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COMPARISON BETWEEN EXPERIMENTAL AND THEORETICAL RESULTS FOR THE FAST-HEAD-TAIL INSTABILITY IN PEP*

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Introduction

The "fast-head-tail" instability has been observed at several storage rings. This is a single-bunch beam instability where the unstable motion can occur in either the horizontal or vertical plane. Kohaupt[1] and Talman[2] have offered a simplified treatment of this instability by modeling the bunch as two rigid macroparticles executing synchrotron oscillations and thus exchanging their longitudinal positions periodically. While the wake field forces which drive the fast-head-tail instability are the same ones which drive the slow-head-tail instability,[3,4] the growth mechanism is considerably different. For the slow-head-tail instability the chromaticity of the storage ring couples the transverse forces and the longitudinal motion such as to produce a net damping or growth when the transverse particle motion is averaged over a longitudinal oscillation period. The slow-head-tail effect on the transverse motion is similar to the resistive effect on an oscillator where the sign and strength of the resistance is determined by the chromaticity. The growth rate is typically slow compared with the synchrotron frequency. On the other hand, the fast-head-tail instability is similar to the case of a parametric oscillator where the particle motion becomes unstable when the oscillation frequency is shifted to a resonant value. As in the case of the parametric oscillation, the coupling at threshold is so strong that a very rapid increase of growth rate with bunch current occurs once threshold is exceeded.

The two particle model describes the particle motion with two normal modes; below a certain stability threshold, these two modes are stable with different frequencies. In the limit of zero beam current only one of these modes has a center-of-charge motion. However, as the current is increased, both modes acquire center-of-charge motions and at threshold the center-of-charge components of their motions become equal in magnitude. thus when the center-of-charge motion is excited by an impulse as by an injection kicker, the relative amplitude of the two modes depends upon the ratio of bunch current to the threshold current. We shall describe the character of this coherent motion both theoretically and experimentally.

Equation of Motion

In formulating the equations of motion for the two macroparticle model we shall use a slightly different development from Kohaupt[1] in order to compare the particle motion with that observed experimentally. The model treats a bunch in the storage ring as a pair of rigid macroparticles each containing half of the population of the bunch and each oscillating longitudinally at the synchrotron frequency exactly out of phase with the other. Twice in each synchrotron period the two macroparticles pass each other longitudinally and interchange roles as head and tail bunch. As the bunch traverses the vacuum chamber, and particularly the rf cavities, each macroparticle generates a transversely deflecting electromagnetic wake field which persists behind the particle for a period which is long compared to the length (in time) of the bunch but short compared to the orbital period. With this assumption about the wake field, at any given moment the tail particle feels the wake of the head particle but not vice versa.

We take the free betatron frequencies of the two macroparticles to be equal and constant which is tantamount to setting the chromaticity equal to zero. When the head particle is executing a betatron oscillation and its transversely deflecting wake is varying accordingly, the tail particle is being driven transversely just at its resonant frequency.

Let us number the particles 1 and 2 and denote by τ_1 and τ_2 their time-displacements from the center of the bunch,

$$\tau_1 = \left(\frac{\tau_m}{2}\right) \sin \nu_s \theta \text{ and } \tau_2 = -\tau_1 \ , \tag{1}$$

where τ_m is the maximum time separation of the particles—related to the bunch length, θ is the azimuthal position of the bunch center, and ν_s is the synchrotron tune.

If ν_β is the betatron tune and $f(\tau)$ describes the time-dependence of the transversely-deflecting wake field, the equations of motion of the two particles when particle 1 leads particle 2 are

$$\begin{aligned} &x_1'' + 2\alpha x_1' + \nu_\beta^2 x_1 = 0 \ , \quad \text{and} \\ &x_2'' + 2\alpha x_2' + \nu_\beta^2 x_2 = f(\tau_1 - \tau_2)\ x_1 \end{aligned} \tag{2}$$

where x' meams $dx/d\theta$. The equations of motion when particle 2 leads particle 1 are

$$\begin{aligned} &x_1'' + 2\alpha x_1' + \nu_\beta^2 x_1 = f(\tau_1 - \tau_2)\ x_1 \ , \quad \text{and} \\ &x_2'' + 2\alpha x_2' + \nu_\beta^2 x_2 = 0 \end{aligned} \tag{3}$$

The damping coefficient α can be taken to include both the radiation damping and any other slow damping* of the coherent motion, such as the slow-head-tail damping.

Equations (2) and (3) may be solved analytically for any function $f(\tau)$. However, for ease of presentation we will restrict ourselves to the case where $f(\tau)$ is a step function, i.e., $f(\tau) = 0$ for $\tau < 0$ and $f(\tau) = K$ for $\tau > 0$ where K is a constant proportional to the bunch current.

In order to simplify the form of the solutions for x_1 and x_2, we use the following definitions:

$$\begin{aligned} &\theta_A = \theta - 2m\pi/\nu_s \\ &\nu_0^2 = \nu_\beta^2 - \alpha^2 \text{ and } P_i = (x_i' + \alpha x_i)/\nu_0 \end{aligned} \tag{4}$$

where m is an integer chosen to give $0 < \theta_A < 2\pi$.

The solutions for x_1 and x_2 can be written in one of two forms depending upon the relative position of the two particles in the bunch. When particle one leads particle two, i.e., when $0 < \theta_A < \pi$, we can write the solution in the following form

$$\begin{bmatrix} x_1 \\ P_1 \\ x_2 \\ P_2 \end{bmatrix}_\theta = \Big[TA(\theta_A) \Big] \begin{bmatrix} x_1 \\ P_1 \\ x_2 \\ P_2 \end{bmatrix}_{2m\pi/\nu_s} \tag{5}$$

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*Slow in this case means damping times long compared to the synchrotron period.

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and when particle 1 trails particle 2, i.e., when $\pi < \theta_A < 2\pi$, the following form is used for the solution

$$\begin{bmatrix} x_1 \\ P_1 \\ x_2 \\ P_2 \end{bmatrix}_\theta = \begin{bmatrix} TB(\theta_A) \end{bmatrix} \begin{bmatrix} x_1 \\ P_1 \\ x_2 \\ P_2 \end{bmatrix}_{(2m+1)\pi/\nu_s} \tag{6}$$

The 4×4 matrices for TA and TB may be written as

$$TA(\theta_A) = \begin{bmatrix} [M_1(\theta_A)] & [\ 0\ ] \\ [M_2(\theta_A)] & [M_1(\theta_A)] \end{bmatrix}$$

$$TB(\theta_A) = \begin{bmatrix} [M_1(\theta_A - \pi)] & [M_2(\theta_A - \pi)] \\ [\ 0\ ] & [M_1(\theta_A - \pi)] \end{bmatrix} \tag{7}$$

where M_1 and M_2 are 2×2 matrices given by

$$M_1(\theta_A) = e^{-\alpha\theta_A} \begin{bmatrix} \cos\nu_0\theta_A & \sin\nu_0\theta_A \\ -\sin\nu_0\theta_A & \cos\nu_0\theta_A \end{bmatrix}$$

and (8)

$$M_2(\theta_A) = e^{-\alpha\theta_A} \begin{bmatrix} \frac{K\theta_A}{2\nu_0}\sin\nu_0\theta_A & \left(-\frac{K\theta_A}{2\nu_0}\cos\nu_0\theta_A + \frac{K}{2\nu_0^2}\sin\nu_0\theta_A\right) \\ \left(\frac{K\theta_A}{2\nu_0}\cos\nu_0\theta_A + \frac{K}{2\nu_0^2}\sin\nu_0\theta_A\right) & \frac{K\theta_A}{2\nu_0}\sin\nu_0\theta_A \end{bmatrix}$$

To obtain the general solution at an arbitrary value of θ in terms of the initial values of x_1, P_1, x_2 and P_2 at $\theta = 0$ is now quite simple. First one starts with the initial values of the vector $\vec{x} = (x_1, P_1, x_2, P_2)$ and repeatedly multiplies it first by the matrix $TA(\pi/\nu_s)$, then by $TB(\pi/\nu_s)$, so that one obtains the value of vector $\vec{x}$ at the end of every half integral synchrotron oscillation. These values for $\vec{x}$ are then used as new initial conditions in Eqs. (5) and (6) to obtain $\vec{x}$ at values of θ during other portions of the synchrotron oscillation.

It should be noted that it is only necessary to evaluate the total matrix for one complete synchrotron oscillation to determine the stability of the motion. However, experimentally the motion is observed on a pickup electrode every revolution (i.e., every time $\theta = 2\pi p + \theta_0$ with p an integer). If the motion was sampled only every synchrotron oscillation period, the observed frequencies of the motion would be different from those actually obtained by observing the motion every revolution. In Appendix A we follow the development of Kohaupt and consider the transverse motion when it is sampled once every synchrotron period. If we use the approximations that $\nu_s \ll \nu_\beta$ and $\alpha \ll 1$, the instability threshold for K is then approximately given by

$$K_{threshold} = \frac{4\nu_0\nu_s}{\pi} \tag{9}$$

An experiment was performed at PEP where the center-of-charge motion of the beam was excited by a single kicker pulse. The resulting motion of the center-of-charge, observed at a pickup electrode, is shown for several values of the beam current in Figs. 1(a), 2(a) and 3(a). The instability threshold current for these series of experiments was 14 mA. We note that as the current approaches threshold, a beat pattern in the usual damped betatron oscillation appears; the amplitude of the beat increases while the beat frequency decreases. In the spectrum analysis of the motion we also observe an extra frequency appearing slightly below the usual betatron frequency.

In Figs. 1(b), 2(b) and 3(b) we show the results of a computer simulation for the two-particle model described above. The quantity plotted is the center-of-charge signal of the two-particle beam sampled every revolution for several values of the coupling parameter K as K approaches threshold. Figure 4 shows the spectrum of the two-particle motion when $K = 0.988$,

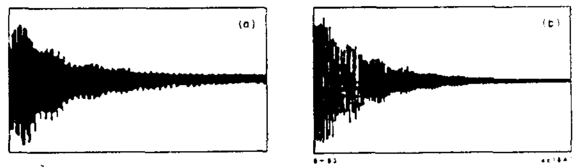

Fig. 1. Comparison of experimental measurement with the calculation using the two-particle model. Figure shows the center-of-charge response to a single kicker excitation as seen by an electrode pick-up. The beam current is 12 mA. The other parameters are $\nu_\beta = 18.19$, $\nu_s = .044$, and $K/K_{threshold} = 0.77$.

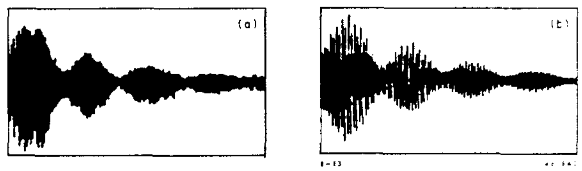

Fig. 2. Same as Fig. 1, but with beam current increased to 13 mA and $K/K_{threshold} = 0.96$.

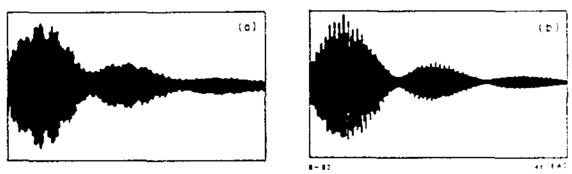

Fig. 3. Same as Fig. 1, but with beam current increased to 13.8 mA and $K/K_{threshold} = 0.99$. The threshold current is 14 mA.

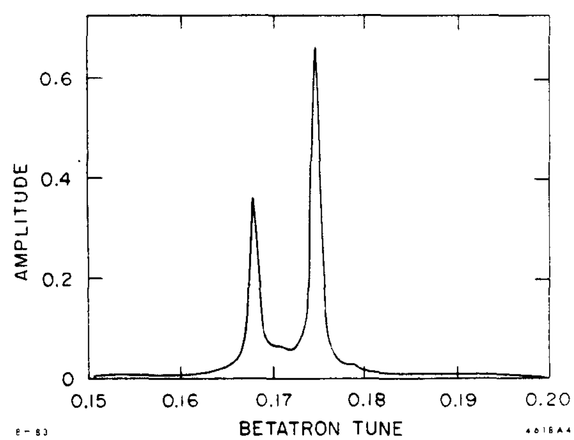

Fig. 4. Spectrum of the two-particle motion near the threshold.

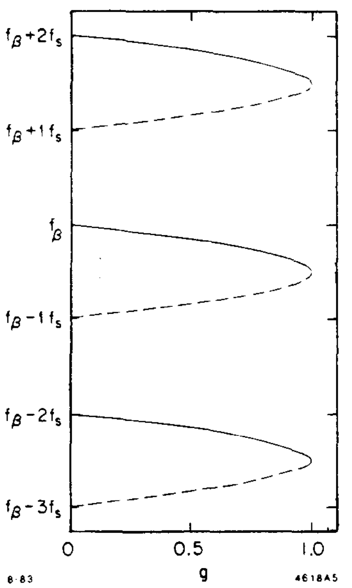

Fig. 5. The frequency spectra of the two fast-head-tail modes in the two-particle model. The solid curves are the spectra for one mode and the dashed curves are for the other mode. The frequencies of the two modes become degenerate at the instability threshold $g = K/K_{threshold} = 1$.

which is close to but still below the instability threshold value of $K = 1.02$. This spectrum can be repeated for other beam currents to yield Fig. 5 (see Appendix A). Note the similarities of the experimental results and the computer simulations although the exact values of K used for the comparison have been adjusted slightly for these comparisons.

For the experiment at PEP the approximate value of $K_{threshold}$ was 1.02 which corresponds to a deflection angle of an electron in one revolution

$$\Delta\left(\frac{dx}{ds}\right)_{tail} = 18\ \mu\text{rad/mm} \times x_{head} \quad . \tag{10}$$

This deflection angle is consistent with the estimate of the PEP rf cavity transverse impedance.

Appendix A

We define the matrix R as the transformation matrix which transforms the vector $(x_1,\ P_1,\ x_2,\ P_2)$ at $\theta_A = 0$ to the vector at $\theta_A = 2\pi/\nu_s$. Therefore

$$R \ = \ TB(\theta_A = 2\pi/\nu_s) \times TA(\theta_A = \pi/\nu_s) \quad . \tag{A1}$$

Next we define the complex phasors ξ_1 and ξ_2 by

$$(x_{1,2} + iP_{1,2}) = e^{-i\nu_0\theta}\, e^{-\alpha\theta}\, \xi_{1,2} \quad . \tag{A2}$$

If we make the approximation that $\nu_s \ll \nu_0$ then the matrix $M_2(\pi/\nu_s)$ from Eq. (8) may be approximated as

$$M_2(\pi/\nu_s) = e^{-\pi\alpha/\nu_s}\begin{bmatrix} 2g\,sin\,(\frac{\pi\nu_0}{\nu_s}) & -2g\,cos\,(\frac{\pi\nu_0}{\nu_s}) \\ 2g\,cos\,(\frac{\pi\nu_0}{\nu_s}) & 2g\,sin\,(\frac{\pi\nu_0}{\nu_s}) \end{bmatrix} \tag{A3}$$

where $g = K\pi\,/\,4\nu_0\nu_s$ is a dimensionless parameter that specifies the strength of the wake force.

The transformation of the phasors through one complete synchrotron oscillation cycle is then

$$\begin{bmatrix} \xi_1(2\pi/\nu_s) \\ \xi_2(2\pi/\nu_s) \end{bmatrix} = \begin{bmatrix} (1-4g^2) & 2ig \\ 2ig & 1 \end{bmatrix} \begin{bmatrix} \xi_1(0) \\ \xi_2(0) \end{bmatrix} \quad . \tag{A4}$$

The motion is stable when $g < 1$ and we can write the phasor eigenvalues as $\lambda_1 = e^{-i\phi}$ and $\lambda_2 = e^{i\phi}$ with $g = sin(\phi/2)$. The corresponding eigenvectors are

$$X_1 = \begin{bmatrix} 1 \\ e^{i\phi/2} \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 1 \\ -e^{-i\phi/2} \end{bmatrix} \quad . \tag{A5}$$

For the case $g > 1$ the eigenvalues are given by

$$\lambda_1 = e^{-\mu} \qquad \text{and} \qquad \lambda_2 = -e^{\mu} \qquad \text{where} \qquad g = cosh(\mu/2) \tag{A6}$$

and the corresponding eigenvectors are

$$X_1 = \begin{bmatrix} 1 \\ i\,e^{\mu/2} \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 1 \\ -i\,e^{-\mu/2} \end{bmatrix} \quad . \tag{A7}$$

For small values of the parameter $2\pi\alpha/\nu_s$, the stability limit is $\mu > 0$, or

$$K < \frac{4\nu_0\,\nu_s}{\pi} \quad \text{or} \quad g < 1 \quad . \tag{A8}$$

In the stable region with $g < 1$, the center-of-charge signal $X_1(\theta) + X_2(\theta)$ can be computed not only at half- synchrotron-period intervals but for all θ as described in the text. A Fourier analysis is performed to yield the spectrum of this signal. The result is that one mode contains the frequencies $f_\beta + (\phi/2\pi)\, f_s + mf_s$ with $m =$ odd integers, while the other mode contains the frequencies f_β - $(\phi/2\pi)\, f_s + mf_s$ with $m =$ even integers where f_β and f_s are the betatron and the synchrotron frequencies. The spectra of both modes are plotted in Fig. 5 versus the parameter g. The amplitude of these spectral lines are such that only the $m = 0$ and $m = 1$ lines are not negligible. These two lines come closer as the beam intensity is increased towards the threshold, in agreement with the experimental observation.

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