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One-Loop n-Point Gauge Theory Amplitudes, Unitarity and Collinear Limits⋆

ZVI BERN
Department of Physics, UCLA, Los Angeles, CA 90024

LANCE DIXON
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

DAVID C. DUNBAR†
Department of Physics, UCLA, Los Angeles, CA 90024

and

DAVID A. KOSOWER
Service de Physique Théorique de Saclay, Centre d'Etudes de Saclay
F-91191 Gif-sur-Yvette cedex, France

ABSTRACT

We present a technique which utilizes unitarity and collinear limits to construct ansätze for one-loop amplitudes in gauge theory. As an example, we obtain the one-loop contribution to amplitudes for n gluon scattering in $N = 4$ supersymmetric Yang-Mills theory with the helicity configuration of the Parke-Taylor tree amplitudes. We prove that our $N = 4$ ansatz is correct using general properties of the relevant one-loop n-point integrals. We also give the "splitting amplitudes" which govern the collinear behavior of one-loop helicity amplitudes in gauge theories.

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†Address after Sept. 1, 1994: University of Swansea, Swansea, SA2 8PP, UK.

1. Introduction

Although of fundamental interest in jet physics, perturbative QCD amplitudes are notoriously difficult to calculate even at tree level [1]. It has nevertheless been possible to derive a set of extremely simple formulae at tree level for "maximally helicity-violating" (MHV) amplitudes with an arbitrary number of external gluons. Parke and Taylor [2] formulated conjectures for these amplitudes in part by using an analysis of collinear limits; they were later proven by Berends and Giele [3,4] using recursion relations. The nonvanishing Parke-Taylor formulae are for amplitudes where two gluons have a given helicity, and the remaining gluons all have the opposite helicity, in the convention where all external particles are treated as outgoing. These amplitudes are called "maximally helicity-violating" because amplitudes with all helicities identical, or all but one identical, vanish at tree-level as a consequence of supersymmetry Ward identities [5]. Analytic results are available for tree amplitudes with all possible helicity configurations and seven or fewer external legs [6,7,8], and numerical implementation of the Berends-Giele recursion relations in principle allows the computation of arbitrary helicity amplitudes with an arbitrary number of external legs. The Parke-Taylor formulae have nonetheless proven useful both as exact results and in approximation schemes [9]. Their simplicity also hints at possible infinite-dimensional symmetries in four-dimensional gauge theories [10]. At one-loop, results for all helicity configurations are known for only up to five external legs; formulae for one-loop amplitudes with special helicity choices but an arbitrary number of external legs are thus perhaps even more desirable in investigations of next-to-leading order QCD corrections to multi-jet cross sections.

At loop level there are two important consistency constraints on amplitudes, collinear behavior and unitarity, which can be used as a guide in constructing ansätze for amplitudes. The use of the collinear limit at one loop is similar to that made by Parke and Taylor at tree level. As the momenta of two external legs become collinear, an n-point amplitude reduces to a sum of $(n-1)$-point amplitudes multiplied by known, singular functions — "splitting amplitudes". (The sum is over the helicity of the fused leg.) At loop level, both tree and loop $(n-1)$-point amplitudes appear, multiplied by loop and tree splitting amplitudes, respectively. The tree-level splitting amplitudes [11,1] are related to the leading-order polarized Altarelli-Parisi coefficients [12]. The loop-level splitting functions can be extracted from the collinear behavior of one-loop five-parton amplitudes. It is then possible to construct ansätze for n-point one-loop amplitudes by a bootstrap approach, in which the correct collinear behavior is imposed on the ansatz, and the procedure is jump-started with known one-loop lower-point results, such as the one-loop five-gluon amplitudes [13] calculated recently using string-based methods [14,15,16].

The second constraint is that of perturbative unitarity. We apply the Cutkosky rules [17] at the amplitude level to determine the absorptive parts (cuts) of the amplitude in all possible channels. The cut amplitude can be written as a tree amplitude on one side of the cut, multiplied by a tree amplitude on the other side of the cut, with the loop integral replaced by an integral over the phase space of the particles crossing the cut. These cuts are generally much simpler to evaluate than the full amplitude. For example, for MHV one-loop amplitudes it turns out that only the MHV tree amplitudes, given by the Parke-Taylor formulae, are required to evaluate the cuts, so that the procedure can be carried out for an arbitrary number of external legs. Furthermore, it is possible to calculate cuts in terms of the imaginary parts of one-loop integrals that would have been encountered in a direct calculation. This makes it straightforward to write down an analytic expression with the correct cuts in all channels, thus avoiding the need to do a dispersion integral directly. The unitarity constraint leaves one with a potential ambiguity at the level of additive polynomial terms in the amplitude. (By 'polynomial terms' we actually mean any cut-free function of the kinematic invariants and spinor products, that is any rational function of these variables.) The collinear constraint can be used to resolve much of this ambiguity.

We expect that the twin constraints of unitarity and the collinear limits will have applicability in generating consistent ansätze for a broad set of one-loop gauge amplitudes; both for general helicity configurations but relatively few external legs, as are required for next-to-leading-order QCD predictions for multi-jet processes at hadron colliders; and also for special helicity configurations with an arbitrary number of external legs. In this paper we focus on the latter application, and more specifically on one-loop MHV amplitudes in $N = 4$ $SU(N_c)$ super-Yang-Mills theory, where we will be able to obtain results for an arbitrary number of external gluons. Supersymmetry Ward identities [5] allow us to obtain additional amplitudes, where certain of the gluons are replaced by fermions or scalars in the $N = 4$ theory. (We use a supersymmetry-preserving regulator [18,14,19,20].) One-loop $N = 4$ amplitudes for four external gluons were first calculated by Green, Schwarz and Brink, as the low-energy limit of a superstring amplitude [21].

The $N = 4$ super-Yang-Mills results presented here can be used as a part of the computation of the corresponding n-gluon helicity amplitude in QCD, where gluons and quarks circulate in the loop. As is manifest in the string-based rules [13,19], one can think of both the gluon and quark contributions to the n-gluon QCD amplitude as different linear combinations of (a) an $N = 4$ supersymmetric amplitude, (b) an $N = 1$ supersymmetric amplitude, and (c) a scalar in the loop. The $N = 4$ super-Yang-Mills results are thus one of the three components of a QCD calculation organized in this manner. There are two advantages of such an organization: on the one hand, supersymmetry cancellations (and explicit four- and five-point results) suggest that the expressions for amplitudes (a) and perhaps (b) should be relatively simple; on the other hand, the remaining scalar loop computation (c) is much easier than a direct gluon (or quark) loop computation, because the scalar carries no spin information around the loop. This decomposition can also reveal structure in gauge-theory amplitudes that would otherwise remain hidden [22].

For the case of the $N = 4$ super-Yang-Mills theory there is a third constraint which allows us to prove that the process outlined above generates the correct amplitude. One can perform a "gedanken calculation" of the loop amplitude, using either superspace techniques [23] or a string-based formalism [14,15,22]. In either approach, one finds in each diagram a manifest cancellation in the numerator of the integrand, such that the numerator loop-momentum polynomial has a degree which is four less than in the pure gluonic case. Standard integral reduction formulae [24,25] (or their equivalents in the language of Feynman-parametrized integrals [26]) allow one to evaluate loop integrals in terms of a linear combination of box integrals, triangles and bubbles. When the reduction formulae are applied to the $N = 4$ integrands, only *scalar* box integrals survive (box integrals where the loop momentum polynomial in the numerator is a constant). Using this fact we can show that the cuts uniquely determine the amplitude, proving the correctness of the ansatz obtained via the unitary-collinear bootstrap. The physical inputs we use to determine the result are summarized in fig. 1.

In a non-supersymmetric theory, such as QCD, one-loop n-gluon amplitudes with all gluon helicities identical, or all but one helicity identical, do not have to vanish. However, such amplitudes are pure polynomials — all cuts vanish. Also, because these amplitudes vanish in supersymmetric theories [5], the contributions from particles of different spin in the loop are equal up to multiplicative constants. For the identical-helicity case, the collinear constraints described above have been used to construct an ansatz [27,28] which has been proven correct by a recursive procedure [29,30]. Mahlon [29] has also constructed an all-n formula for the configuration with one leg of opposite helicity from the rest. Since tree-level amplitudes vanish for these helicity configurations, these amplitudes do not provide a vehicle for studying next-to-leading-order corrections to multi-jet QCD cross-sections.

Other, related examples of n-point loop amplitudes that are known for all n include the n-photon massless QED amplitudes where all photon helicities are identical, or all but one are identical, which have recently been shown to vanish for five or more legs by Mahlon [31]. The QED results can be generalized to amplitudes with external photons and gluons, interacting via a massless quark loop. Amplitudes with five or more legs, where three or more legs are photons instead of gluons, have been shown to vanish when all the helicities are identical [28], and also when one of the photon helicities is reversed [30].

Recently, it has been suggested that the tree amplitudes for jet production grow surprisingly fast for large numbers of external legs [32]. It may be interesting to apply our results to determine whether the one-loop corrections modify this behavior. To do so one would need to cancel the infrared singularities using, for example, the methods of refs. [33,34]. (For other methods, see refs. [35,36].)

This paper is organized as follows: in section 2, we review relevant previous results for tree and one-loop amplitudes. In section 3 we describe the collinear behavior required for a general amplitude and in section 4 we impose this collinear behavior in order to construct an ansatz for the leading-color part of the $N = 4$ super-Yang-Mills n-point MHV amplitude. In section 5 we describe how to calculate the cuts for this amplitude, and show that the ansatz has the correct cuts. In section 6 we prove that the ansatz is correct using the structure of the loop integrals. In section 7 we give a general formula which expresses subleading-color contributions to n-gluon amplitudes in terms of leading-color contributions for adjoint representation particles in the loop. Section 8 contains our conclusions, appendix I collects some needed formulae for scalar box integrals, appendix II contains the splitting amplitudes appearing in the collinear limits, and in appendix III we apply the general formula of section 7 to derive an explicit form for the subleading-color $N = 4$ supersymmetric amplitudes.

2. Review of known results

Tree-level amplitudes for $U(N_c)$ or $SU(N_c)$ gauge theory with n external gluons can be decomposed into color-ordered partial amplitudes, multiplied by an associated color trace. Summing over all non-cyclic permutations reconstructs the full amplitude,

$$\mathcal{A}_n^{\rm tree}(\{k_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\; A_n^{\rm tree}(k_{\sigma(1)}^{\lambda_{\sigma(1)}},\ldots,k_{\sigma(n)}^{\lambda_{\sigma(n)}}) \;, \tag{2.1}$$

where k_i, λ_i, and a_i are respectively the momentum, helicity ($\pm$), and color index of the i-th external gluon, g is the coupling constant, and S_n/Z_n is the set of non-cyclic permutations of $\{1,\ldots,n\}$. The $U(N_c)$ ($SU(N_c)$) generators T^a are the set of hermitian (traceless hermitian) $N_c \times N_c$ matrices, normalized so that ${\rm Tr}\left(T^aT^b\right) = \delta^{ab}$. The color decomposition (2.1) can be derived in conventional field theory simply by using

$$f^{abc} = -\frac{i}{\sqrt{2}}\,{\rm Tr}\left(\left[T^a,T^b\right]T^c\right), \tag{2.2}$$

where the T^a may by either $SU(N_c)$ matrices or $U(N_c)$ matrices. The structure constants f^{abc} vanish when any index belongs to the $U(1)$, which is generated by the matrix $T^{a_{U(1)}} \equiv \mathbf{1}/\sqrt{N_c}$; therefore the partial amplitudes satisfy the $U(1)$ decoupling identities [8,3]

$$\mathcal{A}_n(\{k_i, \varepsilon_i, a_i\}_{i=1}^{n-1}; k_n, \varepsilon_n, a_{U(1)}) = 0 \,. \tag{2.3}$$

An advantage of using $U(N_c)$ matrices in the color decomposition is that the $U(N_c)$ Fierz identities

$$\mathrm{Tr}(T^a X)\,\mathrm{Tr}(T^a Y) = \mathrm{Tr}(XY) \tag{2.4a}$$

$$\mathrm{Tr}(T^a X T^a Y) = \mathrm{Tr}(X)\,\mathrm{Tr}(Y) \tag{2.4b}$$

are simpler than their $SU(N_c)$ counterparts. This is useful both when performing a color decomposition on Feynman diagrams, and when squaring and summing over colors in order to obtain the cross section.

In a supersymmetric theory, amplitudes with all helicities identical, or all but one identical, vanish due to supersymmetry Ward identities [5]. Tree-level gluon amplitudes in super-Yang-Mills and in purely gluonic Yang-Mills are identical (fermions do not appear at this order), so that

$$A_n^{\rm tree}(1^\pm, 2^+, \ldots, n^+) = 0. \tag{2.5}$$

Parity may of course be used to simultaneously reverse all helicities in a partial amplitude. The non-vanishing Parke-Taylor formulae are for maximally helicity-violating (MHV) partial amplitudes, those with two negative helicities and the rest positive,

$$A_{jk}^{\rm tree\ MHV}(1, 2, \ldots, n) \;=\; i\, \frac{\langle j\, k\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \,, \tag{2.6}$$

where we have introduced the notation

$$A_{jk}^{\rm MHV}(1, 2, \ldots, n) \;\equiv\; A_n(1^+, \ldots, j^-, \ldots, k^-, \ldots, n^+), \tag{2.7}$$

for a partial amplitude where j and k are the only legs with negative helicity. Our convention is that all legs are outgoing. The result (2.6) is written in terms of spinor inner-products, $\langle j\, l\rangle = \langle j^- | l^+\rangle = \bar{u}_-(k_j) u_+(k_l)$ and $[j\, l] = \langle j^+ | l^- \rangle = \bar{u}_+(k_j) u_-(k_l)$, where $u_\pm(k)$ is a massless Weyl spinor with momentum k and chirality $\pm$ [37,1].

For one-loop amplitudes, one may perform a similar color decomposition to the tree-level decomposition (2.1); in this case, there are up to two traces over color matrices [38], and one must also sum over the different spins J of the internal particles circulating in the loop. When all internal particles transform as color adjoints, as is the case for $N = 4$ supersymmetric Yang-Mills theory, the result takes the form

$$\mathcal{A}_n\left(\{k_i, \lambda_i, a_i\}\right) = \sum_J n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}\left(\sigma\right)\ A_{n;c}^{[J]}(\sigma), \tag{2.8}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and n_J is the number of particles of spin J. The leading color-structure factor

$$\mathrm{Gr}_{n;1}(1) = N_c\ \ \mathrm{Tr}\left(T^{a_1} \cdots T^{a_n}\right) \tag{2.9}$$

is just N_c times the tree color factor, and the subleading color structures are given by

$$\mathrm{Gr}_{n;c}(1) = \mathrm{Tr}\left(T^{a_1} \cdots T^{a_{c-1}}\right)\ \mathrm{Tr}\left(T^{a_c} \cdots T^{a_n}\right). \tag{2.10}$$

S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant. Once again it is convenient to use $U(N_c)$ matrices; the extra $U(1)$ decouples from all final results [38]. (For internal particles in the fundamental $(N_c + \bar{N}_c)$ representation, only the single-trace color structure ($c = 1$) would be present, and the corresponding color factor would be smaller by a factor of N_c. In this case the $U(1)$ gauge boson will *not* decouple from the partial amplitude, so one should only sum over $SU(N_c)$ indices when color-summing the cross-section.) In each case the massless spin-J particle is taken to have two helicity states: gauge bosons, Weyl fermions, and complex scalars.

In the next-to-leading-order correction to the cross-section, summed over colors, the leading contribution for large N_c comes from $A_{n;1}^{[J]}$; the subleading-in-N_c corrections $A_{n;c}^{[J]}$ ($c > 1$) are down by a factor of $1/N_c^2$ [38]. In section 7 we will show how to obtain $A_{n;c}^{[J]}$ as a sum over permutations of the leading contribution $A_{n;1}^{[J]}$. Therefore, it is sufficient to focus on the calculation of $A_{n;1}^{[J]}$.

Recently, a useful technique based on string theory [14,15] has been developed for calculating one-loop amplitudes explicitly, providing the first calculation of all one-loop five-gluon helicity amplitudes [13]. This method is an alternative to the conventional Feynman diagram expansion, but can be understood as a reorganization of field theory [16] and is also useful in the calculation of effective actions [39]. With this method the gluon amplitudes are most naturally written in a form [13,19] which takes advantage of the simplicity of contributions from supersymmetry multiplets,

$$\begin{aligned}
A_{n;1}^{[0]} &= c_\Gamma \left(V_n^s A_n^{\mathrm{tree}} + i F_n^s\right)\ , \\
A_{n;1}^{[1/2]} &= -c_\Gamma \left((V_n^f + V_n^s) A_n^{\mathrm{tree}} + i(F_n^f + F_n^s)\right)\ , \\
A_{n;1}^{[1]} &= c_\Gamma \left((V_n^g + 4V_n^f + V_n^s) A_n^{\mathrm{tree}} + i(F_n^g + 4F_n^f + F_n^s)\right)\ ,
\end{aligned} \tag{2.11}$$

where the prefactor is

$$c_\Gamma \;=\; \frac{(4\pi)^\epsilon}{16\pi^2}\,\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\ , \tag{2.12}$$

with $\epsilon = (4-D)/2$ the dimensional regularization parameter. The V_n's contain the singular parts of the amplitude (poles as $\epsilon \to 0$), which must be proportional to the tree; the F_n's are finite as $\epsilon \to 0$ and need not be proportional to the tree. (There is some freedom in assigning terms to either V_n or F_n.)

The organization (2.11) in terms of g, f and s pieces amounts to calculating the fermion and gluon loop contributions in terms of the scalar loop contributions plus the contributions from supersymmetric multiplets. For an $N = 4$ super-Yang-Mills theory, summing over the contributions from one gluon, four Weyl fermion and three complex (or six real) scalars, all functions except V_n^g and F_n^g cancel from eq. (2.11) and the amplitudes are

$$A_{n;1}^{N=4} \;\equiv\; A_{n;1}^{[1]} + 4A_{n;1}^{[1/2]} + 3A_{n;1}^{[0]} \;=\; c_\Gamma\,(V_n^g\ A_n^{\rm tree} + iF_n^g). \tag{2.13}$$

For an $N = 1$ chiral multiplet, containing one scalar and one Weyl fermion, only the functions V_n^f and F_n^f survive,

$$A_{n;1}^{N=1\ {\rm chiral}} \;\equiv\; A_{n;1}^{[1/2]}\ + A_{n;1}^{[0]} =\; -c_\Gamma\,\left(V_n^f A_n^{\rm tree} + iF_n^f\right). \tag{2.14}$$

Organization in terms of g, f and s pieces ($N = 4$, $N = 1$ chiral, and scalar contributions) is convenient because in the string-based method or in a superspace approach there are diagram-by-diagram cancellations within a supermultiplet which lead to significant simplifications. In the $N = 1$ chiral contribution, the cancellation is easy to see. The contribution of a complex scalar loop to the effective action is

$$\Gamma_{\rm scalar}[A] = \ln\det{}^{-1}(D_\mu D^\mu), \tag{2.15}$$

where $D_\mu = \partial_\mu + igA_\mu$, while the contribution of a Weyl fermion (coupled non-chirally), in the second-order formalism motivated by the string-based method [16], is

$$\begin{aligned}\Gamma_{\rm fermion}[A] &= \ln\det{}_+ \not{D} \\ &= \ln\det{}^{1/4}(D_\mu D^\mu - \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}),\end{aligned} \tag{2.16}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ is the spin-$\frac{1}{2}$ Lorentz generator. Expanding the scalar operator $D_\mu D^\mu$ about the free operator $\partial_\mu\partial^\mu$ generates derivative interactions, $\bar\phi A^\mu \partial_\mu \phi$, etc. These lead to a loop-momentum polynomial of maximum degree m for an m-point contribution to the effective action for a scalar in the loop. For the $N = 1$ chiral multiplet, however, the terms that come solely from expanding $D_\mu D^\mu$ cancel between scalar and fermion (since the Dirac trace $\mathrm{Tr}(1) = 4$). Surviving terms require at least two insertions of $\sigma^{\mu\nu}F_{\mu\nu}$ in the fermion loop, since $\mathrm{Tr}(\sigma_{\mu\nu}) = 0$. Since $\sigma^{\mu\nu}F_{\mu\nu}$ contains no derivatives with respect to the fermion field, the maximum degree of the loop-momentum polynomial for the $N = 1$ chiral multiplet is two smaller than for a scalar, fermion, or gluon separately, namely $m-2$ for an m-point contribution [19,16,22].

In an $N = 4$ supersymmetric theory there are further cancellations. In the string-based method (as mapped back from the language of Feynman parameters to that of loop momenta [16]), two and three insertions of the operator $\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}$ for a fermion loop cancel against two and three insertions of the corresponding operator $\Sigma^{\mu\nu}F_{\mu\nu}$ for a gluon loop, where $\Sigma^{\mu\nu}$ is the spin-one Lorentz generator. Thus the loop-momentum polynomials entering into the calculation of V_n^g and F_n^g have a maximum degree four smaller than for a scalar in the loop, namely $m-4$ for an m-point contribution [19,22].

This $N = 4$ cancellation may also be seen in superspace, following Gates et al. [23]. In an $N = 1$ superspace calculation in supersymmetric background field gauge, there are three chiral ghost superfields from fixing the gauge symmetry. In the $N = 4$ supersymmetric theory, there are three chiral matter superfields which cancel the ghost contributions to the one-loop effective action, leaving only the contribution of the vector superfield V. The vector superfield couples to external fields via an interaction of the form

$$V\left(W^\alpha D_\alpha + \bar{W}^{\dot\alpha}\bar{D}_{\dot\alpha} \ + \ (D_\alpha, \bar{D}_{\dot\alpha})\text{-independent terms}\right)V, \tag{2.17}$$

where W^α is the supersymmetric background field strength and D_α is the superspace covariant derivative. At least four D's are needed in a loop, so one needs to use at least four insertions of $W^\alpha D_\alpha$ or $\bar{W}^{\dot\alpha}\bar{D}_{\dot\alpha}$. As in the earlier $N = 1$ discussion, each insertion costs a power of the loop momentum, leading to a maximum degree four smaller than the scalar case. The $N = 4$ result is related to the ultraviolet finiteness of $N = 4$ super-Yang-Mills [40], since it forces the potentially divergent two- and three-point contributions to the effective action to vanish.

The simplicity of the g, f, s organization, suggested by supersymmetry, is confirmed by explicit four-gluon [14,20] and five-gluon [13] results. The separate contributions V_n^s, F_n^s, V_n^f, F_n^f, V_n^g and F_n^g are simpler than the full answer for $n = 4, 5$. The $N = 4$ supersymmetric part is the simplest of all — F_4^g and F_5^g vanish, and V_4^g and V_5^g are universal functions, independent of the particular helicity configuration. For the four-point amplitude,

$$V_4^g = -\frac{2}{\epsilon^2}\left[\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon\right] + \ln^2\left(\frac{-s_{12}}{-s_{23}}\right) + \pi^2 \ , \tag{2.18}$$

where $s_{12} = (k_1 + k_2)^2$ and $s_{23} = (k_2 + k_3)^2$ are the usual Mandelstam variables, μ is the renormalization scale, and we have used a supersymmetry-preserving regulator such as dimensional reduction [18,20] or the 'four-dimensional helicity scheme' [14,19]. This four-point $N = 4$ super Yang-Mills amplitude was first calculated using string theory [21]. For the five-point amplitude [13],

$$V_5^g = \sum_{i=1}^{5} -\frac{1}{\epsilon^2}\Big(\frac{\mu^2}{-s_{i,i+1}}\Big)^\epsilon + \sum_{i=1}^{5} \ln\left(\frac{-s_{i,i+1}}{-s_{i+1,i+2}}\right) \ln\left(\frac{-s_{i+2,i+3}}{-s_{i-2,i-1}}\right) + \frac{5}{6}\pi^2 \,. \tag{2.19}$$

As usual, $s_{i,i+1} = (k_i + k_{i+1})^2$.

The remarkable simplicity of these results suggests that it may be possible to find a closed-form expression for certain $N = 4$ supersymmetric amplitudes for all n. Since in a supersymmetric theory $A_n(1^\pm, 2^+, 3^+, \ldots, n^+)$ vanishes [5], we consider in this paper the maximally helicity-violating one-loop amplitudes that do not vanish, $A_{jk}^{N=4 \text{ MHV}}(1, 2, \ldots, n)$.

3. Collinear singularities

In the next section we shall construct an ansatz for the one-loop helicity amplitude $A_{jk}^{N=4 \text{ MHV}}(1, 2, \ldots, n)$. The ansatz is based on examining collinear singularities of amplitudes, starting from the known four- and five-point functions in eqs. (2.18) and (2.19). In this section we describe the collinear behavior of the partial amplitudes $A_{n;1}$ which multiply the single trace structure $N_c \operatorname{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$. (The partial amplitudes $A_{n;c}$, relevant at subleading order in N_c, are determined explicitly in section 7 in terms of $A_{n;1}$, so we need not examine their collinear behavior separately.) We envisage broader applications of these techniques than just to $N = 4$ supersymmetric theories, so in appendix II we give the more general one-loop collinear behavior of (nonsupersymmetric) gauge theory amplitudes with external quarks as well as gluons.

Consider first the n-point tree-level partial amplitude $A_n(1, 2, \ldots, n)$ with an arbitrary helicity configuration. The external legs may be fermions or gluons. There is an implicit color ordering of the vertices $1, 2, \ldots, n$, so that collinear singularities arise only from *neighboring* legs a and b becoming collinear [11,1]. These singularities have the form

$$A_n^{\text{tree}} \xrightarrow{a\parallel b} \sum_{\lambda=\pm} \text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b})\, A_{n-1}^{\text{tree}}(\ldots (a+b)^\lambda \ldots) \,, \tag{3.1}$$

where the non-vanishing splitting amplitudes diverge as $1/\sqrt{s_{ab}}$ in the collinear limit $s_{ab} = (k_a + k_b)^2 \to 0$. In the collinear limit $k_a = z\, P$, $k_b = (1 - z)\, P$, where P is the sum of the collinear momenta; λ is the helicity of the intermediate state with momentum P. The tree splitting amplitudes $\mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ may be found in refs. [2,41,3,1] and are included in appendix II.

The collinear limits of the (color-ordered) one-loop partial amplitudes are expected to have the following form:

$$A_{n;1}^{\mathrm{loop}} \stackrel{a\parallel b}{\longrightarrow} \sum_{\lambda=\pm} \Big(\mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A_{n-1;1}^{\mathrm{loop}}(\ldots (a+b)^{\lambda} \ldots) \\ + \mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A_{n-1}^{\mathrm{tree}}(\ldots (a+b)^{\lambda} \ldots) \Big), \tag{3.2}$$

as shown diagrammatically in fig. 2. The splitting amplitudes $\mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ and $\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ are universal: they depend only on the two legs becoming collinear, and not upon the specific amplitude under consideration. Intuitively, the collinear splitting amplitude describes infrared or long-distance behavior, which is not sensitive to the short-distance details such as the helicities and momenta of the other, hard, external legs. We expect this universal behavior to hold for all one-loop amplitudes, with external (massless) fermions as well as gluons; all one-loop amplitudes that we have inspected do indeed obey eq. (3.2). (A similar equation is expected to govern the limit of one-loop partial amplitudes as one external gluon momentum becomes soft.) The explicit $\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ have been determined from the known four- and five-point one-loop amplitudes [13,42], and are collected in appendix II. An outline of a direct proof of the universality of the splitting amplitudes for the scalar-loop contributions to amplitudes with external gluons was presented in ref. [28]; a more detailed discussion of collinear limits will be given elsewhere. One can also give an indirect universality argument using QCD factorization theorems [43]. From this point of view, the one-loop splitting amplitudes presented here should be related to the spin-polarized versions of virtual corrections to evolution of parton distribution functions [44] and of jet calculus [45].

We may extract the splitting amplitudes $\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ for $g \to gg$ in an $N = 4$ supersymmetric theory rather easily by inspecting the collinear limits of the expressions (2.13), (2.18) and (2.19) for the four- and five-point amplitudes. Since these $N = 4$ scattering amplitudes are proportional to the corresponding tree amplitudes ($F_4^g = F_5^g = 0$), the supersymmetric loop splitting amplitudes must be proportional to the tree splitting amplitudes,

$$\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b}) \;=\; c_\Gamma \times \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b}) \times r_S^{\mathrm{SUSY}}(z, s_{ab}), \tag{3.3}$$

where $s_{ab} = (k_a + k_b)^2$, and the ratio $r_S^{\mathrm{SUSY}}(z, s_{ab})$ is independent of the helicities. Equations (3.1), (3.2) further require the ratio to obey

$$V_5^g \stackrel{a\parallel b}{\longrightarrow} V_4^g \;+ r_S^{\mathrm{SUSY}}(z, s_{ab})\ , \tag{3.4}$$

implying that

$$r_S^{\rm SUSY}(z,s) \;=\; -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s)}\right)^{\epsilon} + 2\ln z\,\ln(1-z) - \frac{\pi^2}{6}\;. \tag{3.5}$$

Equations (3.3), (3.5) turn out to give the collinear behavior of amplitudes in $N=1$ super Yang-Mills (with no matter fields), as well as in the $N=4$ theory, and they hold for external gluinos as well as gluons. (See appendix II.)

The collinear behavior places tight constraints on the possible form of amplitudes. However, one must be aware that there do exist non-vanishing functions which may appear in amplitudes but do not have singular collinear behavior in any channel. The simplest non-trivial example of such a function is the five-point function

$$\frac{\varepsilon(1,2,3,4)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\;, \tag{3.6}$$

since the contracted antisymmetric tensor $\varepsilon(1,2,3,4) \equiv 4i\varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma$ vanishes when any two of the five vectors k_i become parallel ($\sum_{i=1}^5 k_i = 0$). Another example is the six-point function

$$\sum_{P(1,\ldots,5)} \frac{\ln(-s_{12})+\ln(-s_{23})+\cdots+\ln(-s_{61})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle} \tag{3.7}$$

where the summation is over all 120 permutations of legs 1 through 5. Without additional information, functions such as (3.6) and (3.7) represent additive ambiguities in the collinear bootstrap. Functions of the type (3.6) are somewhat more insidious because they cannot be detected by unitarity cuts.

In the following section we apply the constraints on collinear behavior to construct an ansatz for $N=4$ MHV amplitudes with an arbitrary number of external legs. In spite of the potential ambiguities mentioned above, we will prove that at least for these MHV amplitudes, the collinear bootstrap leads naturally to the correct result.

4. $N=4$ supersymmetric amplitudes

In this section we use the collinear limits to construct an ansatz for the one-loop leading-color MHV partial amplitudes in $N=4$ super-Yang-Mills theory. Our starting point will be the assumption that the n-point amplitude has the same structure as that found at the four- and five-point level,

$$A_{n;1}^{N=4\ \rm MHV\ loop} \;=\; c_\Gamma \times A_n^{\rm tree} \times V_n^g\;, \tag{4.1}$$

where V_n^g has no collinear poles but contains the logarithms and dilogarithms found in a loop amplitude. With this assumption, and demanding that this amplitude have the expected collinear limit (3.2), we find that the function V_n^g must satisfy the generalization of equation (3.4),

$$V_n^g \stackrel{a\|b}{\longrightarrow} V_{n-1}^g \ + r_S^{\rm SUSY}(z, s_{ab}). \tag{4.2}$$

In constructing the ansatz it is useful to know the set of integrals that can appear as a result of explicit diagrammatic calculation, since this information restricts the form of the possible logarithms and dilogarithms in the result. As we shall discuss later, this set is precisely the set of scalar box integrals, which are given in appendix I. Using the five-point function V_5^g (2.19) to jump-start the ansatz, and experimenting with functional forms suggested by these integrals for small n, we can find a function with the expected collinear limit (4.2) for all $n \geq 5$,

$$V_n^g = \sum_{i=1}^{n} -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-t_i^{[2]}}\right)^{\epsilon} - \sum_{r=2}^{\lfloor n/2\rfloor -1}\sum_{i=1}^{n} \ln\left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}}\right)\ln\left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}}\right) + D_n + L_n + \frac{n\pi^2}{6}\ , \tag{4.3}$$

where

$$t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2 \tag{4.4}$$

are the momentum invariants, so that $t_i^{[1]} = 0$ and $t_i^{[2]} = s_{i,i+1}$. (All indices are understood to be mod n.) The form of D_n and L_n depends upon whether n is odd or even. For $n = 2m+1$,

$$D_{2m+1} = -\sum_{r=2}^{m-1}\left(\sum_{i=1}^{n} {\rm Li}_2\left[1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right]\right)\ , \tag{4.5}$$

$$L_{2m+1} = -\frac{1}{2}\sum_{i=1}^{n} \ln\left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}}\right)\ , \tag{4.6}$$

whereas for $n = 2m$,

$$D_{2m} = -\sum_{r=2}^{m-2}\left(\sum_{i=1}^{n} {\rm Li}_2\left[1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right]\right) - \sum_{i=1}^{n/2} {\rm Li}_2\left[1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}}\right]\ , \tag{4.7}$$

$$L_{2m} = -\frac{1}{4}\sum_{i=1}^{n} \ln\left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}}\right)\ . \tag{4.8}$$

For $n = 6$ we have verified this result explicitly by direct calculation using the string-based method.

The functions D_n containing the dilogarithms first appear in the six-point amplitude. One must use various dilogarithm identities to show that the function V_n^g has the expected collinear behavior. Note that the explicit cyclic symmetry of V_n^g allows us to verify the behavior (4.2) for just one collinear pair of legs a, b. For example, consider the six-point function where

$$D_6 \;=\; -\mathrm{Li}_2\left[1-\frac{t_1^{[2]}t_6^{[4]}}{t_1^{[3]}t_6^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_2^{[2]}t_1^{[4]}}{t_2^{[3]}t_1^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_3^{[2]}t_2^{[4]}}{t_3^{[3]}t_2^{[3]}}\right], \tag{4.9}$$

and let legs 5 and 6 become collinear: $k_5 \to z k_5'$ and $k_6 \to (1-z)k_5'$. We find

$$D_6 \;\to\; -\mathrm{Li}_2\left[1-\frac{z t_1^{[2]}}{z t_1^{[2]}+(1-z)t_3^{[2]}}\right]-\frac{\pi^2}{6}-\mathrm{Li}_2\left[1-\frac{(1-z)t_3^{[2]}}{z t_1^{[2]}+(1-z)t_3^{[2]}}\right], \tag{4.10}$$

where we have used momentum conservation appropriate for the five point function. Using the dilogarithm identity [46]

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{\pi^2}{6}-\ln(x)\ln(1-x), \tag{4.11}$$

this reduces to

$$-\frac{\pi^2}{3}+\ln\left(\frac{z t_1^{[2]}}{z t_1^{[2]}+(1-z)t_3^{[2]}}\right)\ln\left(\frac{(1-z)t_3^{[2]}}{z t_1^{[2]}+(1-z)t_3^{[2]}}\right). \tag{4.12}$$

These logarithms combine with those arising from other parts of the amplitude to give the correct collinear limit.

For the collinear limit of a seven-point function the dilogarithms become more complicated. Take the dilogarithms in the seven-point function

$$\begin{aligned} D_7 \;=\; & -\mathrm{Li}_2\left[1-\frac{t_1^{[2]}t_7^{[4]}}{t_1^{[3]}t_7^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_2^{[2]}t_1^{[4]}}{t_2^{[3]}t_1^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_3^{[2]}t_2^{[4]}}{t_3^{[3]}t_2^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_4^{[2]}t_3^{[4]}}{t_4^{[3]}t_3^{[3]}}\right] \\ & -\mathrm{Li}_2\left[1-\frac{t_5^{[2]}t_4^{[4]}}{t_5^{[3]}t_4^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_6^{[2]}t_5^{[4]}}{t_6^{[3]}t_5^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_7^{[2]}t_6^{[4]}}{t_7^{[3]}t_6^{[3]}}\right] \end{aligned} \tag{4.13}$$

and consider the limit where legs 6 and 7 become collinear, $k_6 \to z k_6'$, $k_7 \to (1-z)k_6'$. Then

$$\begin{aligned} D_7 \to & -\mathrm{Li}_2\left[1-\frac{t_1^{[2]}((1-z)t_6^{[4]}+z t_1^{[3]})}{t_1^{[3]}((1-z)t_6^{[3]}+z t_1^{[2]})}\right]-\mathrm{Li}_2\left[1-\frac{t_2^{[2]}t_1^{[4]}}{t_2^{[3]}t_1^{[3]}}\right]-\mathrm{Li}_2\left[1-\frac{t_3^{[2]}t_2^{[4]}}{t_3^{[3]}t_2^{[3]}}\right] \\ & -\mathrm{Li}_2\left[1-\frac{t_4^{[2]}((1-z)t_3^{[3]}+z t_3^{[4]})}{t_3^{[3]}((1-z)t_4^{[2]}+z t_4^{[3]})}\right]-\mathrm{Li}_2\left[1-\frac{z t_4^{[3]}}{(1-z)t_4^{[2]}+z t_4^{[3]}}\right]-\frac{\pi^2}{6} \\ & -\mathrm{Li}_2\left[1-\frac{(1-z)t_6^{[3]}}{(1-z)t_6^{[3]}+z t_1^{[2]}}\right]. \end{aligned} \tag{4.14}$$

This expression can be rewritten, using momentum conservation appropriate for the six-point amplitude, as

$$D_6 + \mathrm{Li}_2\Big[1 - \frac{t_1^{[2]} t_4^{[2]}}{t_1^{[3]} t_3^{[3]}}\Big] - \mathrm{Li}_2\Big[1 - \frac{t_1^{[2]}((1-z)t_4^{[2]} + zt_1^{[3]})}{t_1^{[3]}((1-z)t_3^{[3]} + zt_1^{[2]})}\Big] - \mathrm{Li}_2\Big[1 - \frac{t_4^{[2]}((1-z)t_3^{[3]} + zt_1^{[2]}))}{t_3^{[3]}((1-z)t_4^{[2]} + zt_1^{[3]})}\Big]$$
$$- \mathrm{Li}_2\Big[1 - \frac{zt_1^{[3]}}{(1-z)t_4^{[2]} + zt_1^{[3]}}\Big] - \frac{\pi^2}{6} - \mathrm{Li}_2\Big[1 - \frac{(1-z)t_3^{[3]}}{(1-z)t_3^{[3]} + zt_1^{[2]}}\Big] . \tag{4.15}$$

If we define variables $X = (1-z)t_4^{[2]}/((1-z)t_4^{[2]} + zt_1^{[3]})$ and $Y = zt_1^{[2]}/((1-z)t_3^{[3]} + zt_1^{[2]})$ then this collinear limit of D_7 can be expressed in terms of D_6 along with a function of X and Y,

$$\begin{aligned} & D_6 - \frac{\pi^2}{6} + \mathrm{Li}_2\Big[1 - \frac{XY}{(1-X)(1-Y)}\Big] - \mathrm{Li}_2\Big[1 - \frac{Y}{1-X}\Big] \\ & - \mathrm{Li}_2\Big[1 - \frac{X}{1-Y}\Big] - \mathrm{Li}_2\Big[1-(1-X)\Big] - \mathrm{Li}_2\Big[1-(1-Y)\Big] . \end{aligned} \tag{4.16}$$

The dilogarithms in this expression can be eliminated using the following dilogarithm identity,

$$\begin{aligned} \mathrm{Li}_2\Big[1 - \frac{xy}{(1-x)(1-y)}\Big] &= \mathrm{Li}_2\Big[1-(1-x)\Big] + \mathrm{Li}_2\Big[1-(1-y)\Big] + \mathrm{Li}_2\Big[1 - \frac{x}{1-y}\Big] \\ & + \mathrm{Li}_2\Big[1 - \frac{y}{1-x}\Big] + \ln(x)\ln(1-x) + \ln(y)\ln(1-y) - \ln(1-x)\ln(1-y) - \frac{\pi^2}{6} , \end{aligned} \tag{4.17}$$

which is equivalent via (4.11) to a rearrangement of Abel's identity [46]. With this identity we have, in the collinear limit

$$D_7 \ \rightarrow \ D_6 + \text{logarithms} - \frac{\pi^2}{3} . \tag{4.18}$$

The logarithms combine with those arising from other terms in V_7^g to ensure that the ansatz has the correct collinear limit.

An alternate way to write the function V_n^g is in terms of functions related to the scalar box integrals:

$$\begin{aligned} (\mu^2)^{-\epsilon} V^g_{2m+1} &= \sum_{r=2}^{m-1} \sum_{i=1}^{n} F^{\mathrm{2m}\,e}_{n:r;i} + \sum_{i=1}^{n} F^{\mathrm{1m}}_{n:i} , \\ (\mu^2)^{-\epsilon} V^g_{2m} &= \sum_{r=2}^{m-2} \sum_{i=1}^{n} F^{\mathrm{2m}\,e}_{n:r;i} + \sum_{i=1}^{n} F^{\mathrm{1m}}_{n:i} + \sum_{i=1}^{n/2} F^{\mathrm{2m}\,e}_{n:m-1;i} , \end{aligned} \tag{4.19}$$

where the box functions F are defined in appendix I (and are the box integrals multiplied by the dimensionful denominator and a constant). Only certain special types of box functions appear in the expression (4.19). The "easy two-mass" box function $F^{2\mathrm{m}\,e}_{n:r;i}$ has two diagonally-opposite massless legs, with momenta equal to the external momenta k_{i-1} and k_{i+r}; the remaining two diagonally-opposite legs are massive, as they contain the sum of r adjacent external momenta between $i-1$ and $i+r$, and of $n-r-2$ adjacent momenta between $i+r$ and $i-1$. The one-mass box function $F^{1\mathrm{m}}_{n:i}$ is a special case of $F^{2\mathrm{m}\,e}_{n:r;i-2}$ with $r=1$. Box functions with two adjacent external masses, or with three or four external masses, do not appear. This property can be understood from the structure of the unitarity cuts to be calculated directly in the next section; the representation (4.19) is a convenient one for comparing with the results of that calculation.

This representation is also convenient for proving that the ansatz possesses the required collinear limit (4.2) for arbitrary n; consider, for example the limit $k_{n-1} \parallel k_n$ for n odd, and split up the sums in equation (4.19) as follows,

$$
\begin{aligned}
(\mu^2)^{-\epsilon} V^g_{2m+1} \;=\; & \sum_{r=3}^{m-1} \sum_{i=n-r+1}^{n-1} F^{2\mathrm{m}\,e}_{n:r;i} + \sum_{r=2}^{m-2} \sum_{i=2}^{n-r-2} F^{2\mathrm{m}\,e}_{n:r;i} + \sum_{i=2}^{m} F^{2\mathrm{m}\,e}_{n:m-1;i} + \sum_{i=4}^{n-1} F^{1\mathrm{m}}_{n:i} \\
& + \sum_{r=3}^{m-1} \left[F^{2\mathrm{m}\,e}_{n:r;n} + F^{2\mathrm{m}\,e}_{n:r-1;1} \right] + \sum_{r=3}^{m-1} \left[F^{2\mathrm{m}\,e}_{n:r;n-r} + F^{2\mathrm{m}\,e}_{n:r-1;n-r} \right] + \left[F^{2\mathrm{m}\,e}_{n:m-1;1} + F^{2\mathrm{m}\,e}_{n:m-1;m+1} \right] \\
& + \left[F^{2\mathrm{m}\,e}_{n:2;n} + F^{1\mathrm{m}}_{n:3} \right] + \left[F^{2\mathrm{m}\,e}_{n:2;n-2} + F^{1\mathrm{m}}_{n:n} \right] + \left[F^{2\mathrm{m}\,e}_{n:2;n-1} + F^{1\mathrm{m}}_{n:1} + F^{1\mathrm{m}}_{n:2} \right] \; .
\end{aligned}
\tag{4.20}
$$

Using the collinear limits of the F functions detailed in appendix I, these sums reduce to

$$
\begin{aligned}
& \sum_{r=2}^{m-2} \sum_{i=n-r}^{n-1} F^{2\mathrm{m}\,e}_{n-1:r;i} + \sum_{r=2}^{m-2} \sum_{i=2}^{n-r-2} F^{2\mathrm{m}\,e}_{n-1:r;i} + \sum_{i=2}^{m} F^{2\mathrm{m}\,e}_{n-1:m-1;i} + \sum_{i=4}^{n-1} F^{1\mathrm{m}}_{n-1:i} \\
& + \sum_{r=2}^{m-2} \left[F^{2\mathrm{m}\,e}_{n-1:r;1} \right] + \sum_{r=2}^{m-2} \left[F^{2\mathrm{m}\,e}_{n-1:r;n-r-1} \right] + \left[F^{2\mathrm{m}\,e}_{n-1:m-1;1} \right] \\
& + \left[F^{1\mathrm{m}}_{n-1:3} \right] + \left[F^{1\mathrm{m}}_{n-1:1} \right] + \left[F^{1\mathrm{m}}_{n-1:2} + (\mu^2)^{-\epsilon} r_S^{\mathrm{SUSY}}(z, s_{n-1,n}) \right] \\
& = (\mu^2)^{-\epsilon} V^g_{2m} + (\mu^2)^{-\epsilon} r_S^{\mathrm{SUSY}}(z, s_{n-1,n}) \; .
\end{aligned}
\tag{4.21}
$$

The proof is similar for n even.

5. Unitarity constraints

We now show that the unitarity cuts in the expression (4.19) obtained from the collinear limits are correct for the maximally helicity-violating (MHV) configurations (2.7), $A^{\mathrm{MHV}}_{jk}(1,2,\ldots,n)$. We do this by calculating the cuts directly from the Cutkosky rules [17], which turn out to require only the Parke-Taylor (MHV) tree amplitudes [2] for their evaluation. The simple structure of the Parke-Taylor amplitudes (2.6) allows us to evaluate the cuts for all n in terms of a single hexagon integral, or four box integrals. The cuts determine all terms in the amplitude apart from additive polynomial terms without logarithms or dilogarithms. We will fix the remaining polynomial ambiguity in the next section.

We compute the cuts of the MHV amplitudes in all possible channels. We consider the amplitude, not in a physical kinematic configuration, but in a region where exactly one of the momentum invariants is taken to be positive (time-like), and the rest are negative (space-like). In this way we isolate cuts in a single momentum channel. We apply the Cutkosky rules at the amplitude level, rather than at the diagram level. That is, we write the sum of all cut diagrams as the sum of all tree diagrams on one side of the cut, multiplied by the sum of all tree diagrams on the other side of the cut. Thus the cut in the one-loop amplitude is given by the integral over a two-body phase-space of the product of two tree amplitudes, which is then summed over each intermediate helicity configuration that contributes. Since we use the helicity convention that all particles are outgoing, the helicity of each of the two intermediate particles is reversed upon crossing the cut.

There are two distinct cases to consider: (a) the two negative helicity gluons are on the same side of the cut, and (b) the two negative helicity gluons are on opposite sides of the cut. Case (a) is easier, so we consider it first. This cut amplitude is shown in fig. 3. First, we note that the contributions to the cut from intermediate fermions vanish in this case since there is no way to assign intermediate helicity configurations so that fermion helicity is conserved and both the tree amplitudes are non-zero. For example, the intermediate helicity assignment in fig. 3 does not conserve fermion helicity. Since tree amplitudes with two external fermions vanish if the gluons all carry the same helicity [1], the helicity assignments which conserve fermion helicity also vanish. The contributions to the cut from intermediate complex scalars vanish by an analogous argument where the 'helicity' of a complex scalar refers to particle or antiparticle assignment rather than genuine helicity.

Thus in case (a) the only contribution is from intermediate gluons with the helicity assignment shown in fig. 3. The tree amplitudes on either side of the cut are pure-glue MHV tree amplitudes, given in eq. (2.6). Consider the cut in the channel $(k_{m_1} + k_{m_1+1} + \cdots + k_{m_2-1} + k_{m_2})^2$ for the loop amplitude $A_{jk}^{1-\mathrm{loop\ MHV}}(1,2,\ldots,n)$, where $m_1 \le j < k \le m_2$. The cut for this channel is given by

$$\begin{aligned}
&\int d\mathrm{LIPS}(-\ell_1,\ell_2)\; A_{jk}^{\mathrm{tree\ MHV}}(-\ell_1, m_1,\ldots,m_2,\ell_2)\; A_{(-\ell_2)\ell_1}^{\mathrm{tree\ MHV}}(-\ell_2, m_2+1,\ldots,m_1-1,\ell_1) \\
&= \; -iA_{jk}^{\mathrm{tree\ MHV}}(1,2,\ldots,n)\int d\mathrm{LIPS}(-\ell_1,\ell_2)\frac{\langle (m_1-1)\, m_1\rangle\, \langle \ell_1\, \ell_2\rangle^2\, \langle m_2\, (m_2+1)\rangle}{\langle (m_1-1)\, \ell_1\rangle\, \langle \ell_1\, m_1\rangle\, \langle m_2\, \ell_2\rangle\, \langle \ell_2\, (m_2+1)\rangle}\;,
\end{aligned} \tag{5.1}$$

where the spinor inner products are labelled by either loop momenta or external particle labels. We have removed minus signs from the spinor inner products by cancelling constant phases. Here ℓ_1 is the running loop momentum between vertices $(m_1 - 1)$ and m_1, and ℓ_2 is the running loop momentum between vertices m_2 and $(m_2 + 1)$. The arguments in the second amplitude $(m_2 + 1, \ldots, m_1 - 1)$ should be understood to mean $(m_2 + 1, \ldots, n, 1, \ldots, m_1 - 1)$. The $(4 - 2\epsilon)$-dimensional Lorentz-invariant phase space measure is denoted by $d\text{LIPS}(-\ell_1, \ell_2)$. (The factor of $(\mu^2)^\epsilon$ has been suppressed here and in subsequent formulae.) In equation (5.1) the integration momenta ℓ_1 and ℓ_2 appear in only a few of the factors, even though we are considering an arbitrary number of external legs. This simplicity is due to the simple form of the Parke-Taylor amplitudes (2.6). Observe that the integral (5.1) does not depend on the locations j, k of the negative helicity gluons. This result does not require $N = 4$ supersymmetry, but only the assumption that the two negative helicity external states in the MHV amplitude are on the same side of the cut. We will see that the same independence holds in case (b), for the special case of an $N = 4$ super-Yang-Mills theory.

Consider case (b) now. This amplitude is shown in fig. 4 with the possible intermediate helicity assignments. As shown, there are two sets of possible helicity configurations across this cut. The cut has contributions from all possible intermediate states, the scalars, the fermions and the gluons. However, the $N = 4$ supersymmetric sum turns out to be very simple, thanks to the supersymmetric Ward identities [5,1] for the nonvanishing MHV tree amplitudes. These identities relate amplitudes with all external gluons, g, to amplitudes where a pair of gluons is replaced by a pair of gluinos, Λ, or a pair of complex scalars, ϕ. Specifically, if we have the pure gluon MHV tree amplitude $A^{\text{tree}}(g_1^-, g_2^+, \ldots, g_j^-, \ldots g_n^+)$, given explicitly in eq. (2.6), then the amplitudes with gluinos and scalars are

$$\begin{aligned} A^{\text{tree}}(\Lambda_1^-, g_2^+, \ldots, g_j^-, \ldots, \Lambda_n^+) &= \frac{\langle j\, n\rangle}{\langle j\, 1\rangle}\, A^{\text{tree}}(g_1^-, g_2^+, \ldots, g_j^-, \ldots, g_n^+), \\ A^{\text{tree}}(\phi_1^-, g_2^+, \ldots, g_j^-, \ldots, \phi_n^+) &= \frac{\langle j\, n\rangle^2}{\langle j\, 1\rangle^2}\, A^{\text{tree}}(g_1^-, g_2^+, \ldots, g_j^-, \ldots, g_n^+), \end{aligned} \tag{5.2}$$

where '...' denotes positive-helicity gluons, and the helicity assignments on ϕ refer to particle or antiparticle assignments rather than genuine helicity. These and the corresponding relations for the other MHV amplitudes allow us to sum the contributions due to intermediate states. If we consider the contribution due to a single scalar state we have

$$\begin{aligned} & -i\frac{A_{jk}^{\text{tree MHV}}(1, 2, \ldots, n)}{\langle j\, k\rangle^4} \int d\text{LIPS}(-\ell_1, \ell_2) \\ & \qquad \times \frac{\langle (m_1 - 1)\, m_1\rangle \langle m_2\, (m_2 + 1)\rangle \langle \ell_1\, j\rangle^2 \langle \ell_2\, j\rangle^2 \langle \ell_1\, k\rangle^2 \langle \ell_2\, k\rangle^2}{\langle \ell_1\, \ell_2\rangle^2 \langle (m_1 - 1)\, \ell_1\rangle \langle \ell_1\, m_1\rangle \langle m_2\, \ell_2\rangle \langle \ell_2\, (m_2 + 1)\rangle}\,. \end{aligned} \tag{5.3}$$

The contributions for the other states can be obtained using eq. (5.2).

In the $N = 4$ multiplet we have one gluon, four Weyl fermions (both with plus and minus helicities) and three complex scalars (or six real scalars). If we use (5.2) to write the different contributions to the integrand as multiples of the single scalar state contribution (5.3), then the overall factor for the $N = 4$ multiplet is

$$\rho^2 \equiv \left(\frac{\langle \ell_1\, j\rangle \langle \ell_2\, k\rangle}{\langle \ell_2\, j\rangle \langle \ell_1\, k\rangle}\right)^2 - 4\left(\frac{\langle \ell_1\, j\rangle \langle \ell_2\, k\rangle}{\langle \ell_2\, j\rangle \langle \ell_1\, k\rangle}\right) + 6 - 4\left(\frac{\langle \ell_1\, j\rangle \langle \ell_2\, k\rangle}{\langle \ell_2\, j\rangle \langle \ell_1\, k\rangle}\right)^{-1} + \left(\frac{\langle \ell_1\, j\rangle \langle \ell_2\, k\rangle}{\langle \ell_2\, j\rangle \langle \ell_1\, k\rangle}\right)^{-2}, \tag{5.4}$$

where the central term is the contributions from the three complex scalars (the two 'helicity' assignments give equal contributions for complex scalars), the terms flanking the central term are the fermion contributions (one for each possible helicity configuration), and the remaining terms are the contributions of the two gluon helicities. Using the Schouten identity,

$$\langle a\, b\rangle \langle c\, d\rangle = \langle a\, d\rangle \langle c\, b\rangle + \langle a\, c\rangle \langle b\, d\rangle \ , \tag{5.5}$$

and some rearrangements we have

$$\rho^2 = \frac{(\langle \ell_1\, j\rangle \langle \ell_2\, k\rangle - \langle \ell_2\, j\rangle \langle \ell_1\, k\rangle)^4}{\langle \ell_1\, j\rangle^2 \langle \ell_2\, k\rangle^2 \langle \ell_2\, j\rangle^2 \langle \ell_1\, k\rangle^2} = \frac{\langle \ell_1\, \ell_2\rangle^4 \langle j\, k\rangle^4}{\langle \ell_1\, j\rangle^2 \langle \ell_2\, j\rangle^2 \langle \ell_1\, k\rangle^2 \langle \ell_2\, k\rangle^2} \ . \tag{5.6}$$

Using the form (5.6) for ρ^2, we see that the product of ρ^2 with the integrand of the scalar loop contribution (5.3) is identical to the integrand for the cut in case (a), equation (5.1). Thus in both cases (a) and (b), the cut reduces to eq. (5.1). We now evaluate this integral.

The integral (5.1) can be viewed as the cut hexagon loop integral shown in fig. 5. To see this, one may use the on-shell condition $\ell_1^2 = \ell_2^2 = 0$ to rewrite the four spinor product denominators in (5.1) as scalar propagators, multiplied by a numerator factor, for example

$$\frac{1}{\langle \ell_1\, m_1\rangle} = \frac{[m_1\, \ell_1]}{\langle \ell_1\, m_1\rangle\, [m_1\, \ell_1]} = \frac{[m_1\, \ell_1]}{2\ell_1 \cdot k_{m_1}} = \frac{-\,[m_1\, \ell_1]}{(\ell_1 - k_{m_1})^2} \ . \tag{5.7}$$

In addition to these four propagators, there are two cut propagators implicit in the phase-space integral $\int d\mathrm{LIPS}(-\ell_1, \ell_2)$. (For a four- or five-point loop amplitude there are obviously not enough external momenta to make it a genuine hexagon; in this case one can take some of the external momenta to be null.)

Rather than evaluate the cut hexagon integral directly, we can perform a "partial fraction" decomposition of the integrand in order to reduce the number of spinor product factors in the denominator of each term, which will break up the integral into a sum of cut box integrals. (This is equivalent to a Passarino-Veltman reduction [24].) Using the Schouten identity (5.5) we rewrite the integrand of (5.1) as

$$
\begin{aligned}
\mathcal{I} &= -\left(\frac{\langle (m_1-1)\, m_1\rangle\, \langle \ell_1\, \ell_2\rangle}{\langle (m_1-1)\, \ell_1\rangle\, \langle \ell_1\, m_1\rangle}\right)\left(\frac{\langle m_2\, (m_2+1)\rangle\, \langle \ell_2\, \ell_1\rangle}{\langle m_2\, \ell_2\rangle\, \langle \ell_2\, (m_2+1)\rangle}\right) \\
&= -\left(\frac{\langle (m_1-1)\, \ell_2\rangle}{\langle (m_1-1)\, \ell_1\rangle} - \frac{\langle m_1\, \ell_2\rangle}{\langle m_1\, \ell_1\rangle}\right)\left(\frac{\langle m_2\, \ell_1\rangle}{\langle m_2\, \ell_2\rangle} - \frac{\langle (m_2+1)\, \ell_1\rangle}{\langle (m_2+1)\, \ell_2\rangle}\right) \\
&= \frac{\langle m_1\, \ell_2\rangle\, \langle m_2\, \ell_1\rangle}{\langle m_1\, \ell_1\rangle\, \langle m_2\, \ell_2\rangle} \pm \Big[\ m_1 \leftrightarrow (m_1-1)\ ,\ m_2 \leftrightarrow (m_2+1)\ \Big]\ ,
\end{aligned}
\tag{5.8}
$$

antisymmetrizing in each exchange. In terms of propagators,

$$
\begin{aligned}
\mathcal{I} &= -\frac{[\ell_1\, m_1]\, \langle m_1\, \ell_2\rangle\, [\ell_2\, m_2]\, \langle m_2\, \ell_1\rangle}{(\ell_1 - k_{m_1})^2 (\ell_2 + k_{m_2})^2} \pm \Big[\ m_1 \leftrightarrow (m_1-1)\ ,\ m_2 \leftrightarrow (m_2+1)\ \Big] \\
&= -\frac{\mathrm{tr}_-(\not{\ell}_1\ \not{k}_{m_1}\ \not{\ell}_2\ \not{k}_{m_2})}{(\ell_1 - k_{m_1})^2 (\ell_2 + k_{m_2})^2} - \frac{\mathrm{tr}_-(\not{\ell}_1\ \not{k}_{m_1-1}\ \not{\ell}_2\ \not{k}_{m_2})}{(\ell_1 + k_{m_1-1})^2 (\ell_2 + k_{m_2})^2} \\
&\quad - \frac{\mathrm{tr}_-(\not{\ell}_1\ \not{k}_{m_1}\ \not{\ell}_2\ \not{k}_{m_2+1})}{(\ell_1 - k_{m_1})^2 (\ell_2 - k_{m_2+1})^2} - \frac{\mathrm{tr}_-(\not{\ell}_1\ \not{k}_{m_1-1}\ \not{\ell}_2\ \not{k}_{m_2+1})}{(\ell_1 + k_{m_1-1})^2 (\ell_2 - k_{m_2+1})^2}\ ,
\end{aligned}
\tag{5.9}
$$

where the tr_- indicates that a $(1-\gamma_5)/2$ projector has been inserted into the trace. Thus the integral of $\mathcal{I}$ is the sum of four cut box integrals, shown in fig. 6, corresponding to cancelling different pairs of propagators in the cut hexagon integral in fig. 5.

We use a Passarino-Veltman reduction [24] to evaluate the box integrals in terms of scalar boxes, triangles and bubbles. First we evaluate the traces in (5.9) using

$$
\mathrm{tr}_\pm(\not{a}\ \not{b}\ \not{c}\ \not{d}) \ =\ 2(a\cdot b\ c\cdot d - a\cdot c\ b\cdot d + a\cdot d\ b\cdot c) \mp \tfrac{1}{2}\varepsilon(a,b,c,d)\ , \tag{5.10}
$$

where $\varepsilon(a,b,c,d) = 4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma$ with $\varepsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric tensor. First consider the ε terms. The two momenta ℓ_1 and ℓ_2 are related to each other by $\ell_2 = \ell_1 - P - k_{m_1} - k_{m_2}$ with $P = \sum_{i=m_1+1}^{m_2-1} k_i$. Since $\varepsilon(a,b,c,d)$ is antisymmetric the ε terms in the traces reduce to terms linear in ℓ_1. For example, the first term in eq. (5.9), corresponding to fig. 6a, has an ε term

$$
\varepsilon(\ell_1, k_{m_1}, \ell_2, k_{m_2}) = -\varepsilon(\ell_1, k_{m_1}, P, k_{m_2}). \tag{5.11}
$$

Since the box only contains only three independent momenta, which can be taken to be k_{m_1}, k_{m_2} and P, evaluation of the box momentum integral must give zero for the ε term. The γ_5 contribution drops out of the remaining terms in analogous fashion, and so we can replace $\mathrm{tr}_- \to \frac{1}{2}\mathrm{tr}$ in eq.(5.9).

Thus the trace for fig. 6a is

$$-\frac{1}{2}\mathrm{tr}(\not\ell_1\ \not k_{m_1}\ \not\ell_2\ \not k_{m_2}) = -2(\ell_1\cdot k_{m_1})(\ell_2\cdot k_{m_2}) - 2(\ell_1\cdot k_{m_2})(\ell_2\cdot k_{m_1}) + 2(k_{m_1}\cdot k_{m_2})(\ell_1\cdot\ell_2) \tag{5.12}$$

which may be rewritten as

$$\begin{aligned}&\Big(2P\cdot k_{m_1}P\cdot k_{m_2} - P^2 k_{m_1}\cdot k_{m_2}\Big)\\ &+\ (P+k_{m_1})\cdot k_{m_2}\ (\ell_1-k_{m_1})^2\ +\ (P+k_{m_2})\cdot k_{m_1}\ (\ell_2+k_{m_2})^2\\ &+\ (\ell_1-k_{m_1})^2\ (\ell_2+k_{m_2})^2\ .\end{aligned} \tag{5.13}$$

The first term in (5.13) yields a scalar box integral with the coefficient $(2P\cdot k_{m_1}P\cdot k_{m_2} - P^2 k_{m_1}\cdot k_{m_2})$, the next two terms cancel propagators to give triangle integrals, and the last term gives a bubble (two-point) integral. Each of the four cut boxes in (5.9) is reduced in this way into a cut scalar box, two cut scalar triangles and a cut scalar bubble. However, when the four contributions are added together the triangles and bubbles cancel leaving just the cut scalar boxes. This cancellation is expected since, as discussed in sections 2 and 6, the $N=4$ supersymmetric result can be expressed as a sum of scalar boxes with no triangles or bubbles. Note that (up to a factor of 2), the coefficient of the scalar box is precisely its denominator (see eqs. (I.8):

$$(2P\cdot k_{m_1}P\cdot k_{m_2} - P^2 k_{m_1}\cdot k_{m_2}) = \left(\frac{1}{2}\right)\Big((P+k_{m_1})^2(P+k_{m_2})^2 - P^2(P+k_{m_1}+k_{m_2})^2\Big) \tag{5.14}$$

so that the cuts will be given in terms of the scalar box functions F defined in appendix I.

Thus the cuts in the amplitude are given simply by the cuts in the scalar box functions $F^{2\mathrm{m}\,e}_{n:r;i}$ (including the limiting case $F^{1\mathrm{m}}_{n:i+2}$ for $r=1$) with the coefficients

$$c_\Gamma\,(\mu^2)^\epsilon A^{\mathrm{tree\ MHV}}_{jk}(1,2,\ldots,n) \tag{5.15}$$

where $i = m_1+1$ and $r = m_2 - m_1 - 1$. The coefficients are precisely those given in eqs. (4.19), thus confirming that the cuts in the ansatz (4.3) are correct. Although suggestive, this agreement does not yet prove that the ansatz is correct, because the cuts do not necessarily fix possible polynomial terms. In the following section we shall use $N=4$ supersymmetry to show that no ambiguities are present.

For amplitudes other than the MHV amplitudes the evaluation of the cuts is more involved. Furthermore, all-n tree formulas for general non-MHV amplitudes are unknown. For the $N=4$ supersymmetric non-MHV amplitudes the ansatz (4.1), which expresses $A^{N=4\ \mathrm{loop}}_{n;1}$ as a product of the tree amplitude and the universal function V^g_n, still defines a function with perfectly well behaved two-particle collinear limits. However, explicit calculation for the six-point amplitude, $A_{6;1}^{N=4 \text{ loop}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$, by the string-based method shows that the ansatz (4.1) does *not* hold for the general non-MHV case. We have also verified this by evaluating the cut in the channel $(k_1 + k_2 + k_3)^2$ for this amplitude.

6. Fixing remaining ambiguities

The collinear limits do not necessarily provide a tight enough constraint to allow us to prove the uniqueness of our ansatz; unitarity, on the other hand, determines the cuts uniquely — and hence the dilogarithms and logarithms — but does not provide any information about polynomial terms in the amplitude. The $N = 4$ supersymmetric case is however special: a knowledge of the cuts completely determines the amplitude. The set of functions which can appear in the amplitude are all known since they arise from one-loop integrals which are all known [24,25,26]. For the $N = 4$ theory the set of functions is a restricted set. As we will demonstrate, for this restricted set the cuts uniquely determine the coefficients of all functions that may appear in the amplitude, including polynomials.

The key to this result is the property discussed in section 2, that for the $N = 4$ supersymmetric theory the loop-momentum polynomials encountered have a degree that is four less than the purely gluonic case, namely $m - 4$ for an m-point integral. Now, when calculating a general amplitude one may evaluate the tensor integrals (integrals with nontrivial loop-momentum polynomials in the numerator) by Passarino-Veltman reduction [24] to lower-point integrals. Passarino-Veltman reduction takes an m-point ($m \geq 4$) tensor integral of degree d ($d \geq 1$) and reduces it to a sum of m- and $(m-1)$-point integrals of degree $d - 1$. Any scalar m-point integral (an integral with a numerator independent of the loop momentum) can also be reduced to a sum of scalar $(m - 1)$-point integrals for $m \geq 5$ [25]. In this way any one-loop m-point integral can be reduced to a combination of tensor box integrals of degree up to $d + 4 - m$. In a gauge theory the maximum degree of the polynomial in an m-point integral is m, and when one iterates the reduction one arrives at a combination of tensor box integrals of degree four. (To evaluate the tensor boxes explicitly one often performs another iteration to arrive at a combination of known scalar box and tensor triangle integrals.) Since in $N = 4$ super-Yang-Mills the polynomials for the m-point integrals have a maximum degree of $m - 4$, this process will express the m-point integrals in terms of a sum of *scalar* boxes. Thus the full $N = 4$ amplitude can be written simply as a sum over scalar boxes

$$A^{N=4} = \sum_i c_i \, I_i^{\rm box} \tag{6.1}$$

without any triangles or bubbles. (This is why the triangles and bubbles cancelled in the previous calculation of the cuts.) The set of possible scalar box integrals, with massless internal lines, but all possible combinations of external masses, is given explicitly in appendix I. The coefficients c_i may be polynomial functions of the momentum invariants and spinor helicity factors but may not contain logarithms and dilogarithms.

Is this decomposition in terms of scalar boxes determined uniquely, given the cuts? For uniqueness to hold, the equation

$$\sum_i c_i I_i^{\rm box} = \text{polynomial} \tag{6.2}$$

must have only the trivial solution $c_i = 0$. The set of integrals $I_i^{\rm box}$ include the cases where one, two, three, or four legs may be off-shell (massive) as depicted in fig. 7. All these integrals contain logarithms or dilogarithms, which produce cuts that are independent of those produced by other integrals. For example, consider the coefficient of the three-mass box $I_4^{3{\rm m},r,r',i}$ (I.8*d*) appearing in fig. 7. This box integral is a function of five kinematic invariants; two of the invariants, $t_i^{[r]}$ and $t_{i+r+r'}^{[n-r-r'-1]}$, appear together only in this one box, in the term $2\,{\rm Li}_2[1-(t_i^{[r]}t_{i+r+r'}^{[n-r-r'-1]})/(t_{i-1}^{[r+1]}t_i^{[r+r']})]$. Consider the cut in the $t_i^{[r]}$ channel. The cut for this term is proportional to $\ln(-t_i^{[r]})+\ln(-t_{i+r+r'}^{[n-r-r'-1]})-\ln(-t_{i-1}^{[r+1]})-\ln(-t_i^{[r+r']})$. The $\ln(-t_{i+r+r'}^{[n-r-r'-1]})$ part of the cut can only arise from this box, $I_4^{3{\rm m},r,r',i}$, and thus the coefficient of this three mass box in eq. (6.2) must vanish. One can continue in this way to show that eq. (6.2) has only the trivial solution $c_i = 0$ for all coefficients. Thus, for the $N = 4$ supersymmetric case, the coefficients of the scalar boxes in eq. (4.19) are uniquely determined by the cuts, and we have proven that the ansatz is correct.

Since we have a supersymmetric theory, we can use supersymmetry Ward identities to generate amplitudes with external fermions and scalars from n-gluon amplitudes. For supersymmetric MHV amplitudes the Ward identity (5.2) holds for loop amplitudes as well as tree amplitudes, and we obtain the amplitudes with two external fermions trivially from the gluon amplitudes (4.1), (4.3), (4.19).

In general, for theories other than $N = 4$ super-Yang-Mills, the cuts may not uniquely determine the full amplitude. As a simple example, the five-point helicity amplitudes $A_{5;1}(1^-, 2^+, \ldots, 5^+)$ and $A_{5;1}(1^+, 2^+, \ldots, 5^+)$ each have no cuts but are not equal. One cannot reconstruct the full amplitude from the cuts in this case because the amplitude is a more general sum of boxes, triangles, and bubbles, including combinations without branch cuts. In such cases the collinear limits provide restrictions on the form of rational functions that may appear in the amplitudes [28].

7. Remaining partial amplitudes

In this section we show that the one-loop partial amplitudes relevant at subleading orders in N_c, $A_{n;c>1}$, can be obtained by performing appropriate sums over permutations of the leading-color partial amplitudes $A_{n;1}$. The result obtained holds not just in $N=4$ supersymmetry, but also for any one-loop gauge theory amplitude where the external particles and the particles circulating around the loop are both in the adjoint representation of $SU(N_c)$. (In the case of e.g. a quark loop where the internal particle is in the fundamental representation, there are no $A_{n;c>1}$ contributions.) In string theory, the color decomposition is manifest [38,14]; it thus provides a natural framework for discussing this result. We shall also provide the outline of a conventional field theory argument leading to the same result.

As we shall explain, the coefficients of the subleading color structures

$$\mathrm{Gr}_{n;c}(1) = \mathrm{Tr}\,(T^{a_1}\cdots T^{a_{c-1}})\;\mathrm{Tr}\,(T^{a_c}\cdots T^{a_n}) \tag{7.1}$$

are

$$A_{n;c}(1,2,\ldots,c-1;c,c+1,\ldots,n) \;=\; (-1)^{c-1} \sum_{\sigma\in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma) \tag{7.2}$$

where $\alpha_i \in \{\alpha\} \equiv \{c{-}1, c{-}2, \ldots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{c, c{+}1, \ldots, n{-}1, n\}$, and $COP\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1,2,\ldots,n\}$ with n held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i. For example if $\{\alpha\} = \{2,1\}$ and $\{\beta\} = \{3,4,5\}$, then $COP\{\alpha\}\{\beta\}$ contains the twelve elements

$$\begin{array}{llllll} (2,1,3,4,5), & (2,3,1,4,5), & (2,3,4,1,5), & (3,2,1,4,5), & (3,2,4,1,5), & (3,4,2,1,5), \\ (1,2,3,4,5), & (1,3,2,4,5), & (1,3,4,2,5), & (3,1,2,4,5), & (3,1,4,2,5), & (3,4,1,2,5) \end{array} \tag{7.3}$$

(cyclic ordering for a two-element set is meaningless). Note that the ordering of the first sets of indices is reversed with respect to the second.

In an open string theory where the trace structures are just Chan-Paton factors [47] it is easy to see that a formula like (7.2) should be expected. Open string vertex operators corresponding to the two different traces are attached to the two different boundaries of the open string annulus, and the relative orderings of operators from opposite boundaries are summed over in the world-sheet path integral, while the ordering of operators on the same boundary is preserved. (Indeed, for the $N = 4$ super-Yang-Mills result, it would suffice to consider an open superstring in the infinite-tension limit, as its trivial compactification to four dimensions yields precisely this theory.)

More formally, string-rules for generating the subleading-in-N_c partial amplitudes with external gluons are given in ref. [14]. For proving eq. (7.1) we require only a few salient features of the string rules. The rules are in terms of diagrams with only three-point vertices. The tree and loop parts of diagrams are evaluated by a set of substitution rules on a 'master' kinematic expression described in the reference. The right-hand-side of eq. (7.2) contains all the leading-in-N_c diagrams, with legs permuted over $COP\{\alpha\}\{\beta\}$. For the left-hand-side of eq. (7.2) the string-based rules for subleading-in-N_c amplitudes give exactly the same set of diagrams except that two classes are explicitly excluded. These two sets are:

1) diagrams where indices from *both* sets $\{\alpha\}$ and $\{\beta\}$ label leaves of the same tree (attached to the loop), and

2) diagrams where a single tree contains *all* elements of either $\{\alpha\}$ or $\{\beta\}$.

In fig. 8 two examples of diagrams which are excluded from the left-hand-side of eq. (7.2) are given. If we can prove that the two classes of excluded diagrams, when summed over the permutations in $COP\{\alpha\}\{\beta\}$, have vanishing contribution, then eq. (7.2) follows; this is not difficult to do using the string-based rules.

For diagrams of the first type where the tree legs are labelled by the indices from both sets $\{\alpha\}$ and $\{\beta\}$, the diagrams can be arranged so that diagrams cancel pairwise. This follows from the anti-symmetry of the tree substitution rules under the interchange of the ordering of two outer legs of a vertex. (This anti-symmetry is completely analogous to the anti-symmetry of the color-decomposed three-gluon self-interaction [1,19] of field theory.) For example, the pairs of diagrams in figs. 8a and 8b cancel in the string based-rules. Figure 8c is an example of the pairwise cancellation for diagrams of the second type where a single tree contains all elements of $\{\beta\} = \{\beta_1, \beta_2, \beta_3\}$. For all trees in either class we can similarly arrange the diagrams to cancel pairwise when one sums over the permutations in $COP\{\alpha\}\{\beta\}$. This completes the proof of eq. (7.2).

The corresponding analysis in field theory also uses a representation of graphs in terms of trees attached to the loop. Using the trace (or double-line) representation (2.2) of the structure constants f^{abc}, and also eq. (2.4), it is easy to see that the set of all Feynman diagrams (in Feynman gauge) which have only three-point vertices, and no non-trivial trees, feed into both $A_{n;1}$ and $A_{n;c>1}$ in the correct way so that eq. (7.2) is satisfied for this class of diagrams. For diagrams containing non-trivial trees that are attached to the loop, one again needs to show (as in the string-based proof), that the permutation sum over different orderings, $COP\{\alpha\}\{\beta\}$, cancels out right-hand-side contributions to eq. (7.2) of the types 1 and 2 above, since these contributions can be seen to be absent from the left-hand-side. For diagrams with only three-point vertices, the permutation sum drops out from the antisymmetry of the vertices as in the string-based argument. For diagrams with trees containing four-point vertices, the cancellations occur in triplets such as those shown in fig. 9. Diagrams with four-point vertices attached to the loop can be color decomposed into the same color structures encountered above. In this way one can construct a purely field-theoretic proof of eq. (7.2).

Instead of relying on an explicit representation of the vertices to prove the legitimacy of omitting contributions of type 1 and 2, one can use "$U(P) \times U(N_c - P)$ decoupling" of tree amplitudes (even when one leg is off-shell — this is equivalent to the additional decoupling properties of the Berends-Giele current [11,7]). The sum over orderings in $COP\{\alpha\}\{\beta\}$, where some α_i and some β_i belong to the same tree, amounts to computing the color-ordered tree amplitude for those α_i belonging to (say) $U(P)$ for some P, and the β_i belonging to $U(N_c - P)$; but this quantity vanishes. The same is true for the sum over all given orderings when all indices from a given set label leaves on a single tree. The field theory analysis really only relies on the properties of the $U(N_c)$ structure constants f^{abc}, and so it applies to any amplitude containing only such vertices, for example to $N = 4$ or pure $N = 1$ super-Yang-Mills amplitudes with an arbitrary number of external gluinos as well as gluons.

In the $N = 4$ supersymmetric case, the special form of $A_{n;1}^{\mathrm{MHV}}$ in equations (4.1) and (4.19) allows us to simplify the expression for $A_{n;c}$ by explicitly carrying out the permutation sums in eq. (7.2). The computation and results are given in appendix III.

8. Conclusions

Although they are important to the analysis of experimental jet data, few one-loop QCD amplitudes have been calculated. Only four- [48,14,20] and five-point [13] amplitudes relevant for next-to-leading order corrections are known, the latter already made possible only by the development of new techniques. In this paper we have introduced a technique, based on unitarity and collinear limits, which allows one to compute amplitudes without performing explicit diagrammatic calculations. Unitarity, in the form of the Cutkosky rules, fixes the form of the cuts in the amplitudes without ambiguity but imposes no direct constraints on the polynomials in the kinematic variables. The constraints imposed by the collinear limits allow one to guess extrapolations of known results to higher-point functions, and in particular constrain the form of rational functions (lacking cuts) of the invariants and spinor products if present. We presented all one-loop splitting amplitudes for external gluons and fermions required for the collinear bootstrap. Collinear singularities are also very useful in checking results obtained by other means.

We have applied this technique to produce an all-n formula for the non-vanishing maximally helicity-violating amplitudes in an $N = 4$ supersymmetric theory. We fixed the remaining ambiguity in this amplitude by noting that the set of integrals that would appear in a string-based or superspace calculation is only a subset of the usual set of tensor integrals. Within this restricted set of integrals, the cuts uniquely determine the amplitudes, thereby proving that our amplitude is the unique solution to the constraints imposed by unitarity.

For the $N = 4$ supersymmetry case that we have presented here the collinear limits are not actually needed. In contrast, for the all-plus helicity amplitudes the cuts are trivial (they all vanish) and it is the collinear limits that allow one to give an ansatz for the amplitude [27,28], which has subsequently been proven correct [29,30]. In general, the restrictions imposed by collinear behavior and unitarity complement each other.

In the string based-method, it is convenient to organize QCD amplitudes with external gluons into contributions which correspond to an $N = 4$ supersymmetric piece, an $N = 1$ chiral piece, and a scalar piece [13,19,22]. With this type of organization the $N = 4$ supersymmetric amplitude is one of three pieces needed for the QCD loop amplitude. The other pieces are also amenable to the methods described here and will be discussed elsewhere.

Using a "unitary-collinear bootstrap" we have thus constructed a class of one-loop amplitudes with an arbitrary number of external gluons. These amplitudes are one-loop analogs of the Parke-Taylor tree amplitudes. We expect that this method can be used to generate further fixed-n and all-n amplitudes, while bypassing the algebraic barrier usually present in explicit computations.

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Appendix I. Scalar Box Integrals

The scalar box integrals that can arise in principle in the $N = 4$ super-Yang-Mills computation (or QCD computation) have vanishing internal masses, but may have one, two, three or four nonvanishing external masses, and there are two types of two-mass boxes. These integrals are defined and given in ref. [26] (the four-mass boxes are from ref. [49]) and are shown in fig. 7.

The scalar box integral is

$$I_4 = -i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \; \frac{1}{p^2\,(p-K_1)^2\,(p-K_1-K_2)^2\,(p+K_4)^2}\,. \tag{I.1}$$

It is convenient to define the scalar box function

$$F(K_1,K_2,K_3,K_4) = -\frac{2\sqrt{\det S}}{r_\Gamma}\, I_4 \tag{I.2}$$

where

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{I.3}$$

and where the symmetric 4×4 matrix S has components (i, j are $\mathrm{mod}\,4$)

$$S_{ij} = -\frac{1}{2}\,(K_i+\cdots+K_{j-1})^2\,,\quad i\neq j; \qquad S_{ii}=0. \tag{I.4}$$

The external momentum arguments $K_{1\ldots 4}$ in equation (I.2) are sums of external momenta k_i that are the arguments of the n-point amplitude.

With the labelling of legs shown in fig. 7 (that is re-expressing the functions in terms of the invariants $t_i^{[r]}$ of the n-point amplitude), the scalar box function F expanded through order $\mathcal{O}(\epsilon^0)$ for the different cases reduces to

$$\begin{aligned} F^{\rm 1m}_{n:i} = &-\frac{1}{\epsilon^2}\Big[(-t_{i-3}^{[2]})^{-\epsilon} + (-t_{i-2}^{[2]})^{-\epsilon} - (-t_i^{[n-3]})^{-\epsilon}\Big] \\ &+ \mathrm{Li}_2\left(1-\frac{t_i^{[n-3]}}{t_{i-3}^{[2]}}\right) + \mathrm{Li}_2\left(1-\frac{t_i^{[n-3]}}{t_{i-2}^{[2]}}\right) + \frac{1}{2}\ln^2\left(\frac{t_{i-3}^{[2]}}{t_{i-2}^{[2]}}\right) + \frac{\pi^2}{6}\,, \end{aligned} \tag{I.5a}$$

$$\begin{aligned} F^{{\rm 2m}\,e}_{n:r;i} = &-\frac{1}{\epsilon^2}\Big[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+1]})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i+r+1}^{[n-r-2]})^{-\epsilon}\Big] \\ &+ \mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_i^{[r+1]}}\right) + \mathrm{Li}_2\left(1-\frac{t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}}\right) \\ &+ \mathrm{Li}_2\left(1-\frac{t_{i+r+1}^{[n-r-2]}}{t_i^{[r+1]}}\right) - \mathrm{Li}_2\left(1-\frac{t_i^{[r]}t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}t_i^{[r+1]}}\right) + \frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\right)\,, \end{aligned} \tag{I.5b}$$

$$F^{2\mathrm{m}\,h}_{n:r;i} = -\frac{1}{\epsilon^2}\Big[(-t^{[2]}_{i-2})^{-\epsilon} + (-t^{[r+1]}_{i-1})^{-\epsilon} - (-t^{[r]}_{i})^{-\epsilon} - (-t^{[n-r-2]}_{i+r})^{-\epsilon}\Big]$$
$$- \frac{1}{2\epsilon^2}\frac{(-t^{[r]}_{i})^{-\epsilon}(-t^{[n-r-2]}_{i+r})^{-\epsilon}}{(-t^{[2]}_{i-2})^{-\epsilon}} + \frac{1}{2}\ln^2\left(\frac{t^{[2]}_{i-2}}{t^{[r+1]}_{i-1}}\right) \tag{I.5c}$$
$$+ \mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}}{t^{[r+1]}_{i-1}}\right) + \mathrm{Li}_2\left(1-\frac{t^{[n-r-2]}_{i+r}}{t^{[r+1]}_{i-1}}\right) ,$$

$$F^{3\mathrm{m}}_{n:r,r';i} = -\frac{1}{\epsilon^2}\Big[(-t^{[r+1]}_{i-1})^{-\epsilon} + (-t^{[r+r']}_{i})^{-\epsilon} - (-t^{[r]}_{i})^{-\epsilon} - (-t^{[r']}_{i+r})^{-\epsilon} - (-t^{[n-r-r'-1]}_{i+r+r'})^{-\epsilon}\Big]$$
$$- \frac{1}{2\epsilon^2}\frac{(-t^{[r]}_{i})^{-\epsilon}(-t^{[r']}_{i+r})^{-\epsilon}}{(-t^{[r+r']}_{i})^{-\epsilon}} - \frac{1}{2\epsilon^2}\frac{(-t^{[r']}_{i+r})^{-\epsilon}(-t^{[n-r-r'-1]}_{i+r+r'})^{-\epsilon}}{(-t^{[r+1]}_{i-1})^{-\epsilon}} + \frac{1}{2}\ln^2\left(\frac{t^{[r+1]}_{i-1}}{t^{[r+r']}_{i}}\right)$$
$$+ \mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}}{t^{[r+1]}_{i-1}}\right) + \mathrm{Li}_2\left(1-\frac{t^{[n-r-r'-1]}_{i+r+r'}}{t^{[r+r']}_{i}}\right) - \mathrm{Li}_2\left(1-\frac{t^{[r]}_{i}t^{[n-r-r'-1]}_{i+r+r'}}{t^{[r+1]}_{i-1}t^{[r+r']}_{i}}\right) , \tag{I.5d}$$

$$F^{4\mathrm{m}}_{n:r,r',r'';i} = \frac{1}{2}\Bigg(\mathrm{Li}_2\left(\tfrac{1}{2}(1-\lambda_1+\lambda_2+\rho)\right) + \mathrm{Li}_2\left(\tfrac{1}{2}(1-\lambda_1+\lambda_2-\rho)\right)$$
$$+ \mathrm{Li}_2\left(-\tfrac{1}{2\lambda_1}(1-\lambda_1-\lambda_2-\rho)\right) + \mathrm{Li}_2\left(-\tfrac{1}{2\lambda_1}(1-\lambda_1-\lambda_2+\rho)\right) \tag{I.5e}$$
$$- \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2^2}\right)\ln\left(\frac{1+\lambda_1-\lambda_2+\rho}{1+\lambda_1-\lambda_2-\rho}\right)\Bigg) ,$$

where

$$\rho \equiv \sqrt{1-2\lambda_1-2\lambda_2+\lambda_1^2-2\lambda_1\lambda_2+\lambda_2^2} , \tag{I.6}$$

and

$$\lambda_1 = \frac{t^{[r]}_{i}\, t^{[r'']}_{i+r+r'}}{t^{[r+r']}_{i}\, t^{[r'+r'']}_{i+r}} , \qquad \lambda_2 = \frac{t^{[r']}_{i+r}\, t^{[n-r-r'-r'']}_{i+r+r'+r''}}{t^{[r+r']}_{i}\, t^{[r'+r'']}_{i+r}} . \tag{I.7}$$

In terms of these variables, the relations between the scalar box functions and scalar box integrals are given by

$$I^{1\mathrm{m}}_{4:i} = -2r_\Gamma \frac{F^{1\mathrm{m}}_{n:i}}{t^{[2]}_{i-3}t^{[2]}_{i-2}} , \tag{I.8a}$$

$$I^{2\mathrm{m}e}_{4:r;i} = -2r_\Gamma \frac{F^{2\mathrm{m}\,e}_{n:r;i}}{t^{[r+1]}_{i-1}t^{[r+1]}_{i} - t^{[r]}_{i}t^{[n-r-2]}_{i+r+1}} , \tag{I.8b}$$

$$I^{2\mathrm{m}h}_{4:r;i} = -2r_\Gamma \frac{F^{2\mathrm{m}\,h}_{n:r;i}}{t^{[2]}_{i-2}t^{[r+1]}_{i-1}} , \tag{I.8c}$$

$$I^{3\mathrm{m}}_{4:r,r',i} = -2r_\Gamma \frac{F^{3\mathrm{m}}_{n:r,r';i}}{t^{[r+1]}_{i-1}t^{[r+r']}_{i} - t^{[r]}_{i}t^{[n-r-r'-1]}_{i+r+r'}} , \tag{I.8d}$$

$$I^{4\mathrm{m}}_{4:r,r',r'',i} = -2\frac{F^{4\mathrm{m}}_{n:r,r',r'';i}}{t_i^{[r+r']}\, t_{i+r}^{[r'+r'']}\, \rho}\,. \tag{I.8e}$$

We also record here the limits of the functions appearing in V_n^g as two adjacent external momenta, say k_c and k_{c+1}, become collinear. We denote the momentum fraction within the fused leg by z ($k_c = zk_P$ and $k_{c+1} = (1-z)k_P$), replace $P \to c$, and shift the labels of legs $c+2, \ldots, n$ down by one. (The indices on the right-hand sides of the following equations are to be understood mod $(n-1)$ rather than n.) If both k_c and k_{c+1} are part of the sum forming one of the external masses, say the one labelled r in fig. 7, and $r > 2$, then the F functions simply reduce to the corresponding ones with $n \to n-1$,

$$\begin{aligned} F^{1\mathrm{m}}_{n:i} &\to F^{1\mathrm{m}}_{n-1:i}\,, \\ F^{2\mathrm{m}\,e}_{n:r;i} &\to F^{2\mathrm{m}\,e}_{n-1:r-1;i}\,; \end{aligned} \tag{I.9}$$

the behavior is analogous in the case that both external momenta are part of the sum forming a different external mass, so long as three or more external momenta make up that mass. In the case that $r = 2$, the two-mass box $F^{2\mathrm{m}\,e}_{n:r;c}$ reduces to a one-mass box,

$$F^{2\mathrm{m}\,e}_{n:r;c} \to F^{1\mathrm{m}}_{n-1:c+2} + \frac{1}{\epsilon^2}\left(-t_c^{[2]}\right)^{-\epsilon}\,. \tag{I.10}$$

Using the collinear limit

$$t_{c+1}^{[r+1]} \to z t_{c+1}^{[r]} + (1-z) t_c^{[r+1]}\,, \tag{I.11}$$

and Abel's identity [46], one can show that

$$\begin{aligned} F^{2\mathrm{m}\,e}_{n:r;c+1} + F^{2\mathrm{m}\,e}_{n:r-1;c+2} &\to F^{2\mathrm{m}\,e}_{n-1:r-1;c+1}\,, \\ F^{2\mathrm{m}\,e}_{n:r;c-r+1} + F^{2\mathrm{m}\,e}_{n:r-1;c-r+1} &\to F^{2\mathrm{m}\,e}_{n-1:r-1;c-r+1}\,. \end{aligned} \tag{I.12}$$

For the one-mass box function, there are four additional cases to consider, corresponding to $c = i-4, \ldots, i-1$. The latter two are equivalent to the former two under a reflection; the first combines with an $F^{2\mathrm{m}\,e}_{n:2;c+1}$ as follows,

$$F^{2\mathrm{m}\,e}_{n:2;c+1} + F^{1\mathrm{m}}_{n:c+4} \to F^{1\mathrm{m}}_{n-1:c+3} \tag{I.13}$$

while the second reduces to

$$F^{1\mathrm{m}}_{n:c+3} \to -\frac{1}{\epsilon^2}\left(-(1-z)t_c^{[2]}\right)^{-\epsilon} - \mathrm{Li}_2(z)\,. \tag{I.14}$$

Combining the limits of

$$F^{2\mathrm{m}\,e}_{n:2;c} + F^{1\mathrm{m}}_{n:c+3} + F^{1\mathrm{m}}_{n:c+2} \tag{I.15}$$

yields $(\mu^2)^{-\epsilon} r_S^{\mathrm{SUSY}}(z, t_c^{[2]})$ in the form also obtained from gluino amplitudes in the following appendix,

$$\begin{aligned} F^{2\mathrm{m}\,e}_{n:2;c} + F^{1\mathrm{m}}_{n:c+3} + F^{1\mathrm{m}}_{n:c+2} &\longrightarrow F^{1\mathrm{m}}_{n-1:c+2} \\ &+ \frac{1}{\epsilon^2}\left(-t_c^{[2]}\right)^{-\epsilon} - \frac{1}{\epsilon^2}\left(-(1-z)t_c^{[2]}\right)^{-\epsilon} - \mathrm{Li}_2(z) - \frac{1}{\epsilon^2}\left(-z t_c^{[2]}\right)^{-\epsilon} - \mathrm{Li}_2(1-z)\,. \end{aligned} \tag{I.16}$$

Appendix II. Loop Splitting Amplitudes

In this appendix we collect the various splitting amplitudes which are useful for bootstrapping higher-point one-loop amplitudes from known amplitudes. The one-loop splitting amplitudes are obtained by taking the collinear limit of known five parton amplitudes [13,33,42]. The universality of the $(g \to gg)$ splitting amplitudes for arbitrary numbers of legs has been shown for scalar contributions [28], but it is likely to be true in general. All known one-loop amplitudes satisfy eq. (3.2), including amplitudes with external fermions.

In discussing gauge theory amplitudes with external fermions, we distinguish two cases: external fermions in the adjoint representation (gluinos, $\tilde{g}$), and external fermions in the fundamental N_c and $\bar{N}_c$ representations (quarks, q, and antiquarks, $\bar{q}$). The color decomposition of scattering amplitudes with external gluons and gluinos are identical to the n-gluon color decompositions (2.1) and (2.8). The tree and loop splitting amplitudes for $g \to \tilde{g}\tilde{g}$ and $\tilde{g} \to \tilde{g}g$ may therefore be defined via the same equations (3.1), (3.2) used to define $g \to gg$.

The color decomposition of amplitudes with external quarks as well as gluons is somewhat different, but the collinear behavior (3.1), (3.2) again holds for the tree partial amplitudes A_n, and is expected to hold for the leading-in-N_c one-loop partial amplitudes $A_{n;1}$, with an appropriate definition of A_n and $A_{n;1}$. (For amplitudes with four or more external quarks, one must restrict to the leading-in-N_c contributions even at tree level, in order to obtain the simple color-adjacent collinear behavior (3.1).) For example, tree amplitudes with two external quarks and $n-2$ gluons have the decomposition

$$\mathcal{A}_n^{\rm tree}(1_{\bar{q}}, 2_q, 3, \ldots, n) \;=\; \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}})_{i_2}^{\;\bar{\imath}_1} \; A_n^{\rm tree}(1_{\bar{q}}, 2_q; \sigma(3), \ldots, \sigma(n)). \qquad \text{(II.1)}$$

The A_n obey (3.1) for all four possibilities: $g \to gg$, $q \to qg$, $\bar{q} \to g\bar{q}$ and $g \to \bar{q}q$. The $g \to \bar{q}q$ collinear limit produces an $(n-1)$-gluon partial amplitude on the right-hand-side of (3.1), whereas the remaining limits produce two-quark-$(n-3)$-gluon partial amplitudes. The one-loop amplitudes can be decomposed as follows,

$$\mathcal{A}_n^{1-\rm loop}(1_{\bar{q}}, 2_q, 3, \ldots, n) \;=\; \sum_{j=1}^{n-1} \sum_{\sigma \in S_{n-2}/S_{n;j}} {\rm Gr}_{n;j}^{(\bar{q}q)}(\sigma(3, \ldots, n)) \; A_{n;j}(1_{\bar{q}}, 2_q; \sigma(3, \ldots, n)) \;, \qquad \text{(II.2)}$$

where

$$
\begin{aligned}
\mathrm{Gr}_{n;1}^{(\bar{q}q)}(3,\ldots,n) &= N_c\ (T^{a_3}\ldots T^{a_n})_{i_2}^{\;\bar{\imath}_1}\ , \\
\mathrm{Gr}_{n;2}^{(\bar{q}q)}(3;4,\ldots,n) &= 0\ , \\
\mathrm{Gr}_{n;j}^{(\bar{q}q)}(3,\ldots,j+1;j+2,\ldots,n) &= \mathrm{Tr}(T^{a_3}\ldots T^{a_{j+1}})\ \ (T^{a_{j+2}}\ldots T^{a_n})_{i_2}^{\;\bar{\imath}_1}\ , \quad j=3,\ldots,n-2, \\
\mathrm{Gr}_{n;n-1}^{(\bar{q}q)}(3,\ldots,n) &= \mathrm{Tr}(T^{a_3}\ldots T^{a_n})\ \ \delta_{i_2}^{\bar{\imath}_1}\ ,
\end{aligned}
\tag{II.3}
$$

and $S_{n;j}$ is the subset of the permutation group S_{n-2} that leaves $\mathrm{Gr}_{n;j}^{(\bar{q}q)}$ invariant. The color-ordered partial amplitudes $A_{n;1}$ give the leading-in-N_c one-loop contribution to the color-summed cross-section, and for $n=5$ ($\bar{q}qggg$) the $A_{n;1}$ have the expected one-loop collinear behavior (3.2) in all channels [42], with the splitting amplitudes given below.

Before presenting the explicit loop splitting amplitudes, we first review the tree-level splitting amplitudes, since they enter into the collinear behavior of loop amplitudes as well. Also, most of the one-loop splitting amplitudes are proportional to the tree-level ones.

The splitting amplitudes for the process $g \to gg$, when the gluon momenta k_a and k_b become collinear are [2,41,3,1]

$$
\begin{aligned}
\mathrm{Split}_{-}^{\mathrm{tree}}(a^-,b^-) &= 0, \\
\mathrm{Split}_{-}^{\mathrm{tree}}(a^+,b^+) &= \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}, \\
\mathrm{Split}_{-}^{\mathrm{tree}}(a^+,b^-) &= -\frac{z^2}{\sqrt{z(1-z)}\,[a\,b]}, \\
\mathrm{Split}_{-}^{\mathrm{tree}}(a^-,b^+) &= -\frac{(1-z)^2}{\sqrt{z(1-z)}\,[a\,b]},
\end{aligned}
\tag{II.4}
$$

where $k_a = zP$ and $k_b = (1-z)P$ with $P = (k_a + k_b)$. The remaining $\mathrm{Split}_{+}^{\mathrm{tree}}$ can be obtained from these by parity.

The $g \to \bar{q}q$ splitting amplitudes are

$$
\begin{aligned}
\mathrm{Split}_{+}^{\mathrm{tree}}(\bar{q}^+,q^-) &= \frac{z^{1/2}(1-z)^{3/2}}{\sqrt{z(1-z)}\,\langle \bar{q}\,q\rangle}, \\
\mathrm{Split}_{+}^{\mathrm{tree}}(\bar{q}^-,q^+) &= -\frac{z^{3/2}(1-z)^{1/2}}{\sqrt{z(1-z)}\,\langle \bar{q}\,q\rangle}, \\
\mathrm{Split}_{-}^{\mathrm{tree}}(\bar{q}^+,q^-) &= \frac{z^{3/2}(1-z)^{1/2}}{\sqrt{z(1-z)}\,[\bar{q}\,q]}, \\
\mathrm{Split}_{-}^{\mathrm{tree}}(\bar{q}^-,q^+) &= -\frac{z^{1/2}(1-z)^{3/2}}{\sqrt{z(1-z)}\,[\bar{q}\,q]},
\end{aligned}
\tag{II.5}
$$

and the $q \to qg$ and $\bar{q} \to g\bar{q}$ splitting amplitudes are

$$\begin{aligned}
\mathrm{Split}^{\mathrm{tree}}_{-}(q^+, a^+) &= \frac{z^{1/2}}{\sqrt{z(1-z)}\,\langle q\,a\rangle}, \\
\mathrm{Split}^{\mathrm{tree}}_{-}(q^+, a^-) &= -\frac{z^{3/2}}{\sqrt{z(1-z)}\,[q\,a]}, \\
\mathrm{Split}^{\mathrm{tree}}_{-}(a^+, \bar{q}^+) &= \frac{(1-z)^{1/2}}{\sqrt{z(1-z)}\,\langle a\,\bar{q}\rangle}, \\
\mathrm{Split}^{\mathrm{tree}}_{-}(a^-, \bar{q}^+) &= -\frac{(1-z)^{3/2}}{\sqrt{z(1-z)}\,[a\,\bar{q}]}.
\end{aligned} \tag{II.6}$$

Again the remaining ones can be obtained by parity. The tree-level splitting amplitudes with gluinos are identical to those with quarks; simply replace $q \to \tilde{g}$, $\bar{q} \to \tilde{g}$ in the above expressions.

The loop splitting functions have a structure similar to the tree splitting amplitudes, so it is useful to express them in terms of a proportionality constant r_S defined by

$$\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b}) \;=\; c_\Gamma \times \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b}) \times r_S(-\lambda, a^{\lambda_a}, b^{\lambda_b}) \tag{II.7}$$

for general partons a and b. The only exception to eq. (II.7) is for $g^- \to g^+g^+$ (and its parity conjugate $g^+ \to g^-g^-$), where $\mathrm{Split}^{\mathrm{tree}}$ vanishes but $\mathrm{Split}^{\mathrm{loop}}$ does not. In general $r_S(-\lambda, a^{\lambda_a}, b^{\lambda_b})$ depends on the parton helicities, although in a supersymmetric theory it turns out to be helicity-independent.

The loop splitting amplitudes also depend on the particles circulating in the loop. The contribution to the $g \to gg$ loop splitting amplitudes, $\mathrm{Split}^{\mathrm{loop}}(a, b)$, for an adjoint spin-$J$ particle (with two helicity states) are denoted by $\mathrm{Split}^{[J]}$, and the corresponding proportionality constant by $r_S^{[J]}$. The splitting amplitudes with internal particles in the fundamental representation are the same but multiplied by $1/N_c$. The splitting amplitudes we present describe the collinear behavior before subtraction of the ultraviolet pole, that is of unrenormalized amplitudes. These are slightly simpler than the corresponding ones for physical ('renormalized') amplitudes, but it is easy to convert from the former to the latter; in the $\overline{MS}$ subtraction scheme one simply adds to r_S the helicity-independent term $-\frac{1}{2\epsilon}\hat{\beta}_0$, where $\hat{\beta}_0 = \frac{11}{3} - \frac{2}{3}\frac{n_f}{N_c}$ in QCD with n_f quark flavors.

The $g \to gg$ splitting amplitudes may be directly obtained from the four- [14,20] and five-point [13] helicity amplitudes. The $\mathrm{Split}^{[J]}_{+}(a^+, b^+)$ obey the supersymmetry relation $\mathrm{Split}^{[1]} = -\,\mathrm{Split}^{[1/2]} = \mathrm{Split}^{[0]}$, where

$$\mathrm{Split}^{[1]}_{+}(a^+, b^+) \;=\; -\frac{1}{48\pi^2}\sqrt{z(1-z)}\;\frac{[a\,b]}{\langle a\,b\rangle^2}\;. \tag{II.8}$$

We present the remaining $g \to gg$ loop splitting amplitudes in terms of r_S:

$$
\begin{aligned}
r_S^{[1]}(-, a^+, b^+) &= -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s_{ab})}\right)^{\epsilon} + 2\ln z\,\ln(1-z) + \frac{1}{3}z(1-z) - \frac{\pi^2}{6}\ , \\
r_S^{[1/2]}(-, a^+, b^+) &= -\frac{1}{3}z(1-z)\ , \\
r_S^{[0]}(-, a^+, b^+) &= +\frac{1}{3}z(1-z)\ , \\
r_S^{[1]}(+, a^\pm, b^\mp) &= -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s_{ab})}\right)^{\epsilon} + 2\ln z\,\ln(1-z) - \frac{\pi^2}{6}\ , \\
r_S^{[1/2]}(+, a^\pm, b^\mp) &= 0\ , \\
r_S^{[0]}(+, a^\pm, b^\mp) &= 0\ .
\end{aligned}
\tag{II.9}
$$

One can extract the loop-level $q \to qg$ and $\bar{q} \to \bar{q}g$ splitting amplitudes from two independent sources: Giele and Glover's expressions for the one-loop $(\gamma^*, Z) \to q\bar{q}$ and $(\gamma^*, Z) \to qg\bar{q}$ helicity amplitudes [33], and a calculation of $\bar{q}q \to ggg$ at one-loop [42]. Both methods agree, and we find

$$
\begin{aligned}
r_S(q^-, a^+) &= r_S(q^+, a^-) = f(1-z, s_{qa})\ , \\
r_S(q^-, a^-) &= r_S(q^+, a^+) = f(1-z, s_{qa}) + \left(1 + \frac{1}{N^2}\right)\frac{1-z}{2}\ , \\
r_S(a^+, \bar{q}^+) &= r_S(a^-, \bar{q}^-) = f(z, s_{a\bar{q}}) + \left(1 + \frac{1}{N^2}\right)\frac{z}{2}\ , \\
r_S(a^-, \bar{q}^+) &= r_S(a^+, \bar{q}^-) = f(z, s_{a\bar{q}})\ ,
\end{aligned}
\tag{II.10}
$$

where the function $f(z,s)$ is

$$
\begin{aligned}
f(z,s) = & -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(-s)}\right)^{\epsilon} - \mathrm{Li}_2(1-z) \\
& - \frac{1}{N_c^2}\left[-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{(1-z)(-s)}\right)^{\epsilon} + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{(-s)}\right)^{\epsilon} - \mathrm{Li}_2(z)\right]\ .
\end{aligned}
\tag{II.11}
$$

We have also extracted the factor $r_S^{[J]}$ for the loop-level $g \to \bar{q}q$ splitting amplitudes from the one-loop $\bar{q}q \to ggg$ helicity amplitudes [42]. By symmetry, $r_S^{[J]}$ is the same here for every helicity configuration. The results are:

$$\begin{aligned}
r_S^{[1]}(\pm,\bar{q}^{\mp},q^{\pm}) &= -\frac{1}{\epsilon^2}\left[\left(\frac{\mu^2}{z(-s)}\right)^{\epsilon}+\left(\frac{\mu^2}{(1-z)(-s)}\right)^{\epsilon}-2\left(\frac{\mu^2}{(-s)}\right)^{\epsilon}\right] \\
&\quad +\frac{13}{6\epsilon}\left(\frac{\mu^2}{(-s)}\right)^{\epsilon}+\ln(z)\,\ln(1-z)-\frac{\pi^2}{6}+\frac{83}{18}-\frac{\delta_R}{6} \\
&\quad -\frac{1}{N_c^2}\left[-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{(-s)}\right)^{\epsilon}-\frac{3}{2\epsilon}\left(\frac{\mu^2}{(-s)}\right)^{\epsilon}-\frac{7}{2}-\frac{\delta_R}{2}\right], \qquad \text{(II.12)} \\
r_S^{[1/2]}(\pm,\bar{q}^{\mp},q^{\pm}) &= -\frac{2}{3\epsilon}\left(\frac{\mu^2}{-s_{\bar{q}q}}\right)^{\epsilon}-\frac{10}{9}, \\
r_S^{[0]}(\pm,\bar{q}^{\mp},q^{\pm}) &= -\frac{1}{3\epsilon}\left(\frac{\mu^2}{-s_{\bar{q}q}}\right)^{\epsilon}-\frac{8}{9}.
\end{aligned}$$

Here δ_R is a parameter controlling the variant of dimensional regularization used. In a supersymmetric scheme such as dimensional reduction [18,20] or four-dimensional helicity [14] with 2 physical gluon helicity states, $\delta_R = 0$; in a "conventional" scheme [48] with $2-2\epsilon$ physical gluon helicity states, $\delta_R = 1$.

To convert the external quark results (II.10), (II.11) and (II.12) into external gluino results, one must correct for the different $SU(N_c)$ group theory factors; however, this simply amounts to replacing $1/N_c^2 \to -1$ in the expressions. Making these replacements, and setting $\delta_R = 0$, we find that in $N = 1$ super-Yang-Mills theory, for every possible helicity configuration and every choice of adjoint external states ($g \to gg$, $\tilde{g} \to \tilde{g}g$ or $g \to \tilde{g}\tilde{g}$), the proportionality constant r_S is given by

$$\begin{aligned}
r_S^{\rm SUSY}(z,s) &= -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(-s)}\right)^{\epsilon}-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{(1-z)(-s)}\right)^{\epsilon}+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{(-s)}\right)^{\epsilon}-\mathrm{Li}_2(1-z)-\mathrm{Li}_2(z) \\
&= -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s)}\right)^{\epsilon}+\ln z\,\ln(1-z)-\mathrm{Li}_2(1-z)-\mathrm{Li}_2(z) \\
&= -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s)}\right)^{\epsilon}+2\ln z\,\ln(1-z)-\frac{\pi^2}{6}.
\end{aligned} \qquad \text{(II.13)}$$

The independence of $r_S^{\rm SUSY}$ from the external states is consistent with the supersymmetry Ward identities [5], when the regulator is consistent with supersymmetry ($\delta_R = 0$) [18,14,19,20].

Due to supersymmetry, the same result (II.13) should also hold in $N = 4$ super-Yang-Mills theory. This is easiest to verify directly for the $g \to gg$ splitting amplitudes, because the difference between the $N = 4$ and $N = 1$ contributions to n-gluon amplitudes is just the contribution of three chiral multiplets (fermions plus scalars). From the supersymmetry relation for $\mathrm{Split}_{+}^{[J]}(a^{+},b^{+})$, and equation (II.9) for $r_S^{[J]}$, we see that the chiral multiplet contribution ($[1/2]+[0]$) to the loop splitting amplitudes vanishes for every helicity configuration.

The $\tilde{g} \to \tilde{g}g$ and $g \to \tilde{g}\tilde{g}$ splitting amplitudes are slightly subtler, because amplitudes with external fermions in $N=4$ super-Yang-Mills theory have contributions with virtual scalars coupling to fermion lines via Yukawa interactions, in addition to the gauge interactions assumed in the above results. We have calculated these additional Yukawa contributions to five-point amplitudes with two external fermions (they are a natural intermediate step in the string-based gauge-theory calculation), and have extracted the corresponding extra terms in the loop splitting amplitudes. They produce an extra term $r_S^{\rm Yukawa}$ which should be added to the proportionality constant r_S in the case of external fermions.

For $q \to qg$ and $\bar{q} \to g\bar{q}$, with the virtual scalar in the adjoint representation, the correction is

$$\begin{aligned}
r_S^{\rm Yukawa}(q^-,a^+) &= r_S^{\rm Yukawa}(q^+,a^-) = r_S^{\rm Yukawa}(a^-,\bar{q}^+) = r_S^{\rm Yukawa}(a^+,\bar{q}^-) = 0, \\
r_S^{\rm Yukawa}(q^-,a^-) &= r_S^{\rm Yukawa}(q^+,a^+) = \left(1+\frac{1}{N_c^2}\right)\frac{1-z}{2}\,, \\
r_S^{\rm Yukawa}(a^+,\bar{q}^+) &= r_S^{\rm Yukawa}(a^-,\bar{q}^-) = \left(1+\frac{1}{N_c^2}\right)\frac{z}{2}\,.
\end{aligned} \tag{II.14}$$

The corresponding result where quarks are replaced by gluinos can be found by letting $1/N_c^2 \to -1$; it vanishes in every case. Thus we see that $r_S^{\rm SUSY}$ controls the $\tilde{g} \to \tilde{g}g$ behavior for $N=4$ as well as $N=1$ super-Yang-Mills.

Finally, the Yukawa contribution to $g \to \bar{q}q$ is the same for all helicities,

$$r_S^{\rm Yukawa}(\pm,\bar{q}^{\mp},q^{\pm}) = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{\bar{q}q}}\right)^{\epsilon} + \frac{3}{2} - \frac{1}{N_c^2}\left[\frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{\bar{q}q}}\right)^{\epsilon} + \frac{1}{2}\right]\,. \tag{II.15}$$

Again making the substitution $1/N_c^2 \to -1$ in order to get the gluino result, we find that the Yukawa contribution $r_S^{\rm Yukawa}$ cancels the extra chiral loop contribution $r_S^{[1/2]} + r_S^{[0]}$ in (II.12), verifying that $r_S^{\rm SUSY}$ also governs $g \to \tilde{g}\tilde{g}$ in $N=4$ super-Yang-Mills.

Appendix III. Explicit Summation of Subleading-Color Terms

In this appendix we carry out the sum over permutations in eq. (7.2) to obtain the explicit form of the subleading-in-N_c partial amplitudes $A_{n;c}$. We calculate the coefficient of the subleading color structure

$$\mathrm{Gr}_{n;c}(1) = \mathrm{Tr}\left(T^{a_1}\cdots T^{a_{c-1}}\right)\,\mathrm{Tr}\left(T^{a_c}\cdots T^{a_n}\right)\,. \tag{III.1}$$

and define $\alpha_i \in \{\alpha\} \equiv \{c-1, c-2, \ldots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{c, c+1, \ldots, n-1, n\}$,

The partial amplitude $A_{n;1}$ in eqs. (4.1), (4.19) is a sum of the one-mass and easy two-mass box integral functions with the appropriate cyclic ordering. Since $A_{n;c}$ is given by a sum of permuted $A_{n;1}$ it will also be a sum of these integrals but now with all the orderings specified by $COP\{\alpha\}\{\beta\}$. Because many orderings of the external momenta appear, in this section we use the more explicit notation for the arguments of the scalar box function F. The arguments $k_{i_1}, P_1, k_{i_2}, P_2$ denote the four external momenta of the box diagram, in cyclic order around the box; in the cases we shall encounter here, k_{i_1} and k_{i_2} are massless external momenta, while P_1 and P_2 are in general massive vectors (sums of external momenta). In $A_{n;1}$, P_1 and P_2 were always sums of cyclicly consecutive momenta, but in $A_{n;c}$ this is no longer the case. If either P_1 or P_2 consists of a single external momentum, then F reduces to a rescaled one-mass box.

In terms of these rescaled boxes, the function V_n^g of eqs. (4.19) is

$$V_n^g \;=\; \mu^{2\epsilon} \sum_{i_1, i_2} F(k_{i_1}, P_{i_1+1, i_2-1}, k_{i_2}, P_{i_2+1, i_1-1}), \tag{III.2}$$

where we define the sum of consecutive momenta

$$P_{i,j} \;\equiv\; k_i + k_{i+1} + \cdots + k_j. \tag{III.3}$$

The sum in equation (III.2) (and analogous sums in future equations) runs over all distinct i_1, i_2 such that P_{i_1+1, i_2-1} and P_{i_2+1, i_1-1} are nonzero; the indices in (III.2) are all treated mod n.

Now we sum the expression $A_{n;1} = c_\Gamma A_{jk}^{\text{tree MHV}} V_n^g$ over the permutations in $COP\{\alpha\}\{\beta\}$, to obtain $A_{n;c}$. The α_i indices are all treated mod $c-1$ (i.e. cyclicly in $\{c-1, c-2, \ldots, 2, 1\}$), while the β_i indices are treated mod $n-(c-1)$ (cyclicly in $\{c, c+1, \ldots, n\}$). Focus first on the (rescaled) box integral where β_1 and β_2 have been "pulled out", that is, where the two massless legs have labels β_1 and β_2,

$$F(k_{\beta_1}, P_{\beta_1+1, \beta_2-1} + P_{\alpha_2, \alpha_1-1}, k_{\beta_2}, P_{\beta_2+1, \beta_1-1} + P_{\alpha_1, \alpha_2-1}). \tag{III.4}$$

Here P_{α_i, α_j} and P_{β_i, β_j} are sums of consecutive momenta within the respective $\{\alpha\}$ and $\{\beta\}$ sets,

$$\begin{aligned} P_{\alpha_1, \alpha_2-1} \;&\equiv\; k_{\alpha_1} + k_{\alpha_1+1} + \cdots + k_{\alpha_2-1}, \qquad \text{(indices mod } c-1\text{)}, \\ P_{\beta_1+1, \beta_2-1} \;&\equiv\; k_{\beta_1+1} + k_{\beta_1+2} + \cdots + k_{\beta_2-1}, \qquad \text{(indices mod } n-(c-1)\text{)}, \end{aligned} \tag{III.5}$$

etc. Note that both the $\{\alpha\}$ and $\{\beta\}$ sets have been partitioned in two in a specific way in the integral (III.4). Denote the coefficient which multiplies this rescaled box in $A_{n;c}$ by $(\mu^2)^\epsilon c(\alpha_1, \alpha_2; \beta_1, \beta_2)$.

In general, quite a few permutations in $COP\{\alpha\}\{\beta\}$ contribute to $c(\alpha_1,\alpha_2;\beta_1,\beta_2)$, with different spinor product denominators from permuting $A_{jk}^{\text{tree MHV}}$. Define $M(\beta_1,\beta_2;\alpha_1-1,\alpha_2)$ to be the set of all *mergings* of the two ordered sets $\{\beta_1+1,\beta_1+2,\ldots,\beta_2-1\}$ and $\{\alpha_1-1,\alpha_1-2,\ldots,\alpha_2\}$ within the range $[\beta_1,\beta_2]$. The coefficient $c(\alpha_1,\alpha_2;\beta_1,\beta_2)$ is obtained by summing over mergings in $M(\beta_1,\beta_2;\alpha_1-1,\alpha_2)$ on one side of the pulled-out legs β_1 and β_2, and over mergings in $M(\beta_2,\beta_1;\alpha_2-1,\alpha_1)$ on the other side:

$$
\begin{aligned}
c(\alpha_1,\alpha_2;\beta_1,\beta_2) \;=\; (-1)^{c-1}\, c_\Gamma\, i\, \langle j\, k\rangle^4 & \sum_{\{i_1,i_2,\ldots,i_m\}\in M(\beta_1,\beta_2;\alpha_1-1,\alpha_2)} \frac{1}{\langle\beta_1\, i_1\rangle\,\langle i_1\, i_2\rangle\cdots\langle i_m\,\beta_2\rangle} \\
\times & \sum_{\{i_1,i_2,\ldots,i_{m'}\}\in M(\beta_2,\beta_1;\alpha_2-1,\alpha_1)} \frac{1}{\langle\beta_2\, i_1\rangle\,\langle i_1\, i_2\rangle\cdots\langle i_{m'}\,\beta_1\rangle}\,.
\end{aligned}
\tag{III.6}
$$

Spinor product identities can be used to simplify the permutation sum. The standard "eikonal" identity reads

$$
\sum_{j=j_1+1}^{j_2} \mathcal{S}_n(j-1,j) \;=\; \mathcal{S}_n(j_1,j_2), \tag{III.7}
$$

where the soft factor $\mathcal{S}_n(i,j)$ is

$$
\mathcal{S}_n(i,j) \;=\; \frac{\langle i\, j\rangle}{\langle i\, n\rangle\,\langle n\, j\rangle}\,. \tag{III.8}
$$

With the help of (III.7), one can show that

$$
\begin{aligned}
& \sum_{\{i_1,i_2,\ldots,i_m\}\in M(\beta_1,\beta_2;\alpha_1-1,\alpha_2)} \frac{1}{\langle\beta_1\, i_1\rangle\,\langle i_1\, i_2\rangle\cdots\langle i_m\,\beta_2\rangle} \\
&= \frac{1}{\langle\alpha_1-1\;\alpha_1-2\rangle\cdots\langle\alpha_2+1\;\alpha_2\rangle}\,\frac{1}{\langle\beta_1\;\beta_1+1\rangle\cdots\langle\beta_2-1\;\beta_2\rangle}\left(\frac{\langle\beta_1\,\beta_2\rangle}{\langle\beta_1\;\alpha_1-1\rangle\,\langle\alpha_2\,\beta_2\rangle}\right).
\end{aligned}
\tag{III.9}
$$

The proof is by induction on α_1, i.e. on the number of α's. For each merging in $M(\beta_1,\beta_2;\alpha_1-1,\alpha_2)$ there are several mergings in $M(\beta_1,\beta_2;\alpha_1,\alpha_2)$, from inserting α_1 all possible places between β_1 and α_1-1. Due to (III.7) they all produce the same multiplicative factor, namely

$$
\begin{aligned}
&\mathcal{S}_{\alpha_1}(\beta_1,\beta_1+1)+\mathcal{S}_{\alpha_1}(\beta_1+1,\beta_1+2)+\cdots+\mathcal{S}_{\alpha_1}(\beta_s,\alpha_1-1) \\
&= \mathcal{S}_{\alpha_1}(\beta_1,\alpha_1-1) \;=\; \frac{\langle\beta_1\;\alpha_1-1\rangle}{\langle\beta_1\,\alpha_1\rangle\,\langle\alpha_1\;\alpha_1-1\rangle}\,,
\end{aligned}
\tag{III.10}
$$

which is just the factor needed to go from α_1-1 to α_1 on the right-hand side of equation (III.9). It is also easy to see from (III.7) that the induction starts correctly when there is only one α.

Applying equation (III.9) twice, the coefficient $c(\alpha_1,\alpha_2;\beta_1,\beta_2)$ is given by the product

$$
\begin{aligned}
& c(\alpha_1,\alpha_2;\beta_1,\beta_2) \\
&= \frac{(-1)^{c-1}\, c_\Gamma\, i\, \langle j\, k\rangle^4}{\langle \alpha_1-1\, \alpha_1-2\rangle \cdots \langle \alpha_2+1\, \alpha_2\rangle} \frac{1}{\langle \beta_1\, \beta_1+1\rangle\cdots\langle\beta_2-1\,\beta_2\rangle}\left(\frac{\langle\beta_1\,\beta_2\rangle}{\langle\beta_1\,\alpha_1-1\rangle\,\langle\alpha_2\,\beta_2\rangle}\right) \\
&\quad\times \frac{1}{\langle\beta_2\,\beta_2+1\rangle\cdots\langle\beta_1-1\,\beta_1\rangle}\frac{1}{\langle\alpha_2-1\,\alpha_2-2\rangle\cdots\langle\alpha_1+1\,\alpha_1\rangle}\left(\frac{\langle\beta_2\,\beta_1\rangle}{\langle\beta_2\,\alpha_2-1\rangle\,\langle\alpha_1\,\beta_1\rangle}\right) \\
&= c_\Gamma \frac{i\,\langle j\,k\rangle^4}{\langle 1\,2\rangle\cdots\langle c-1,\,1\rangle\,\langle c\,,c+1\rangle\cdots\langle n\,c\rangle}(-1)\langle\beta_1\,\beta_2\rangle^2\mathcal{S}_{\beta_1}(\alpha_1-1,\alpha_1)\mathcal{S}_{\beta_2}(\alpha_2-1,\alpha_2)\ .
\end{aligned}
\tag{III.11}
$$

The same analysis works also for the integrals where α_1,β_2, etc., are "pulled out". Also, if β_1,β_2 are pulled out, and all the α variables are on one side, then we only get one factor of the type (III.9) instead of two; this generates the terms in $S_{n;c}$ in eq. (III.13).

Altogether, we find that $A_{n;c}$ for the MHV $N=4$ supersymmetric amplitudes becomes for $c\geq 3$,

$$
A_{n;c} \;=\; c_\Gamma\ (\mu^2)^\epsilon i\, \frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\cdots\langle c-1,\,1\rangle\ \ \langle c,\,c+1\rangle\cdots\langle n\,c\rangle}\,(G_{n;c}+S_{n;c})\ \ , \tag{III.12}
$$

where,

$$
\begin{aligned}
G_{n;c} \;=\; & \sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1\neq\alpha_2}}^{c-1}\ \sum_{\substack{\beta_1,\beta_2=c\\ \beta_1<\beta_2}}^{n}\Big[-\langle\beta_1\,\beta_2\rangle^2\mathcal{S}_{\beta_1}(\alpha_1-1,\alpha_1)\mathcal{S}_{\beta_2}(\alpha_2-1,\alpha_2) \\
&\qquad \times F(k_{\beta_1},P_{\beta_1+1,\beta_2-1}+P_{\alpha_2,\alpha_1-1},k_{\beta_2},P_{\beta_2+1,\beta_1-1}+P_{\alpha_1,\alpha_2-1})\Big] \\
&+\sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1<\alpha_2}}^{c-1}\ \sum_{\substack{\beta_1,\beta_2=c\\ \beta_1\neq\beta_2}}^{n}\Big[-\langle\alpha_1\,\alpha_2\rangle^2\mathcal{S}_{\alpha_1}(\beta_1-1,\beta_1)\mathcal{S}_{\alpha_2}(\beta_2-1,\beta_2) \\
&\qquad \times F(k_{\alpha_1},P_{\alpha_1+1,\alpha_2-1}+P_{\beta_2,\beta_1-1},k_{\alpha_2},P_{\alpha_2+1,\alpha_1-1}+P_{\beta_1,\beta_2-1})\Big] \\
&+\sum_{\alpha_1,\alpha_2=1}^{c-1}\ \sum_{\beta_1,\beta_2=c}^{n}\Big[+\langle\alpha_1\,\beta_2\rangle^2\mathcal{S}_{\alpha_1}(\beta_1-1,\beta_1)\mathcal{S}_{\beta_2}(\alpha_2-1,\alpha_2) \\
&\qquad \times F(k_{\alpha_1},P_{\beta_1,\beta_2-1}+P_{\alpha_2,\alpha_1-1},k_{\beta_2},P_{\beta_2+1,\beta_1-1}+P_{\alpha_1+1,\alpha_2-1})\Big]\ , \\
S_{n;c} \;=\; & -\sum_{\alpha_1=1}^{c-1}\ \sum_{\substack{\beta_1,\beta_2=c\\ \beta_1\neq\beta_2}}^{n}\frac{\langle\beta_1\,\beta_2\rangle\,\langle\alpha_1-1\,\alpha_1\rangle}{\langle\beta_1\,\alpha_1\rangle\,\langle\beta_2\,\alpha_1-1\rangle}\ F(k_{\beta_1},P_{\beta_1+1,\beta_2-1},k_{\beta_2},P_{\beta_2+1,\beta_1-1}+P_{\{\alpha\}}) \\
&-\sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1\neq\alpha_2}}^{c-1}\ \sum_{\beta_1=c}^{n}\frac{\langle\alpha_1\,\alpha_2\rangle\,\langle\beta_1-1\,\beta_1\rangle}{\langle\alpha_1\,\beta_1\rangle\,\langle\alpha_2\,\beta_1-1\rangle}\ F(k_{\alpha_1},P_{\alpha_1+1,\alpha_2-1},k_{\alpha_2},P_{\alpha_2+1,\alpha_1-1}+P_{\{\beta\}})\ ,
\end{aligned}
\tag{III.13}
$$

and

$$P_{\{\alpha\}} \;\equiv\; \sum_{\alpha_i\in\{\alpha\}} k_{\alpha_i}\ , \qquad P_{\{\beta\}} \;\equiv\; \sum_{\beta_i\in\{\beta\}} k_{\beta_i}\ ; \tag{III.14}$$

We define $F(k_{i_1}, P_1, k_{i_2}, P_2)$ to vanish if either $P_1^\mu = 0$ or $P_2^\mu = 0$. We also set $P_{\alpha_1+1,\alpha_1} \equiv P_{\beta_1+1,\beta_1} \equiv 0$.

For $c = 2$, we have, with $\{\alpha\} = \{1\}$:

$$\begin{aligned} A_{n;2} \;=\; & c_\Gamma\ (\mu^2)^\epsilon i\, \frac{\langle j\,k\rangle^4}{\langle 2\,3\rangle\cdots\langle n\,2\rangle} \sum_{\substack{\beta_1,\beta_2=2\\ \beta_1\neq\beta_2}}^{n} \Big[\mathcal{S}_1(\beta_1,\beta_2)\ F(k_{\beta_1}, P_{\beta_1+1,\beta_2-1}, k_{\beta_2}, P_{\beta_2+1,\beta_1-1} + k_1) \\ & -\mathcal{S}_1(\beta_1-1,\beta_1)\ F(k_1, P_{\beta_1,\beta_2-1}, k_{\beta_2}, P_{\beta_2+1,\beta_1-1})\Big]\ . \end{aligned} \tag{III.15}$$

We have also verified directly using the Cutkosky rules that the expression in eq. (III.12) has all the correct cuts.

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Figure Captions

Fig. 1: To obtain all-n expressions we impose a variety of constraints summarized here.

Fig. 2: In the collinear limit of a one-loop amplitude we obtain two type of terms: tree splitting amplitudes multiplying one-loop amplitudes and loop splitting amplitudes multiplying tree amplitudes.

Fig. 3: The possible intermediate helicities when both negative helicity gluons lie on the same side of the cut.

Fig. 4: The two possible intermediate helicities when the negative helicity gluons lie on opposite sides of the cut.

Fig. 5: After sewing the MHV tree amplitudes together, the cut can be rearranged to be the cut of the hexagon integral shown.

Fig. 6: The cut hexagon integral can be reduced to the sum of the four cut box integrals shown.

Fig. 7: The different types of box integrals given in eq. (I.8).

Fig. 8: These pairs of diagrams cancel by the antisymmetry of the three-point vertex.

Fig. 9: Diagrams with four-point vertices cancel in triplets.

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