# The two-loop five-point amplitude in $\mathcal{N}=4$ super-Yang-Mills theory 

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#### Abstract

We compute the symbol of the two-loop five-point scattering amplitude in $\mathcal{N}=4$ super-YangMills theory, including its full color dependence. This requires evaluating the symbol of all two-loop five-point nonplanar massless master integrals, for which we give explicit results.


A great deal of progress in calculating scattering amplitudes has been driven by the fruitful interplay between new formal ideas and the need for increasingly precise theoretical predictions at collider experiments. For instance, techniques such as generalized unitarity [1] and the symbol calculus [2] were first introduced in the realm of maximally supersymmetric Yang-Mills theory ( $\mathcal{N}=4 \mathrm{sYM}$ ) and went on to have a large impact on precision collider physics. In this letter, we use cutting-edge techniques to take a first look at the analytic form of the two-loop five-point amplitude in $\mathcal{N}=4 \mathrm{sYM}$ beyond the planar, $N_{c} \rightarrow \infty$, limit of $S U\left(N_{c}\right)$ gauge theory.

Amplitudes in $\mathcal{N}=4 \mathrm{sYM}$ possess rigid analytic properties that make them easier to compute than their pure Yang-Mills counterparts, the state of the art being the three-loop four-gluon $\mathcal{N}=4 \mathrm{sYM}$ amplitude [3]. Historically, calculations in $\mathcal{N}=4 \mathrm{sYM}$ have therefore preceded analogous computations in QCD. The planar five-point amplitude at two loops in $\mathcal{N}=4 \mathrm{sYM}$ was first obtained numerically [4], confirming the prediction of [5]. In pure Yang-Mills, the first planar two-loop five-point amplitude, evaluated numerically, was for the all-plus helicity configuration [6]. Since then, a flurry of activity in planar multi-leg two-loop amplitudes has seen the analytic calculation of the all-plus amplitude [7], the numerical evaluation of all five-parton QCD amplitudes [811], and recently the computation of analytic expressions for all five-gluon scattering amplitudes [12, 13]. These achievements were made possible by the development of efficient ways to reduce amplitudes to master integrals using integration-by-parts (IBP) relations [14, 15] automated by Laporta's algorithm [16] or modern reformulations based on unitarity cuts and computational algebraic geometry [ $10,17-20]$, and to compute master integrals from their differential equations [21, 22]. Indeed, all planar five-point master integrals have now been computed [23, 24], and substantial progress has been made in the nonplanar sectors as well [25-27].

In this work, we first discuss the integrand of the twoloop five-point amplitude in $\mathcal{N}=4 \mathrm{sYM}$, and how it can be reduced to a form involving only so-called pure integrals (i.e., integrals satisfying a differential equation in canonical form [22]). We then use the aforemen-
tioned new techniques for integral reduction and differential equations (most notably the method introduced in [26]) to compute the symbols [2] (see also [28, 29]) of all nonplanar massless two-loop five-point master integrals. From these integrals we finally assemble the symbol of the complete two-loop five-point $\mathcal{N}=4 \mathrm{sYM}$ amplitude and discuss consistency checks of our result. Throughout, we work at the level of the symbol where transcendental constants are set to zero. While such contributions are important for the numerical evaluation of an amplitude, the symbol itself contains a major part of the non-trivial analytic structure of the amplitude.

Our result constitutes the first analytic investigation of two-loop five-point amplitudes in any gauge or gravity theory beyond the planar limit. Just as the one-loop five-gluon amplitude [30] did, our two-loop result should provide valuable theoretical data for further exploring the properties of structurally complex amplitudes, as well as the proposed duality between scattering amplitudes and Wilson loops at subleading color [31]. Furthermore, the methods will impact precision collider phenomenology: the master integrals are directly applicable to QCD amplitudes, opening the way to computing three-jet production at hadron colliders at next-to-next-to-leading order.

## Construction of the amplitude

In any $S U\left(N_{c}\right)$ gauge theory with all states in the adjoint representation, the trace-based color decomposition [32, 33] of any two-loop five-point amplitude is ${ }^{1}$

$$
\begin{align*}
& \mathcal{A}_{5}^{(2)}=\sum_{S_{5} /\left(S_{3} \times Z_{2}\right)} \frac{\operatorname{tr}[15](\operatorname{tr}[234]-\operatorname{tr}[432])}{N_{c}} A^{\mathrm{DT}}[15 \mid 234]  \tag{1}\\
& +\sum_{S_{5} / D_{5}}(\operatorname{tr}[12345]-\operatorname{tr}[54321])\left(A^{\mathrm{ST}}[12345]+\frac{A^{\mathrm{SLST}}[12345]}{N_{c}^{2}}\right) .
\end{align*}
$$

[^0]
(a)




FIG. 1. Diagram topologies entering the local representation of the two-loop five-point integrand of $\mathcal{N}=4 \mathrm{sYM}$ [43]. Each diagram has an associated color structure and numerator which we suppress.

Here, single-trace (ST), subleading-color single-trace (SLST) and double-trace (DT) denote different partial amplitudes. $S_{n}\left(Z_{n}\right)$ is the (cyclic) permutation group, and $D_{n}$ the dihedral group.

It is a powerful fact about MHV scattering amplitudes in $\mathcal{N}=4 \mathrm{sYM}$ that all leading singularities [34] are given in terms of different permutations of ParkeTaylor tree-(super-)amplitudes [35, 36]. This highly nontrivial result has been derived from a dual formulation of leading singularities in terms of the Grassmannian [37]. Furthermore, $\mathcal{N}=4 \mathrm{sYM}$ amplitudes are conjectured to be of uniform transcendental weight [5, 38-40]. A representation of the four-dimensional integrand has been given in [41], where this Parke-Taylor structure together with further special analytic properties of $\mathcal{N}=4 \mathrm{sYM}-$ logarithmic singularities and no residues at infinite loop momentum-are manifest. In this representation, the full, color-dressed amplitude splits into 3 distinct parts

$$
\begin{equation*}
\mathcal{A}_{n}^{(2)}=C \otimes \mathrm{PT} \otimes g_{\text {pure }} \tag{2}
\end{equation*}
$$

where $C$ schematically denotes the color structure of the gauge theory. For a five-point scattering amplitude, the space of Parke-Taylor factors is spanned by a set of 3 ! Kleiss-Kuijf (KK) independent elements [42] that we denote by $\mathrm{PT}\left[1 \sigma_{2} \sigma_{3} \sigma_{4} 5\right]$, where

$$
\begin{equation*}
\operatorname{PT}\left[\sigma_{1} \sigma_{2} \sigma_{3} \sigma_{4} \sigma_{5}\right]=\frac{\delta^{8}(Q)}{\left\langle\sigma_{1} \sigma_{2}\right\rangle\left\langle\sigma_{2} \sigma_{3}\right\rangle\left\langle\sigma_{3} \sigma_{4}\right\rangle\left\langle\sigma_{4} \sigma_{5}\right\rangle\left\langle\sigma_{5} \sigma_{1}\right\rangle} . \tag{3}
\end{equation*}
$$

The super-momentum conserving delta-function, $\delta^{8}(Q)$, encodes the supersymmetric Ward identities relating the ( --+++ )-helicity five-gluon amplitude to all other fiveparticle amplitudes. The third part, $g_{\text {pure }}$, denotes a pure function of transcendental weight 4 .

The goal of this section will be to compute the partial amplitudes in (1). Our starting point is the integrand of [43] which is valid in $d=4-2 \epsilon$ space-time dimensions and is given in terms of the six topologies in Fig. 1,

$$
\begin{equation*}
\mathcal{A}_{5}^{(2)}=\sum_{S_{5}}\left(\frac{I^{(a)}}{2}+\frac{I^{(b)}}{4}+\frac{I^{(c)}}{4}+\frac{I^{(d)}}{2}+\frac{I^{(e)}}{4}+\frac{I^{(f)}}{4}\right) . \tag{4}
\end{equation*}
$$

The sum is over all 5 ! permutations of external legs and the rational numbers correspond to diagram-symmetry factors.

For each of the topologies in Fig. 1, we construct a basis of pure master integrals, on which the amplitude (4) can be decomposed, so that the separation into color, rational, and transcendental parts (2) becomes manifest. Most required master integrals are already known in pure form $[7,23,26,27,44]$. The one missing topology, which we discuss momentarily, is the nonplanar doublepentagon (diagram (c) of Fig. 1). The integrals we are concerned with are functions of five Mandelstam invariants, $s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$, with $s_{i j}=\left(k_{i}+k_{j}\right)^{2}$. We also encounter the parity-odd $\varepsilon$-tensor contraction

$$
\begin{equation*}
\operatorname{tr}_{5}=4 i \varepsilon_{\mu \nu \rho \sigma} k_{1}^{\mu} k_{2}^{\nu} k_{3}^{\rho} k_{4}^{\sigma}=\operatorname{tr}\left(\gamma^{5} \not k_{1} \not k_{2} \not k_{3} \not k_{4}\right) \tag{5}
\end{equation*}
$$

To find a basis of pure master integrals for the top-level (eight-propagator) topology of Fig. 1(c) it was necessary to construct nine independent numerators. Specifically, we chose the following set of master integrals:

1. The parity-even part of the integral with numerator $N_{1}^{(a)}$ identified in [41], rewritten as spinor traces in Eq. (21) of [45]. By deleting $\gamma^{5}$ from the spinor traces, we obtain the parity-even parts in a form that is valid in $d$ dimensions. Two more pure integrals are obtained from it by using the diagram's $Z_{2} \times Z_{2}$ symmetry.
2. (6-2 $\epsilon$ )-dimensional scalar integrals with any of the eight propagators squared, normalized by a factor of $\operatorname{tr}_{5}$ and a homogeneous linear function of the $s_{i j}$ variables. Six such integrals, which we have converted to integrals in $(4-2 \epsilon)$ dimensions [46-49], are included in our basis.

Explicit expressions for these new pure master integrals can be found in the ancillary file masters.m.

Next, we construct differential equations in canonical form [22] for the master integrals. The (iterated) branchcut structure of the integrals is encoded in the symbol letters which are algebraic functions of the kinematic invariants. It is convenient to parametrize the five-point kinematics in terms of variables that rationalize all letters of the alphabet. This can be accomplished via momentumtwistors [50] and the $x_{i}$-parametrization proposed in [6]. ${ }^{2}$ For the nonplanar double-pentagon integral, we find that the complete system contains 108 masters and depends on the $31 W_{\alpha}$-letters suggested in [25]:

$$
\begin{equation*}
\partial_{x_{i}} \mathcal{I}_{a} \equiv \frac{\partial \mathcal{I}_{a}}{\partial x_{i}}=\epsilon \sum_{\alpha=1}^{31} \frac{\partial \log W_{\alpha}}{\partial x_{i}} M_{\alpha}^{a b} \mathcal{I}_{b}, 1 \leq a, b \leq 108 \tag{6}
\end{equation*}
$$

[^1]Ten of the letters $(\alpha \in\{1, \ldots, 5\} \cup\{16, \ldots, 20\})$ are simple Mandelstam invariants $s_{i j}, 15$ further letters $(\alpha \in$ $\{6, \ldots, 15\} \cup\{21, \ldots, 25\})$ are differences of Mandelstam invariants $s_{i j}-s_{k l}$, the 5 parity-odd letters $(\alpha \in\{26, \ldots, 30\})$ can be expressed as ratios of spinor-brackets such as $\frac{\langle 12\rangle[15]\langle 45\rangle[24]}{[12]\langle 15\rangle[45]\langle 24\rangle}$ which invert under complex conjugation $\langle\cdot\rangle \leftrightarrow[\cdot]$ or $\operatorname{tr}_{5} \rightarrow-\operatorname{tr}_{5}$, and the final, parity-even letter $(\alpha=31)$ is $\operatorname{tr}_{5}$. The $31 M_{\alpha}$-matrices consist of simple rational numbers.

Computing the $M_{\alpha}$-matrices in (6) requires performing IBP reduction on differentials of the original masters $\partial_{x_{i}} \mathcal{I}_{a}$ with respect to the kinematic variables in order to re-express them in terms of the original basis $\mathcal{I}_{a}$. We use the efficient approach introduced in [26], which builds on the modern formulation of IBP relations in terms of unitarity cuts and computational algebraic geometry [10, 17-20]. The method requires IBP reduction at only 30 rational, numerical phase-space points to fix all the $M_{\alpha}$, dramatically reducing the computation time compared to analytic IBP reduction. Combined with the first-entry condition [51], which restricts integrals to only have branch-cut singularities at physical thresholds, we obtain solutions to the differential equations at the symbol level for all master integrals. As a check, we verified that we reproduce (at symbol level) all known results for descendant integrals ( $\leq 7$ propagators). The full results are included in the ancillary file masters.m.

Having established a basis and computed the master integrals required for massless two-loop five-point amplitudes, we can now write the $\mathcal{N}=4 \mathrm{sYM}$ amplitude in that basis. As already stated, we use the $d$-dimensional representation of the integrand given in [43]. While this representation has the advantage of being in the so-called Bern-Carrasco-Johansson (BCJ) form [52], which allows for the immediate construction of the gravity integrand via the 'double-copy' prescription, it obscures some of the simplicity of the final result. For instance, each individual diagram in Fig. 1 introduces spurious rational factors. Applying Fierz color-identities [32] to decompose the integrand (4) into the partial amplitudes in (1) and using IBP reduction to rewrite those in our pure basis, we can obtain a representation that is manifestly in the form of (2). In particular, we find a simple rational kinematic dependence for all partial amplitudes via at most six KK-independent Parke-Taylor factors:

$$
\begin{align*}
A^{\mathrm{ST}}[12345] & =\mathrm{PT}[12345] \mathrm{M}_{(2)}^{\mathrm{BDS}}, \\
A^{\mathrm{DT}}[15 \mid 234] & =\sum_{\sigma(234) \in S_{3}} \mathrm{PT}\left[1 \sigma_{2} \sigma_{3} \sigma_{4} 5\right] g_{\sigma_{2} \sigma_{3} \sigma_{4}}^{\mathrm{DT}},  \tag{7}\\
A^{\mathrm{SLST}}[12345] & =\sum_{\sigma(234) \in S_{3}} \mathrm{PT}\left[1 \sigma_{2} \sigma_{3} \sigma_{4} 5\right] g_{\sigma_{2} \sigma_{3} \sigma_{4}}^{\mathrm{SLST}},
\end{align*}
$$

where $\mathrm{M}_{(2)}^{\mathrm{BDS}}$ is the two-loop BDS ansatz [5] and $g_{\vec{\sigma}}^{\mathrm{X}}$ are pure functions. Both $\mathrm{M}_{(2)}^{\mathrm{BDS}}$ and $g_{\vec{\sigma}}^{\mathrm{X}}$ can be written as $\mathbb{Q}$-linear combinations of our pure master integrals. The IBP reduction was done following the same strategy al-
ready discussed for the differential equations. Given the simple kinematic dependence of the result it was sufficient to perform the reduction at 6 numerical kinematic points. Furthermore, we were able to achieve a computational speedup by performing all calculations in a finite field with a 10-digit cardinality, before reconstructing the simple rational numbers from their finite-field images using Wang's algorithm [53-55].

Inserting the symbol of the master integrals, we directly obtain the symbol of the two-loop five-point $\mathcal{N}=$ 4 sYM amplitude. The amplitude is naturally decomposed into parity-even and parity-odd parts under a signflip of ' $\operatorname{tr}_{5}$ ' defined in (5). At symbol level, the parity grading can be determined by counting the number of parity-odd letters, $W_{26}, \ldots, W_{30}$, in a given symbol tensor. The parity-odd part of our result is highly constrained by the first- and second-entry conditions, as well as the integrability of the symbol [2], leading to a much simpler structure than the even part. It is important to note that in all collinear limits the parity-odd parts of the amplitude vanish since the external momenta span only a 3 -dimensional space and hence $\operatorname{tr}_{5}=0$. We attach the explicit symbol-level results for the partial amplitudes in the ancillary file amplitudes.m.

## Validation

In the previous section we described the assembly of the two-loop five-point amplitude in $\mathcal{N}=4 \mathrm{sYM}$ in terms of pure master integrals. In this section we validate our final result by checking nontrivial identities between different terms and comparing to known kinematic limits. We focus our discussion on verifying collinear factorization when two external momenta become parallel [56]. Aside from this check, we also verified that:

- The planar amplitude matches the BDS ansatz [5] stating that four- and five-particle amplitudes in planar $\mathcal{N}=4 \mathrm{sYM}$ are given to all orders by exponentiating the one-loop amplitude [30].
- The partial amplitudes satisfy the group-theoretic Edison-Naculich relations [57], allowing us to write all subleading single-trace partial amplitudes $A^{\mathrm{SLST}}$ in terms of linear combinations of planar $A^{\mathrm{ST}}$ and double-trace $A^{\mathrm{DT}}$ amplitudes, e.g.

$$
\begin{align*}
A^{\mathrm{SLST}}[12345]= & 5 A^{\mathrm{ST}}[13524]  \tag{8}\\
& +\sum_{\text {cyclic }}\left[A^{\mathrm{ST}}[12435]-2 A^{\mathrm{ST}}[12453]\right. \\
& \left.+\frac{1}{2}\left(A^{\mathrm{DT}}[12 \mid 345]-A^{\mathrm{DT}}[13 \mid 245]\right)\right]
\end{align*}
$$

where the 5 cyclic permutations are generated by the relabeling $i \rightarrow i+1(\bmod 5)$. Thus we need not discuss $A^{\text {SLST }}$ further, and the amplitude is fully specified by two functions, $\mathrm{M}_{(2)}^{\mathrm{BDS}}$ and $g_{234}^{\mathrm{DT}}$.

- The infrared poles of the amplitude match the universal pole structure predicted by Catani [58], see also e.g. [56, 59], where the poles of two-loop amplitudes can be computed in terms of known treeand one-loop amplitudes.

Several of these checks require the one-loop five-point amplitudes expanded through order $\epsilon^{2}$. An exact expression for the integrand of this amplitude is known [60]. The box integrals are known to all orders in $\epsilon$ [47]. The only integral that is not known to all orders is the (parity-odd) six-dimensional scalar pentagon, $\mathcal{I}_{5}^{d=6-2 \epsilon}$, whose symbol can either be computed to any order in $\epsilon$ from [61] or by direct evaluation of the integral with HyperInt [62].

The test we discuss in more detail is the collinear limit of the double-trace partial amplitudes $A^{\mathrm{DT}}$. As already stated, all parity-odd contributions of any partial amplitude vanish in this limit since $\operatorname{tr}_{5}=0$. For concreteness, in the rest of this section we focus on $A^{\mathrm{DT}}[15 \mid 234]$, which in our conventions is symmetric in the (15) indices and totally antisymmetric in (234). All other double-trace amplitudes are given by simple relabelling. Scattering amplitudes obey a universal collinear factorization equation $[1,56]$. Here, we discuss the fivepoint limit $2 \| 3$ where two momenta, $k_{2}$ and $k_{3}$, become collinear $k_{2}=\tau P, k_{3}=(1-\tau) P$ with collinear splitting fraction $\tau$. The two-loop amplitude factorizes into $\sum_{\ell=0}^{2} \operatorname{Split}_{23}^{(\ell)}(\epsilon) \times \mathcal{A}_{4}^{(2-\ell)}(\epsilon)$ :


The empty blobs on the left of each diagram denote the collinear splitting functions and the filled blobs on the right are the four-point amplitudes depending only on $P, k_{1}, k_{4}$ and $k_{5}$. The color part of the splitting function is very simple: in the example above it is directly proportional to $f^{23 P}$. Kinematic expressions for the one- and two-loop splitting functions can be found in $[1,56]$. Furthermore, the one- and two-loop four-point amplitudes [1, 33], and relevant integrals [63, 64], are also known to the required order in the $\epsilon$-expansion. To approach the collinear limit, we map from the generic five-dimensional kinematic space (parametrized in terms of the $x_{i}$ of [6]) to the collinear limit. This can be done via the following substitution (see footnote 2):

$$
\begin{equation*}
x_{1} \mapsto s \tau, x_{2} \mapsto c s \delta, x_{3} \mapsto r_{2} c s \delta, x_{4} \mapsto \delta, x_{5} \mapsto-\frac{1}{c \delta} \tag{10}
\end{equation*}
$$

where $s$ characterizes the overall scale of all Mandelstam invariants, $\delta \rightarrow 0$ corresponds to the collinear limit, $\tau$ is the aforementioned collinear splitting fraction, $r_{2}=\frac{s_{15}}{s_{45}}$ the ratio of Mandelstam invariants of the underlying fourpoint process and $c \sim \frac{[23]}{\langle 23\rangle}$ corresponds to an azimuthal phase. Expanding the 31 letter alphabet to leading order
in $\delta$, we find 14 multiplicatively independent letters in the collinear limit: 7 physical $\left\{\delta, s, \tau, 1-\tau, r_{2}, 1+r_{2}, c\right\}$ (in fact this number reduces to 6 because $c \delta^{2}$ always appear in this fixed combination) and 7 spurious letters that cannot be part of the (leading power) limit. When comparing the collinear limit of our result to the factorization formula (9), we note that only Parke-Taylor factors where legs 2 and 3 are adjacent become singular. For instance, while $\mathrm{PT}[12345] \mapsto \frac{1}{\sqrt{\tau(1-\tau)}\langle 23\rangle} \mathrm{PT}[1 P 45]$, $\mathrm{PT}[12435]$ has no collinear singularity in the $2 \| 3$ limit. We find that our result exactly matches the collinear factorization formula (9). Besides this limit, there are two further inequivalent collinear limits we can check for $A^{\mathrm{DT}}[15 \mid 234]$ : when $1 \| 5$ and $1 \| 2$. When looking at the color factors of the appropriate relabelling of (9) it becomes clear that neither of them contains $\operatorname{tr}[15](\operatorname{tr}[234]-\operatorname{tr}[432])$ so $A^{\mathrm{DT}}[15 \mid 234]$ is forced to vanish in these limits. We have checked that our result indeed reproduces this behavior.

## Discussion of the result and outlook

After discussing various consistency checks of our answer for the two-loop five-point amplitude in $\mathcal{N}=4 \mathrm{sYM}$, let us briefly summarize some of its analytic features. First, we highlight in Tab. I that a number of terms in the $\epsilon$-expansion vanish, which is of course predicted by the Catani formula. We note that some of the two-loop master integrals have weight-two odd terms, but this contribution is absent from the amplitude.

|  | $1 / \epsilon^{4} w_{0}$ | $1 / \epsilon^{3} w_{1}$ | $1 / \epsilon^{2} w_{2}$ | $1 / \epsilon^{1} w_{3}$ | $\epsilon^{0} w_{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{\text {even }}^{\text {ST }}$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| $A_{\text {odd }}^{\text {ST }}$ | 0 | 0 | 0 | $\checkmark$ | $\checkmark$ |
| $A_{\text {even }}^{\text {DT }}$ | 0 | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| $A_{\text {odd }}^{\mathrm{DT}}$ | 0 | 0 | 0 | $\checkmark$ | $\checkmark$ |
| $A_{\text {even }}^{\text {SLST }}$ | 0 | 0 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| $A_{\text {odd }}^{\text {SLST }}$ | 0 | 0 | 0 | 0 | $\checkmark$ |

TABLE I. Summary of vanishing on non-vanishing terms in the $\epsilon$-expansion of the different partial amplitudes.

We also note that our answers for the amplitude, as well as individual pure master integrals, are compatible with the empirical second-entry-conditions first observed for individual integrals in [7, 24, 25, 27]. It would be very interesting to understand the underlying physical reason for this property perhaps from the point of view of a diagrammatic coaction principle [61, 65, 66].

Our full result is too lengthy to print in this letter. However, it has very restricted analytic structure. For instance, the parity-odd transcendental part of any derivative of any weight 4 function in the amplitude belongs to a 12-dimensional subspace of the 111-dimensional space of weight 3 parity-odd functions that obey integrability and the second-entry condition of [25]. This 12-dimensional subspace is spanned by the 12 inequivalent permutations,
$\Sigma_{j}$, of the $\mathcal{O}\left(\epsilon^{0}\right)$ part of the pure, parity-odd scalar pentagon in $d=6, \mathcal{I}_{5}^{d=6}\left(\Sigma_{j}\right)$. (Due to the dihedral $D_{5^{-}}$ invariance of the integral, there are only $5!/ 10=12$ inequivalent permutations.) The parity-odd part of the $1 / \epsilon$ coefficient of $\mathrm{M}_{(2)}^{\mathrm{BDS}}$ is just $-5 \mathcal{I}_{5}^{d=6}(\{12345\})$.

Let us recall that the amplitude is fully specified by $g_{234}^{\mathrm{DT}}$ and the previously-known $\mathrm{M}_{(2)}^{\mathrm{BDS}}$. We may write the odd transcendental part of the derivative of the odd part of $g_{234}^{\mathrm{DT}}$ using this $\mathcal{I}_{5}^{d=6}$ basis, as

$$
\begin{equation*}
\left.\partial_{x_{i}}\left[g_{234}^{\mathrm{DT}, \text { odd }}\right]\right|_{\mathrm{odd}}=\sum_{j, \gamma} \mathcal{I}_{5}^{d=6}\left(\Sigma_{j}\right) m_{j \gamma} \frac{\partial \log W_{\gamma}}{\partial x_{i}}, \tag{11}
\end{equation*}
$$

where $j$ labels the 12 inequivalent pentagon-permutations $\{12543\},\{12453\},\{13524\},\{12534\},\{13254\},\{12354\}$, $\{14325\},\{13425\},\{14235\},\{12435\},\{13245\},\{12345\}$, and $\gamma \in\{1, \ldots, 5\} \cup\{16, \ldots, 20\} \cup\{31\}$ are the nonzero final entries. In these conventions, the matrix $m_{j \gamma}$ is
$m_{j \gamma}=\left(\begin{array}{ccccccccccc}-\frac{17}{4} & -\frac{5}{4} & -6 & -\frac{17}{4} & -\frac{7}{2} & -\frac{17}{4} & -\frac{7}{4} & \frac{1}{2} & -1 & -\frac{17}{4} & 10 \\ \frac{17}{4} & \frac{5}{4} & \frac{5}{4} & \frac{17}{4} & 4 & \frac{17}{4} & \frac{11}{2} & \frac{17}{4} & \frac{1}{2} & \frac{1}{2} & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ -\frac{17}{4} & -6 & -\frac{5}{4} & -\frac{17}{4} & -\frac{7}{2} & \frac{1}{2} & -\frac{7}{4} & -\frac{17}{4} & -\frac{17}{4} & -1 & 10 \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & 0 & 1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{17}{4} & 6 & 6 & \frac{17}{4} & 9 & -\frac{1}{2} & 4 & -\frac{1}{2} & -\frac{5}{4} & -\frac{5}{4} & -10\end{array}\right)$
which has rank 8 , so only eight independent combinations of final entries appear.

While the first derivatives are quite constrained, the second derivatives (actually the $\{2,1,1\}$-coproducts) of the $\epsilon^{0}$-terms of the amplitude span the entire 79dimensional space identified in [25].

Building on this first analytic result for a nonplanar two-loop five-point amplitude, there are a number of avenues for future research. The upcoming work of [67] will explore the analytic structure of the factorization of the amplitude when one of the external gluons becomes soft. For this limit, there exists an eikonal semi-infinite Wilson line picture. Starting at two loops the possibility of coupling three hard lines via nontrivial color connections opens up, which leads to an interesting parity-odd component of the soft-emission function which is compatible with the soft limit of our symbol-level result. Furthermore, it would be interesting to explore the subleading-in-color behavior of this scattering amplitude in multiRegge kinematics [68-70]. With our result, it now also becomes possible to test the proposed relation between scattering amplitudes and Wilson loops beyond the leading term in the large $N_{c}$ limit [31], and it would be interesting to match our result to a future near-collinear OPE computation on the Wilson-loop side.

Since we have now computed the symbol of all relevant Feynman integrals for massless two-loop five-point scattering, we can in principle discuss other theories, such as $\mathcal{N} \leq 4 \mathrm{sYM}$ as well as $\mathcal{N} \geq 4$ supergravity. In particular, it would be interesting to investigate the uniform transcendentality (UT) property of two-loop five-point amplitudes in $\mathcal{N}=8$ supergravity. According to [71], this integrand only has logarithmic singularities and no poles at infinity, so one would expect a UT result. Finding such a result would lend further credence to the empirical relation between logarithmic poles of the integrand and transcendentality properties of amplitudes.

## Acknowledgments

L.D. and E.H. thank Falko Dulat and Hua Xing Zhu for valuable discussions, as well as Huan-Hang Chi and Yang Zhang for initial collaboration on a related project. We thank Harald Ita for useful comments on the manuscript. The work of S.A. is supported by the Fonds de la Recherche Scientifique-FNRS, Belgium. The work of L.D. and E.H. is supported by the U.S. Department of Energy (DOE) under contract DE-AC0276 SF 00515 . The work of M.Z. is supported by the Swiss National Science Foundation under contract SNF200021 179016 and the European Commission through the ERC grant pertQCD. The work of B.P. is supported by the French Agence Nationale pour la Recherche, under grant ANR-17-CE31-0001-01. We thank the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support. L.D. acknowledges support by a grant from the Simons Foundation (341344, LA).
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[^0]:    ${ }^{1}$ In our conventions, we factor out $\frac{e^{-\gamma \epsilon} g^{2} N_{c}}{(4 \pi)^{2-\epsilon}}$ per loop in a standard perturbative expansion of the amplitude. The generators of the fundamental representation of $S U\left(N_{c}\right)$ are normalized as $\operatorname{tr}\left[T^{a} T^{b}\right]=\delta^{a b}$, and $\operatorname{tr}\left[i_{1} i_{2} \ldots i_{k}\right] \equiv \operatorname{tr}\left[T^{a_{i_{1}}} T^{a_{i}} \cdots T^{a_{k}}\right]$.

[^1]:    2 Explicit expressions for our kinematics can be found in the ancillary file kinematics.m.

