_5 + 16\,\zeta_2\zeta_3\right)\left(\frac{C_A\,\alpha_s}{4\pi}\right)^3 + \dots .
\tag{3.8}
$$

We now use eq. (2.9) to construct the eikonal quantity

$$\mathcal{G}_{0,\,\text{eik}} = \mathcal{G}_0 - 2\,B = -2\,\gamma_q - 2\,B \tag{3.9}$$

in $\mathcal{N} = 4$ SYM for a Wilson line in the fundamental "F" representation through three loops, using eqs. (3.3) and (3.7):

$$\mathcal{G}_{0,\,\text{eik},F} = C_F \left\{ -28\,C_A\,\zeta_3 \left(\frac{\alpha_s}{4\pi}\right)^2 + \left(192\,\zeta_5 + \frac{176}{3}\,\zeta_2\zeta_3\right) C_A^2 \left(\frac{\alpha_s}{4\pi}\right)^3 + \ldots \right\}. \tag{3.10}$$

We see that all the C_F terms cancel, except for the overall one. This result reflects non-abelian exponentiation for this type of Wilson line [60, 61]. These results agree with the leading transcendental part of the results for f_L^q in ref. [31].

The threshold soft anomalous dimension γ^s defined in refs. [33, 34] (called γ^W in ref. [32]) is the same as $\mathcal{G}_{0,\,\text{eik},F}$ up to a conventional minus sign, $\gamma^s = -\mathcal{G}_{0,\,\text{eik},F}$, and eq. (3.10) agrees with the leading transcendental part of the QCD result in refs. [33, 34]. The rapidity anomalous dimension γ^r, which enters the SCET description of transverse momentum resummation, has also been computed to three loops [34]. The result agrees with the threshold soft anomalous dimension, up to terms that are proportional to coefficients of the QCD beta function. This result was explained in ref. [37] by mapping the appropriate configurations of Wilson lines for the two computations into each other using a conformal transformation. Hence we will obtain the four-loop values of both the threshold soft and rapidity anomalous dimensions in planar $\mathcal{N} = 4$ SYM from

$$\gamma^{s,\,\text{planar }\mathcal{N}=4} = \gamma^{r,\,\text{planar }\mathcal{N}=4} = -\mathcal{G}_{0,\text{eik}}^{\text{planar }\mathcal{N}=4}. \tag{3.11}$$

In planar $\mathcal{N} = 4$ SYM, the natural Wilson line is in the adjoint representation, not the fundamental. In the large N_c limit, this collinear anomalous dimension can be obtained from eq. (3.10) simply by multiplying by an overall factor of 2, since $C_A = 2C_F$ in the large N_c limit. What about at four loops? At this order, quadratic Casimir scaling might be violated. That is, inspecting the color factors of all the Feynman diagrams that contribute at this order, we see that $\mathcal{G}_{0,\,\text{eik},F}$ might contain — besides C_F times a polynomial in C_A — a color factor of

$$\frac{d_F^{abcd} d_A^{abcd}}{N_F} = \frac{(N_c^2 - 1)(N_c^2 + 6)}{48}. \tag{3.12}$$

(See e.g. eq. (2.14) of ref. [80].) If so, the corresponding term in the case of an adjoint Wilson line would have the same numerical coefficient multiplying

$$\frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N_c^2(N_c^2 + 36)}{24}. \tag{3.13}$$

However, the latter factor is precisely twice the former factor in the large N_c limit, which is the same factor as for the conversion $C_F \to C_A$ in this limit. Given that there are no C_F terms in $\mathcal{G}_{0,\,\text{eik},F}$ except for the overall C_F, $\mathcal{G}_{0,\,\text{eik}}$ in the large N_c limit of $\mathcal{N} = 4$ SYM can be

extracted from the leading transcendality terms of the corresponding eikonal quantity in the large N_c limit of QCD. (The beta-function correction terms to eq. (2.9) for a non-conformal theory are also subleading in transcendentality.)

In summary, the eikonal collinear anomalous dimension for planar $\mathcal{N} = 4$ SYM can be obtained from the large-N_c QCD results for γ_q and B_q through four loops, using

$$\mathcal{G}_{0,\text{eik}}^{\text{planar } \mathcal{N}=4} = 2\left(-2\gamma_q|_{\text{L.C.L.T.}} - 2B_q|_{\text{L.C.L.T.}}\right). \tag{3.14}$$

Inserting eqs. (3.2) and (3.6), we obtain

$$\mathcal{G}_{0,\text{eik}}^{\text{planar } \mathcal{N}=4} = -28\,\zeta_3\,g^4 + \left(192\,\zeta_5 + \frac{176}{3}\,\zeta_2\zeta_3\right)g^6 - \left(1700\,\zeta_7 + 416\,\zeta_2\,\zeta_5 + 720\,\zeta_3\,\zeta_4\right)g^8. \tag{3.15}$$

The virtual anomalous dimension in planar $\mathcal{N} = 4$ SYM is known to all orders from integrability [62, 63, 64]:

$$B^{\text{planar } \mathcal{N}=4} = 12\,\zeta_3\,g^4 - \left(80\,\zeta_5 + 16\,\zeta_2\zeta_3\right)g^6 + \left(700\,\zeta_7 + 80\,\zeta_2\,\zeta_5 + 168\,\zeta_3\,\zeta_4\right)g^8 + \ldots \tag{3.16}$$

We set $L = 2$ in eq. (3.16) of ref. [62], and multiply by $-1/2$ to account for the different normalization convention.

The non-eikonal collinear anomalous dimension in planar $\mathcal{N} = 4$ SYM is then:

$$\begin{aligned} \mathcal{G}_0^{\text{planar } \mathcal{N}=4} &= \mathcal{G}_{0,\text{eik}}^{\text{planar } \mathcal{N}=4} + 2B^{\text{planar } \mathcal{N}=4} \\ &= -4\,\zeta_3\,g^4 + \left(32\,\zeta_5 + \frac{80}{3}\,\zeta_2\zeta_3\right)g^6 - \left(300\,\zeta_7 + 256\,\zeta_2\,\zeta_5 + 384\,\zeta_3\,\zeta_4\right)g^8. \end{aligned} \tag{3.17}$$

The numerical value of the four-loop coefficient is

$$-1238.747717254773533291898\ldots \tag{3.18}$$

which can be compared with the number from ref. [27]:

$$-1240.9(3). \tag{3.19}$$

The two results are within about 0.2%, although they are not within the error budget of 0.3 reported in ref. [27]. It would be very nice to check the analysis in this paper with an improved numerical value.

The first order at which $\mathcal{G}_0^{\mathcal{N}=4}$ can have a subleading-color term is four loops. Recently this term has been computed numerically [28],

$$\mathcal{G}_{0,\text{NP}}^{(4),\mathcal{N}=4} = -384 \times (-17.98 \pm 3.47)\,\frac{1}{N_c^2}. \tag{3.20}$$

Could one try to get an analytic value for this quantity using the methods in this paper?

One issue is that the principle of maximal transcendentality has not really been tested yet for cases where there is a subleading-color contribution to $\mathcal{N} = 4$ SYM, but one could try nevertheless. The good news is that the simple relation (2.9) continues to hold at subleading color — whereas in a non-conformal theory it would receive additional corrections depending on the infrared-finite part of a form factor [59]. The bad news is that there are not yet analytic values for the subleading-color terms in QCD at four loops, for either γ_q or B_q. (Approximate numerical values are available for B_q [40].) Once they become available, it will be possible to compute eq. (3.20) analytically, if it is not already known by then. In fact, the eikonal bypass of using eq. (2.9) should become unnecessary at that point, once the full color dependence of the QCD result for γ_q is known.

In summary, in this paper we obtained an analytical value (3.17) for the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ SYM, which also provides the Regge trajectory, threshold soft anomalous dimension and rapidity anomalous dimension at this order. We hope that this additional data point will inspire those versed in integrability methods to try to compute this quantity to all loop orders!

Acknowledgments

References

[1] L. Brink, J. H. Schwarz and J. Scherk, Nucl. Phys. B **121**, 77 (1977).

[2] F. Gliozzi, J. Scherk and D. I. Olive, Nucl. Phys. B **122**, 253 (1977).

[3] J. A. Minahan and K. Zarembo, JHEP **0303**, 013 (2003) [hep-th/0212208].

[4] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B **664**, 131 (2003) [hep-th/0303060].

[5] N. Beisert, B. Eden and M. Staudacher, J. Stat. Mech. **0701**, P01021 (2007) [hep-th/0610251].

[6] N. Beisert *et al.*, Lett. Math. Phys. **99**, 3 (2012) [arXiv:1012.3982 [hep-th]].

[7] B. Basso, A. Sever and P. Vieira, Phys. Rev. Lett. **111** (2013) 091602 [arXiv:1303.1396 [hep-th]].

[8] N. Gromov, V. Kazakov, S. Leurent and D. Volin, Phys. Rev. Lett. **112** (2014) 011602 [arXiv:1305.1939 [hep-th]].

[9] L. F. Alday and J. M. Maldacena, JHEP **0706**, 064 (2007) [arXiv:0705.0303 [hep-th]].

[10] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **795**, 385 (2008) [arXiv:0707.0243 [hep-th]].

[11] A. Brandhuber, P. Heslop and G. Travaglini, Nucl. Phys. B **794**, 231 (2008) [arXiv:0707.1153 [hep-th]].

[12] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **795**, 52 (2008) [arXiv:0709.2368 [hep-th]].

[13] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **826**, 337 (2010) [arXiv:0712.1223 [hep-th]].

[14] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. D **78**, 045007 (2008) [arXiv:0803.1465 [hep-th]].

[15] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **815**, 142 (2009) [arXiv:0803.1466 [hep-th]].

[16] V. Del Duca, C. Duhr and V. A. Smirnov, JHEP **1003**, 099 (2010) [arXiv:0911.5332 [hep-ph]].

[17] V. Del Duca, C. Duhr and V. A. Smirnov, JHEP **1005**, 084 (2010) [arXiv:1003.1702 [hep-th]].

[18] A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605 [arXiv:1006.5703 [hep-th]].

[19] L. J. Dixon, J. M. Drummond and J. M. Henn, JHEP **1111**, 023 (2011) [arXiv:1108.4461 [hep-th]].

[20] L. J. Dixon, J. M. Drummond, M. von Hippel and J. Pennington, JHEP **1312**, 049 (2013) [arXiv:1308.2276 [hep-th]].

[21] L. J. Dixon, J. M. Drummond, C. Duhr and J. Pennington, JHEP **1406**, 116 (2014) [arXiv:1402.3300 [hep-th]].

[22] S. Caron-Huot, L. J. Dixon, A. McLeod and M. von Hippel, Phys. Rev. Lett. **117**, 241601 (2016) [arXiv:1609.00669 [hep-th]].

[23] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005) [hep-th/0505205].

[24] G. P. Korchemsky and A. V. Radyushkin, Nucl. Phys. B **283**, 342 (1987).

[25] G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. B **279**, 359 (1992) [hep-ph/9203222].

[26] S. Moch, J. A. M. Vermaseren and A. Vogt, JHEP **0508**, 049 (2005) [hep-ph/0507039].

[27] F. Cachazo, M. Spradlin and A. Volovich, Phys. Rev. D **76**, 106004 (2007) [arXiv:0707.1903 [hep-th]].

[28] R. H. Boels, T. Huber and G. Yang, arXiv:1711.08449 [hep-th].

[29] S. G. Naculich and H. J. Schnitzer, Nucl. Phys. B **794**, 189 (2008) [arXiv:0708.3069 [hep-th]].

[30] V. Del Duca and E. W. N. Glover, JHEP **0805**, 056 (2008) [arXiv:0802.4445 [hep-th]].

[31] S. Moch, J. A. M. Vermaseren and A. Vogt, Phys. Lett. B **625**, 245 (2005) [hep-ph/0508055].

[32] T. Becher, M. Neubert and G. Xu, JHEP **0807**, 030 (2008) [arXiv:0710.0680 [hep-ph]].

[33] Y. Li, A. von Manteuffel, R. M. Schabinger and H. X. Zhu, Phys. Rev. D **91**, 036008 (2015) [arXiv:1412.2771 [hep-ph]].

[34] Y. Li and H. X. Zhu, Phys. Rev. Lett. **118**, 022004 (2017) [arXiv:1604.01404 [hep-ph]].

[35] J. Y. Chiu, A. Jain, D. Neill and I. Z. Rothstein, JHEP **1205**, 084 (2012) [arXiv:1202.0814 [hep-ph]].

[36] Y. Li, D. Neill and H. X. Zhu, [arXiv:1604.00392 [hep-ph]].

[37] A. A. Vladimirov, Phys. Rev. Lett. **118**, 062001 (2017) [arXiv:1610.05791 [hep-ph]].

[38] M. Spradlin, A. Volovich and C. Wen, Phys. Rev. D **78**, 085025 (2008) [arXiv:0808.1054 [hep-th]].

[39] J. Henn, A. V. Smirnov, V. A. Smirnov, M. Steinhauser and R. N. Lee, JHEP **1703**, 139 (2017) [arXiv:1612.04389 [hep-ph]].

[40] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP **1710**, 041 (2017) [arXiv:1707.08315 [hep-ph]].

[41] A. V. Kotikov and L. N. Lipatov, hep-ph/0112346.

[42] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. B **661**, 19 (2003) Erratum: [Nucl. Phys. B **685**, 405 (2004)] [hep-ph/0208220].

[43] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, Phys. Lett. B **595**, 521 (2004) Erratum: [Phys. Lett. B **632**, 754 (2006)] [hep-th/0404092].

[44] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher and V. N. Velizhanin, J. Stat. Mech. **0710**, P10003 (2007) [arXiv:0704.3586 [hep-th]].

[45] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15**, 725 (2000) [hep-ph/9905237].

[46] A. Brandhuber, G. Travaglini and G. Yang, JHEP **1205**, 082 (2012) [arXiv:1201.4170 [hep-th]].

[47] F. Loebbert, D. Nandan, C. Sieg, M. Wilhelm and G. Yang, JHEP **1510**, 012 (2015) [arXiv:1504.06323 [hep-th]].

[48] A. Brandhuber, M. Kostacinska, B. Penante, G. Travaglini and D. Young, JHEP **1608**, 134 (2016) [arXiv:1606.08682 [hep-th]].

[49] F. Loebbert, C. Sieg, M. Wilhelm and G. Yang, JHEP **1612**, 090 (2016) [arXiv:1610.06567 [hep-th]].

[50] A. Brandhuber, M. Kostacinska, B. Penante and G. Travaglini, Phys. Rev. Lett. **119**, 161601 (2017) [arXiv:1707.09897 [hep-th]].

[51] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **70**, 2677 (1993) [hep-ph/9302280].

[52] L. F. Alday, J. M. Henn, J. Plefka and T. Schuster, JHEP **1001**, 077 (2010) [arXiv:0908.0684 [hep-th]].

[53] J. M. Henn, S. G. Naculich, H. J. Schnitzer and M. Spradlin, JHEP **1004**, 038 (2010) [arXiv:1001.1358 [hep-th]].

[54] J. M. Henn, S. G. Naculich, H. J. Schnitzer and M. Spradlin, JHEP **1008**, 002 (2010) [arXiv:1004.5381 [hep-th]].

[55] J. M. Drummond and J. M. Henn, JHEP **1105**, 105 (2011) [arXiv:1008.2965 [hep-th]].

[56] J. M. Henn, S. Moch and S. G. Naculich, JHEP **1112**, 024 (2011) [arXiv:1109.5057 [hep-th]].

[57] J. L. Bourjaily, S. Caron-Huot and J. Trnka, JHEP **1501**, 001 (2015) [arXiv:1303.4734 [hep-th]].

[58] J. Bourjaily, L. J. Dixon, F. Dulat and E. Panzer, to appear.

[59] L. J. Dixon, L. Magnea and G. F. Sterman, JHEP **0808**, 022 (2008) [arXiv:0805.3515 [hep-ph]].

[60] J. G. M. Gatheral, Phys. Lett. **133B**, 90 (1983).

[61] J. Frenkel and J. C. Taylor, Nucl. Phys. B **246**, 231 (1984).

[62] L. Freyhult, A. Rej and M. Staudacher, J. Stat. Mech. **0807**, P07015 (2008) [arXiv:0712.2743 [hep-th]].

[63] L. Freyhult and S. Zieme, Phys. Rev. D **79**, 105009 (2009) [arXiv:0901.2749 [hep-th]].

[64] D. Fioravanti, P. Grinza and M. Rossi, Phys. Lett. B **675**, 137 (2009) [arXiv:0901.3161 [hep-th]].

[65] N. Kidonakis, G. Oderda and G. F. Sterman, Nucl. Phys. B **525**, 299 (1998) [hep-ph/9801268].

[66] N. Kidonakis, G. Oderda and G. F. Sterman, Nucl. Phys. B **531**, 365 (1998) [hep-ph/9803241].

[67] S. Catani, Phys. Lett. B **427**, 161 (1998) [hep-ph/9802439].

[68] G. F. Sterman and M. E. Tejeda-Yeomans, Phys. Lett. B **552**, 48 (2003) [hep-ph/0210130].

[69] A. H. Mueller, Phys. Rev. D **20**, 2037 (1979).

[70] J. C. Collins, Phys. Rev. D **22**, 1478 (1980).

[71] A. Sen, Phys. Rev. D **24**, 3281 (1981).

[72] J. C. Collins, Adv. Ser. Direct. High Energy Phys. **5**, 573 (1989) [hep-ph/0312336].

[73] G. P. Korchemsky, Phys. Lett. B **217**, 330 (1989).

[74] L. Magnea and G. F. Sterman, Phys. Rev. D **42**, 4222 (1990).

[75] V. Ravindran, J. Smith and W. L. van Neerven, Nucl. Phys. B **704**, 332 (2005) [hep-ph/0408315].

[76] T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli and C. Studerus, JHEP **1006**, 094 (2010) [arXiv:1004.3653 [hep-ph]].

[77] J. M. Henn, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, JHEP **1605**, 066 (2016) [arXiv:1604.03126 [hep-ph]].

[78] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, Phys. Lett. B **557**, 114 (2003) [hep-ph/0301021].

[79] S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B **688**, 101 (2004) [hep-ph/0403192].

[80] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP **1702**, 090 (2017) [arXiv:1701.01404 [hep-ph]].