SLAC-PUB-6525
hep-ph/9406286
June 1994
T/E

On the QCD corrections to $b \to s\gamma$ in supersymmetric models*

Harald Anlauf†

Stanford Linear Accelerator Center
Stanford University, Stanford, CA 94309

Abstract

We reinvestigate the leading QCD corrections to the radiative decay $b \to s\gamma$ for supersymmetric extensions of the Standard Model. Although the major contributions to the corrections originate from the running of the effective Lagrangian from the W scale down to the b scale, additional corrections are expected from large mass splittings between the particles running in the loops, as well as from integrating out heavy particles at scales different from the W mass. The calculation is performed in the framework of effective field theories.

Submitted to Nuclear Physics B

---

*Work supported in part by the Department of Energy under contract No. DE–AC03–76SF00515.

†Supported by a fellowship of the Deutsche Forschungsgemeinschaft.

1 Introduction

Among the rare decays of B mesons, the recently observed radiative weak decays B $\to X_s\gamma$ [1], where X_s is a hadronic state with total strangeness $S = -1$, have received much attention. As a loop-induced flavor-changing neutral current (FCNC) process this decay is in particular sensitive to contributions from physics beyond the Standard Model (SM).

Since the b quark mass is much larger than the QCD scale Λ, one assumes that the inclusive decay rate is well described by the spectator model, where the b quark undergoes a radiative decay. The transition amplitude for this process is given by the matrix element of an effective magnetic moment operator. The coefficient of this operator is obtained to lowest order [2] by integrating out all heavy particles (t quark, W boson, ...), leaving one with an effective field theory describing the transition $b \to s\gamma$ on the parton level at the weak scale. The QCD corrections to this coefficient[1] have been calculated to leading logarithmic accuracy in [4, 5, 6, 7, 8, 9] and are known to enhance the rate within the SM by a factor of 2–4, depending on the masses of the b and of the t quarks. This enhancement is, however, subject to large uncertainties due to the inaccuracy of some input parameters like the strong coupling constant α_S, and due to the residual renormalization scale dependence which we will not address in this work (for a recent discussion in the context of $b \to s\gamma$ see also [10, 11, 12]).

On the other hand, if the particles in the loop have vastly different masses, one expects sizeable corrections to the Wilson coefficients already at the weak scale. These contributions, which are usually considered as a next-to-leading order effect, have been discussed in ref. [13] for the Standard Model where, in the case of a top quark much heavier than the W, they were found to give an additional enhancement on the order of 20%.

In the case of the Minimal Supersymmetric Standard Model (MSSM) [14], the situation is even more complicated. First, due to the richer particle content, more diagrams contribute to the magnetic moment operators, and the larger number of free parameters supply many potential additional sources of flavor-changing neutral currents [15]. However, if one assumes further that the MSSM is a low-energy effective theory from minimal supergrav-

[1]For a recent review and earlier work see e.g. ref. [3].

ity [16] with radiative breaking of the electroweak symmetry, it is known [17, 11, 18, 19, 20, 21, 22, 23, 24, 25] that, besides the SM contribution mediated by the W, only the charged Higgs ($H^{\pm}$) and the chargino ($\chi^{+}_{1,2}$) exchange diagrams provide significant contributions. With the present experimental lower limits on the supersymmetry (SUSY) spectrum the gluino contribution is then small and the neutralino contribution is always negligible.[2] The W and the $H^{\pm}$ contributions always have the same sign in the MSSM, but the chargino contribution can have either sign and may, for instance, cancel the $H^{\pm}$ contribution or even dominate the amplitude (for small chargino masses and large $\tan\beta$).

Since SUSY has to be broken, the mass splitting between the various particles running in the loop can be very large, leading to additional important QCD corrections. We advocate that, in a parameter space analysis in the MSSM, one should not simply add up the contributions of all diagrams at the W scale and use the renormalization group evolution to run down this sum to the b scale, but rather consider the individual contributions separately. This consideration is especially important for the chargino contribution, since the lightest chargino can be significantly lighter than the W.

For the reasons mentioned above, we will ignore the contributions from diagrams with gluinos and neutralinos in the present work; they may easily be included. Corrections of the type considered in this work will, however, always be numerically unimportant.

Our strategy will be similar to the work by Cho and Grinstein [13]. Starting from the full theory at sufficiently high scales, we will construct a series of effective theories that are well suited for the description of the low-energy physics of interest. We shall give all ingredients that are necessary to obtain the leading QCD corrections to the $b \to s\gamma$ inclusive rate and discuss some simple estimates for an MSSM scenario with the assumptions mentioned above. However, a full parameter space analysis, which depends on the details of the implementation of the soft SUSY breaking, is beyond the scope of the present paper and will be discussed elsewhere.

[2] This assumption is supported by present experimental data on $B \to X_s\gamma$ as well as the lack of evidence for large contributions beyond the SM to other FCNC processes, e.g. $\bar{K}^0K^0$ mixing and rare K decays. For a discussion of a scenario with a light gluino in the mass range 2–5 GeV see [26] (and references quoted therein).

This paper is organized as follows: In section 2 we briefly review the elements of effective field theories needed for the present work. Section 3 explains in detail the calculation for a type-II two-Higgs doublet model, which is contained in the MSSM, while the contributions of SUSY particles are discussed in section 4. Our results are presented in section 5, followed by the conclusions.

2 Effective field theory and $b \to s\gamma$

The basic idea of effective field theories is by now well established, and many excellent reviews have appeared in the literature [27, 28, 29, 30, 31]. Starting from some underlying full theory, one integrates out the heavy degrees of freedom, thereby producing a tower of non-renormalizable interactions (with couplings proportional to inverse powers of the heavy particle mass) that contain the virtual heavy particle effects. One then runs the resulting effective field theory down to the appropriate scale of interest using the renormalization group. If additional heavy particle thresholds are crossed during the renormalization group running, these particles will then be integrated out. The major advantages of using an effective theory for the calculation of low-energy observables are convenience, since calculations are usually simpler than in the full theory, and the insight gained.

A nontrivial feature of the effective field theory framework is the automatic summation of large logarithms that originate from perturbatively calculable short-distance physics by the renormalization group. As explained in detail in [31], the renormalization scale μ in a dimensional scheme (e.g., $\overline{\mathrm{MS}}$) serves to separate short-distance from long-distance physics. The Appelquist–Carazzone decoupling theorem [32] can be implemented properly by hand, in the $\overline{\mathrm{MS}}$ scheme, which is mass-independent, matching the effective theories below and above thresholds. The advantage of having a mass-independent scheme is that the renormalization group β functions do not explicitly depend on the scale μ, while the validity of the decoupling theorem guarantees that all intuitive reasoning based on a so-called physical renormalization scheme remains true.

When the effective theory contains two heavy mass scales m_1, m_2 of comparable magnitude, it is usually a good approximation to integrate out both particles at a common scale. On the other hand, if the ratio $x \equiv (m_1/m_2)^2$ is very small (i.e., $x \ll 1$), even if the coupling constant is small, the product $\alpha \ln x$ may become of order one, forcing the summation of all powers of this product, while corrections to the sum are suppressed by powers of α or x. Sometimes the situation is less favorable and lies somewhere in between, as is the case for the SM with a heavy top quark [13] of, say, 175 GeV. For the process $b \to s\gamma$, the most important correction is the QCD running between the W and the b scale, whose size is (parametrically) given by

$$\alpha_S(m_{\mathrm{W}}) \cdot \ln(m_{\mathrm{W}}/m_{\mathrm{b}})^2 \simeq 0.7\,,$$

while

$$\alpha_S(m_{\mathrm{W}}) \cdot \ln(m_{\mathrm{t}}/m_{\mathrm{W}})^2 \simeq (m_{\mathrm{W}}/m_{\mathrm{t}})^2 \simeq 0.2$$

indicates that one might miss numerically important pieces if either of the latter were neglected, compared to next-to-leading order corrections that are of order $\alpha_S \simeq 0.1$.[3] What one can achieve with reasonable effort is to take into account the resummation of the leading terms in the limit of a heavy top quark, and then simply add in the nonleading terms, thus neglecting terms which are (up to logarithms) $O(\alpha_s \cdot (m_{\mathrm{W}}/m_{\mathrm{t}})^2)$. At this stage, the scale at which the nonleading terms are added is completely arbitrary, and can only be determined by a calculation of the power corrections. The remaining uncertainty is, however, less important than neglected next-to-leading order corrections.

Let us now turn to the application to the $b \to s\gamma$ transition. The effective Hamiltonian of interest may be written as a sum of $\Delta B = 1$, $\Delta S = 1$ operators:

$$H_{\mathrm{eff}} = \frac{4G_{\mathrm{F}}}{\sqrt{2}} K_{\mathrm{tb}} K^*_{\mathrm{ts}} \sum_i C_i(\mu) \mathcal{O}_i(\mu)\ . \tag{1}$$

A suitable operator basis $\{\mathcal{O}_i\}$ will be given below.

In general the definition of the operators in (1) requires the specification of a renormalization scheme. One derives renormalization group equations for the

[3] Of course a full calculation of the next-to-leading order corrections is necessary to resolve the well-known ambiguity in the choice of scales in leading-order calculations. However, this calculation would require the computation of three-loop anomalous dimensions for the process under consideration.

composite operators $\mathcal{O}_i(\mu)$ and the coefficient functions $C_i(\mu)$ from the fact that the effective Hamiltonian is independent of the renormalization scale. The renormalization of a composite operator is formally defined in terms of the divergent renormalization constants Z_{ij}, which relate renormalized and bare operators:

$$\mathcal{O}_i^{\mathrm{bare}} = Z_{ij}(\mu)\mathcal{O}_j(\mu) \tag{2}$$

Since the bare operators are μ independent, the renormalized operators depend on the subtraction scale via the μ dependence of the Z_{ij}:

$$\mu\frac{d}{d\mu}\mathcal{O}_i = \left(\mu\frac{d}{d\mu}Z_{ij}^{-1}\right)\mathcal{O}_j^{\mathrm{bare}} = -\gamma_{ik}\mathcal{O}_k, \tag{3}$$

where

$$\gamma_{ik} = Z_{ij}^{-1}\mu\frac{d}{d\mu}Z_{jk} \tag{4}$$

is the so-called anomalous dimension matrix.

The renormalization group equation for the Wilson coefficients C_i then reads:

$$\mu\frac{d}{d\mu}C_i(\mu) = \sum_j \left(\gamma^T\right)_{ij} C_j(\mu)\ . \tag{5}$$

If QCD corrections are neglected, the solution to this differential equation is straightforward. When QCD corrections are included, it is favorable to eliminate the derivative with respect to the renormalization scale in favor of a derivative with respect to the coupling constant:

$$\beta\frac{dC_i}{dg_3} = \sum_j \left(\gamma^T\right)_{ij} C_j \tag{6}$$

Here (and in the following) g_3 denotes the QCD coupling constant, and $\beta = \mu(dg_3/d\mu)$ is the QCD beta function.

The most convenient way to calculate the anomalous dimension matrix γ is to consider Green functions with insertions of composite operators. Denote by $\Gamma^{(n)}_{\mathcal{O}_i}$ a renormalized n-point 1PI Green function with one insertion of the operator $\mathcal{O}_i$. The anomalous dimension γ_{ij} that determines the mixing of $\mathcal{O}_i$

into $\mathcal{O}_j$ may then be simply read from the renormalization group equation for $\Gamma^{(n)}_{\mathcal{O}_i}$,

$$\gamma_{ij}\Gamma^{(n)}_{\mathcal{O}_j} = -\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m} - n\gamma_{ext}\right)\Gamma^{(n)}_{\mathcal{O}_i} \ . \tag{7}$$

Here $\gamma_m = (\mu/m)(dm/d\mu)$, and $n\gamma_{\rm ext}$ accounts for the wave-function anomalous dimensions arising from radiative corrections to the external lines of the Green functions.

We use dimensional regularization with minimal subtraction ($\overline{\rm MS}$), $d = 4-2\epsilon$. The $\mathrm{SU}(3)_{\rm C}\times \mathrm{U}(1)_{\rm em}$ covariant derivative then reads:

$$D_\mu = \partial_\mu - i\mu^\epsilon g_3 G^a_\mu X^a - i\mu^\epsilon e A_\mu Q \ . \tag{8}$$

We use the background field R_ξ gauge [33] throughout this work. The anomalous dimensions of the fields in this case are given by:

$$\begin{aligned} &\gamma_{\rm quark} = \frac{2}{3}\frac{g_3^2}{8\pi^2}, \quad \gamma_{\rm squark} = -\frac{4}{3}\frac{g_3^2}{8\pi^2}, \quad \gamma_{\rm gluon} = \frac{\beta}{g_3}, \\ &\gamma_{\rm m} = -4\frac{g_3^2}{8\pi^2}, \quad \gamma_{\tilde{\rm m}} = -6\frac{g_3^2}{8\pi^2}, \quad \beta = b\frac{g_3^3}{8\pi^2}. \end{aligned} \tag{9}$$

The coefficient appearing in the β-function has the value

$$b = -\frac{11}{2} + \frac{1}{3}n_f + \frac{1}{12}n_{\tilde{q}} + n_{\tilde{g}}, \tag{10}$$

where $n_f, n_{\tilde{q}}$ and $n_{\tilde{g}}$ are the number of active quark flavors, squarks and gluinos, respectively.

The solution to the differential equation (6) is given by

$$C(\mu) = T_g\left[\exp\int_{g_3(\mu_0)}^{g_3(\mu)} dg\,\frac{\gamma^T(g)}{\beta(g)}\right]C(\mu_0), \tag{11}$$

where T_g means an ordering in the coupling such that g increases from right to left (for $\mu < \mu_0$). Since our anomalous dimension matrices will be g_3^2 times a purely numerical matrix,

$$\gamma = \frac{g_3^2}{8\pi^2}\hat{\gamma} + O(g_3^4) \ ,$$

the g ordering is superfluous, and the g integration is trivial:

$$C(\mu) = \exp\left[\left(\frac{1}{2b}\ln\frac{\alpha_3(\mu)}{\alpha_3(\mu_0)}\right)\hat{\gamma}^T\right] C(\mu_0) \ . \tag{12}$$

In all cases considered below, the operator basis of choice contains the following set of operators involving only light degrees of freedom (that is, photons, gluons, and light quarks with masses below $m_{\rm W}$):[4]

Dimension $d+1$:

$$\begin{aligned}
O^1_{\rm LR} &= -\frac{1}{16\pi^2} m_{\rm b}\bar{s}_{\rm L} D^2 b_{\rm R} \\
O^2_{\rm LR} &= \mu^\epsilon \frac{g_3}{16\pi^2} m_{\rm b}\bar{s}_{\rm L}\sigma^{\mu\nu} X^a b_{\rm R}\ G^a_{\mu\nu} \\
O^3_{\rm LR} &= \mu^\epsilon \frac{eQ_{\rm b}}{16\pi^2} m_{\rm b}\bar{s}_{\rm L}\sigma^{\mu\nu} b_{\rm R}\ F_{\mu\nu} \ .
\end{aligned}$$

Dimension $d+2$:

$$\begin{aligned}
P^{1,A}_{\rm L} &= -\frac{i}{16\pi^2}\bar{s}_{\rm L} T^A_{\mu\nu\sigma} D^\mu D^\nu D^\sigma b_{\rm L} \\
P^2_{\rm L} &= \mu^\epsilon \frac{eQ_{\rm b}}{16\pi^2}\bar{s}_{\rm L}\gamma^\mu b_{\rm L}\ \partial^\nu F_{\mu\nu} \\
P^4_{\rm L} &= i\mu^\epsilon \frac{eQ_{\rm b}}{16\pi^2}\bar{s}_{\rm L}\gamma^\mu\gamma^5 D^\nu b_{\rm L}\ \tilde{F}_{\mu\nu} \ .
\end{aligned} \tag{13}$$

The tensors $T^A_{\mu\nu\sigma}$ appearing in $P^{1,A}_{\rm L}$, $A = 1,\ldots 4$, are defined by:

$$\begin{aligned}
T^1_{\mu\nu\sigma} &= g_{\mu\nu}\gamma_\sigma, \qquad & T^2_{\mu\nu\sigma} &= g_{\mu\sigma}\gamma_\nu, \\
T^3_{\mu\nu\sigma} &= g_{\nu\sigma}\gamma_\mu, \qquad & T^4_{\mu\nu\sigma} &= -i\epsilon_{\mu\nu\sigma\tau}\gamma^\tau\gamma_5 \ .
\end{aligned} \tag{14}$$

In order to apply the procedure outlined above to the MSSM case, we first consider the extension of the calculation by Cho and Grinstein [13] to the case of a type-II two-Higgs doublet model. Two cases already exist to consider, namely that the charged Higgs can be either much lighter or much heavier than the t quark. We shall explain in detail how the contributions induced by the chargino loops add to this picture .

[4] Note that our normalization differs from ref. [13]. We have omitted their operator $P^3_{\rm L}$, since the corresponding Wilson coefficients will always be zero, and none of the operators under consideration will mix back into it.

3 Two-Higgs doublet model

3.1 $m_{\mathrm{t}} > m_{\mathrm{H}^\pm}$

If the top quark is heavier than the W and the charged Higgs, then the first step is to integrate out the top quark at the scale $\mu = m_{\mathrm{t}}$. This procedure leads to an effective field theory for $\mu < m_{\mathrm{t}}$ without the t, but with new vertices of dimension larger than four that contain the virtual t effects. For the process under consideration we need, in addition to the operator basis (13), further operators.

In general, in the range $m_{\mathrm{t}} > \mu > m_{\mathrm{W}}$, one has to consider higher dimensional operators that contain the Ws, the would-be Goldstone bosons $\phi_\pm$ and the charged Higgs field. By naive dimensional analysis, we expect that higher dimensional operators are suppressed by inverse powers of the ratio $x_{\mathrm{tW}} \equiv (m_{\mathrm{t}}/m_{\mathrm{W}})^2$. Since this ratio is not very large for phenomenologically acceptable top quark masses, the effects of the higher dimensional operators are not necessarily small, compared to the leading dimension 5 and dimension 6 operators. Also, the matching conditions at threshold in general are combinations of rational functions and polynomials in x_{tW}.

Nevertheless, we shall take the approach motivated in chapter two and keep only the leading operators and the leading terms in the matching contributions. Although we are unable to calculate the power corrections, we shall later add the subleading terms in $1/x_{\mathrm{tW}}$, so that we get the same result in the limit of neglecting strong corrections for $\mu > m_{\mathrm{W}}$, as when all heavy particles are integrated out simultaneously at the W scale.

The relevant part of the interaction Lagrangian in the charged current sector reads

$$\begin{aligned} L_{\mathrm{CC}} &= \frac{g_2}{\sqrt{2}} W_\mu^+ \bar{U}\gamma^\mu K P_{\mathrm{L}} D + \mathrm{h.c.} \\ &+ \frac{g_2}{\sqrt{2} m_{\mathrm{W}}} \left(\phi_+ \bar{U} \left[M_U K P_{\mathrm{L}} - K M_D P_{\mathrm{R}} \right] D + \mathrm{h.c.} \right) , \end{aligned} \tag{15}$$

where $U = (u, c, t)$ and $D = (d, s, b)$ represent up-type and down-type quarks, respectively; $M_U = \mathrm{diag}(m_{\mathrm{u}}, m_{\mathrm{c}}, m_{\mathrm{t}})$, $M_D = \mathrm{diag}(m_{\mathrm{d}}, m_{\mathrm{s}}, m_{\mathrm{b}})$ are the quark mass matrices; $g_2 = e/\sin\theta_{\mathrm{W}}$ is the gauge coupling of $\mathrm{SU}(2)_{\mathrm{W}}$; $P_{\mathrm{L,R}}$ are projectors on the left- and right-handed components of the fermions; and K

is the Kobayashi-Maskawa matrix. In the present work, we shall neglect the masses of the quarks of the first two generations whenever appropriate.

From these expressions one can see that the leading terms for $x_{\mathrm{tW}} \gg 1$ come from vertices that involve the charged would-be Goldstone bosons $\phi_{\pm}$ and the top quark, since they are proportional to the top quark mass. For this reason, in the range $m_{\mathrm{t}} > \mu > m_{\mathrm{W}}$, we shall need (analogous to the findings in [13]) the following operators with external would-be Goldstone bosons in addition to the operator basis (13):

$$\begin{aligned}
Q_{\mathrm{LR}} &= \mu^{2\epsilon}\frac{g_3^2}{16\pi^2} m_{\mathrm{b}}\phi_+\phi_-\,\bar{s}_{\mathrm{L}}b_{\mathrm{R}} \\
R_{\mathrm{L}}^1 &= i\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\phi_+\phi_-\,\bar{s}_{\mathrm{L}}\not{D}b_{\mathrm{R}} \\
R_{\mathrm{L}}^2 &= i\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}(D^\mu\phi_+)\phi_-\,\bar{s}_{\mathrm{L}}\gamma_\mu b_{\mathrm{R}} \\
R_{\mathrm{L}}^3 &= i\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\phi_+(D^\mu\phi_-)\bar{s}_{\mathrm{L}}\gamma_\mu b_{\mathrm{R}}\ .
\end{aligned} \tag{16}$$

The inclusion of explicit factors g_3^2 into these operators is motivated by the Gilman-Wise trick [34], which allows all one-loop contributions to the anomalous dimension matrices to be of $O(g_3^2)$, so that the diagonalization of these matrices is scale independent. We will freely use this trick later on.

The interaction Lagrangian for the charged Higgs with the quarks reads:

$$L_{\mathrm{H}^\pm f\bar{f}'} = \frac{g_2}{\sqrt{2}m_{\mathrm{W}}}\left(H^+\bar{U}\left[\cot\beta M_U K P_{\mathrm{L}} + \tan\beta K M_D P_{\mathrm{R}}\right]D + \mathrm{h.c.}\right), \tag{17}$$

with $\tan\beta = v_1/v_2$ being the ratio of the vacuum expectation values of the Higgs fields that give rise to the masses of up- and down-type quarks, respectively.

The interaction (17) has the same structure and quark-mass dependence of the couplings as the interaction of the would-be Goldstone bosons $\phi_\pm$ [see (15)]. In the limit $x_{\mathrm{tH}} \equiv (m_{\mathrm{t}}/m_{\mathrm{H}^\pm})^2 \gg 1$, keeping only the leading terms in $1/x_{\mathrm{tH}}$, we are lead to the following operators with charged Higgs bosons we have to add to our operator basis in the range $m_{\mathrm{t}} > \mu > m_{\mathrm{H}^\pm}$:

$$Q'_{\mathrm{LR}} = \mu^{2\epsilon}\frac{g_3^2}{16\pi^2} m_{\mathrm{b}} H^+ H^-\,\bar{s}_{\mathrm{L}}b_{\mathrm{R}}$$

$$
\begin{aligned}
R_{\mathrm{L}}^{1\,\prime} &= i\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}H^+H^-\,\bar{s}_{\mathrm{L}}\not{D}b_{\mathrm{R}} \\
R_{\mathrm{L}}^{2\,\prime} &= i\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}(D^\mu H^+)H^-\,\bar{s}_{\mathrm{L}}\gamma_\mu b_{\mathrm{R}} \\
R_{\mathrm{L}}^{3\,\prime} &= i\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}H^+(D^\mu H^-)\bar{s}_{\mathrm{L}}\gamma_\mu b_{\mathrm{R}}\ .
\end{aligned} \tag{18}
$$

3.1.1 Matching at $\mu = m_{\mathrm{t}}$

For $\mu > m_{\mathrm{t}}$, our effective theory is a fully renormalizable theory, which still contains all particles and interactions, so that in this case all coefficients of our effective Hamiltonian are zero:

$$
C_i(\mu = m_{\mathrm{t}}^+) = 0 \qquad \text{for all } i\ . \tag{19}
$$

When we cross the t threshold from above, i.e., when we integrate out the top quark at $\mu = m_{\mathrm{t}}$, we obtain the following changes to the coefficients of the effective Hamiltonian, as a result of the interactions of the would-be Goldstone bosons from matching the three-point functions $\Gamma^{bs\gamma}$ and Γ^{bsg} [13] [(here $\Delta C_i = C_i(m_{\mathrm{t}}^-) - C_i(m_{\mathrm{t}}^+)$)]:

$$
\begin{aligned}
\Delta_{(\phi,W)}C_{O^1_{\mathrm{LR}}} &= -\frac{1}{2} \\
\Delta_{(\phi,W)}C_{O^2_{\mathrm{LR}}} &= \frac{Q_{\mathrm{b}}}{Q_{\mathrm{t}}}\Delta_{(\phi,W)}C_{O^3_{\mathrm{LR}}} = -\frac{1}{2} \\
\Delta_{(\phi,W)}C_{P^{1,1}_{\mathrm{L}}} &= \Delta_{(\phi,W)}C_{P^{1,3}_{\mathrm{L}}} = \frac{11}{18} \\
\Delta_{(\phi,W)}C_{P^{1,2}_{\mathrm{L}}} &= -\frac{8}{9} \\
\Delta_{(\phi,W)}C_{P^{1,4}_{\mathrm{L}}} &= -Q_{\mathrm{b}}\Delta_{(\phi,W)}C_{P^4_{\mathrm{L}}} = \frac{1}{2} \\
\Delta_{(\phi,W)}C_{P^2_{\mathrm{L}}} &= \frac{3}{4Q_{\mathrm{b}}} \\
\Delta_{(\phi,W)}C_{Q_{\mathrm{LR}}} &= -\frac{16\pi^2}{g_3^2} \\
\Delta_{(\phi,W)}C_{R^1_{\mathrm{L}}} &= \Delta_{(\phi,W)}C_{R^2_{\mathrm{L}}} = \frac{16\pi^2}{g_3^2} \\
\Delta_{(\phi,W)}C_{R^3_{\mathrm{L}}} &= 0\ .
\end{aligned} \tag{20}
$$

Similarly, there are contributions from the interactions with the charged Higgs bosons:[5]

$$\begin{aligned}
\Delta_{(H)} C_{O^1_{\mathrm{LR}}} &= \frac{1}{2} \\
\Delta_{(H)} C_{O^2_{\mathrm{LR}}} &= \frac{Q_{\mathrm{b}}}{Q_{\mathrm{t}}} \Delta_{(H)} C_{O^3_{\mathrm{LR}}} = \frac{1}{2} \\
\Delta_{(H)} C_{P^{1,1}_{\mathrm{L}}} &= \Delta_{(H)} C_{P^{1,3}_{\mathrm{L}}} = \frac{11}{18} \cot^2 \beta \\
\Delta_{(H)} C_{P^{1,2}_{\mathrm{L}}} &= -\frac{8}{9} \cot^2 \beta \\
\Delta_{(H)} C_{P^{1,4}_{\mathrm{L}}} &= -Q_{\mathrm{b}} \Delta_{(H)} C_{P^4_{\mathrm{L}}} = \frac{1}{2} \cot^2 \beta \\
\Delta_{(H)} C_{P^2_{\mathrm{L}}} &= \frac{3}{4 Q_{\mathrm{b}}} \cot^2 \beta \\
\Delta_{(H)} C_{Q'_{\mathrm{LR}}} &= \frac{16\pi^2}{g_3^2} \\
\Delta_{(H)} C_{R^1_{\mathrm{L}}{}'} &= \Delta_{(H)} C_{R^2_{\mathrm{L}}{}'} = \frac{16\pi^2}{g_3^2} \cot^2 \beta \\
\Delta_{(H)} C_{R^3_{\mathrm{L}}{}'} &= 0 \,.
\end{aligned} \tag{21}$$

At this point it is worthwhile to note that had we not matched at the scale $\mu = m_{\mathrm{t}}$ but at a different scale (or used a different subtraction scheme), we would have found logarithmic contributions in the matching corrections to the coefficient of P^2_{L}:

$$\Delta_{(\phi,W)} C_{P^2_{\mathrm{L}}} = \frac{1}{Q_{\mathrm{b}}} \left[\frac{3}{4} + \frac{1}{6} \ln \frac{\mu^2}{m_{\mathrm{t}}^2} \right], \quad \Delta_{(H)} C_{P^2_{\mathrm{L}}} = \frac{1}{Q_{\mathrm{b}}} \left[\frac{3}{4} + \frac{1}{6} \ln \frac{\mu^2}{m_{\mathrm{t}}^2} \right] \cot^2 \beta \,. \tag{22}$$

These logarithms that vanish for $\mu = m_{\mathrm{t}}$ are regenerated at lower scales by the renormalization group for the effective theory below m_{t}. It is therefore not surprising that they are present in the full expressions for this coefficient given in the appendix, when both particles in a loop are integrated out at the same scale; in the full expressions it appears as an unsuppressed logarithm of the mass ratio of the particles in the loop.

[5] We prefer to keep the contributions from each interaction separate, for there are different cases below.

3.1.2 Running below m_{t}

The anomalous dimension matrices for the mixing of the operators O_i and P_i have already been given in ref. [13]. For completeness, we quote the result obtained in this work.

First, the operators Q and R, with would-be Goldstone boson fields, mix into the operators without $(O,\, P)$:

$$\hat{\gamma} = \begin{array}{c c} & \begin{array}{cccc} O_{\mathrm{LR}} & P_{\mathrm{L}}^{1,A} & P_{\mathrm{L}}^{2} & P_{\mathrm{L}}^{4} \end{array} \\ \begin{array}{c} Q_{\mathrm{LR}} \\ R_{\mathrm{L}}^{1} \\ R_{\mathrm{L}}^{2} \\ R_{\mathrm{L}}^{3} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6Q_{\mathrm{b}} & 0 \\ 0 & 0 & -1/6Q_{\mathrm{b}} & 0 \end{pmatrix} \end{array} . \tag{23}$$

Note that this mixing back is of order $O(g_3^2)$ due to our choice of the coefficients in front of the operators Q, R, and not due to “proper” QCD corrections.

For the QCD-induced entries in the anomalous dimension matrix, one has

$$\hat{\gamma} = \begin{array}{c c} & \begin{array}{ccccccccc} O_{\mathrm{LR}}^{1} & O_{\mathrm{LR}}^{2} & O_{\mathrm{LR}}^{3} & P_{\mathrm{L}}^{1,1} & P_{\mathrm{L}}^{1,2} & P_{\mathrm{L}}^{1,3} & P_{\mathrm{L}}^{1,4} & P_{\mathrm{L}}^{2} & P_{\mathrm{L}}^{4} \end{array} \\ \begin{array}{c} O_{\mathrm{LR}}^{1} \\ O_{\mathrm{LR}}^{2} \\ O_{\mathrm{LR}}^{3} \\ P_{\mathrm{L}}^{1,1} \\ P_{\mathrm{L}}^{1,2} \\ P_{\mathrm{L}}^{1,3} \\ P_{\mathrm{L}}^{1,4} \\ P_{\mathrm{L}}^{2} \\ P_{\mathrm{L}}^{4} \end{array} & \begin{pmatrix} \frac{20}{3} & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & \frac{2}{3} & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{16}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & -1 & \frac{2}{3} & 2 & -2 & -2 & 0 & 0 \\ 4 & \frac{3}{2} & 0 & -\frac{113}{36} & \frac{137}{18} & -\frac{113}{36} & -\frac{4}{3} & \frac{9}{4} & 0 \\ 2 & 1 & 1 & -2 & 2 & \frac{2}{3} & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & -\frac{113}{36} & \frac{89}{18} & -\frac{113}{36} & \frac{4}{3} & \frac{9}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} . \tag{24}$$

Similarly, the mixing among the operators with would-be Goldstone boson fields is known to be:

$$\hat{\gamma} = \begin{array}{c c} & \begin{array}{cccc} Q_{\mathrm{LR}} & R_{\mathrm{L}}^{1} & R_{\mathrm{L}}^{2} & R_{\mathrm{L}}^{3} \end{array} \\ \begin{array}{c} Q_{\mathrm{LR}} \\ R_{\mathrm{L}}^{1} \\ R_{\mathrm{L}}^{2} \\ R_{\mathrm{L}}^{3} \end{array} & \begin{pmatrix} -2b & 0 & 0 & 0 \\ 0 & -2b & 0 & 0 \\ 0 & 0 & -2b & 0 \\ 0 & 0 & 0 & -2b \end{pmatrix} \end{array} . \tag{25}$$

Obviously, the same mixing matrices are found when one considers mixing of the operators with charged Higgs fields, i.e., when one replaces $Q_{\rm LR} \to Q'_{\rm LR}$, $R^i_{\rm L} \to R^i_{\rm L}{}'$ in eqs. (23) and (25).

3.1.3 Matching at $\mu = m_{\rm H^\pm}$ and $\mu = m_{\rm W}$

In the process of scaling down, when we encounter the charged Higgs or W threshold, we have to integrate out the $\rm H^\pm$ or W and would-be Goldstone bosons, respectively. Due to decoupling that has to take place below threshold, we remove the operators Q' and R' from our operator basis for $\mu < m_{\rm H^\pm}$ and Q and R for $\mu < m_{\rm W}$. Again we obtain the finite changes of the coefficients of the operators O and P by matching Green functions calculated in the theories above and below threshold.

Since we neglect small terms proportional to $m_{\rm u}$ or $m_{\rm c}$, we find no non-vanishing contribution from the matching of the effective theories above and below $\mu = m_{\rm H^\pm}$, i.e., our Wilson coefficients are continuous:

$$C_i(m^+_{\rm H^\pm}) = C_i(m^-_{\rm H^\pm}) \ . \tag{26}$$

Matching the effective theories above and below $\mu = m_{\rm W}$, we find the following changes in the coefficients of the effective Hamiltonian (here $\Delta C = C(m^-_{\rm W}) - C(m^+_{\rm W})$):

$$\begin{aligned}
\Delta_{(\phi,W)} C_{O^1_{\rm LR}} &= \Delta_{(\phi,W)} C_{O^2_{\rm LR}} = \Delta_{(\phi,W)} C_{O^3_{\rm LR}} = 0 \\
\Delta_{(\phi,W)} C_{P^{1,1}_{\rm L}} &= \Delta_{(\phi,W)} C_{P^{1,3}_{\rm L}} = \frac{2}{9} \\
\Delta_{(\phi,W)} C_{P^{1,2}_{\rm L}} &= -\frac{7}{9} \\
\Delta_{(\phi,W)} C_{P^{1,4}_{\rm L}} &= 1 \\
\Delta_{(\phi,W)} C_{P^{2}_{\rm L}} &= \frac{1}{2Q_{\rm b}} \\
\Delta_{(\phi,W)} C_{P^{4}_{\rm L}} &= -\frac{3}{Q_{\rm b}} \ .
\end{aligned} \tag{27}$$

Again, had we matched at a different scale $\mu \neq m_{\rm W}$, we would have found different matching contributions for some of the coefficients:

$$\begin{aligned}
\Delta_{(\phi,W)} C_{P_{\rm L}^{1,1}} &= \Delta_{(\phi,W)} C_{P_{\rm L}^{1,3}} = \frac{2}{9} + \frac{2}{3} \ln \frac{\mu^2}{m_{\rm W}^2} \\
\Delta_{(\phi,W)} C_{P_{\rm L}^{1,2}} &= -\frac{7}{9} - \frac{4}{3} \ln \frac{\mu^2}{m_{\rm W}^2} \\
\Delta_{(\phi,W)} C_{P_{\rm L}^{2}} &= \frac{1}{Q_{\rm b}} \left(\frac{1}{2} + \frac{2}{3} \ln \frac{\mu^2}{m_{\rm W}^2} \right) \ .
\end{aligned} \tag{28}$$

But the coefficients of $\ln \mu^2$ are just the coefficients of those logarithms in eq. (77) that give the leading (divergent) contribution to the C_i in the limit of small quark masses. These logarithms are regenerated by the renormalization group running in the low-energy effective theory valid at scales $\mu < m_{\rm W}$ and therefore need not be discussed here any further.

We are now use free to add subleading terms in $1/x_{\rm tW}$, $1/x_{\rm tH}$ to the coefficients C_i. In order to see how this is accomplished, let us for the moment neglect the proper QCD corrections, so that we have to take into account only the entries in the anomalous dimension matrix given in (23). Solving the renormalization group equations (5), we find that only one coefficient runs below $m_{\rm t}$,

$$C_{P_{\rm L}^2}(\mu) = C_{P_{\rm L}^2}(m_{\rm t}) + \left(\frac{1}{6Q_{\rm b}} + \frac{1}{6Q_{\rm b}} \cot^2 \beta \right) \log \frac{\mu^2}{m_{\rm t}^2} \ , \tag{29}$$

where the first term in parentheses is due to the mixing of $R_{\rm L}^2$ into $P_{\rm L}^2$, and the second due to $R_{\rm L}^2{}'$. We see that the renormalization group reproduces the logarithmic terms already discussed in eq. (22), which would have been there had we done the matching at a different scale.

The subleading contributions are found by taking the standard one-loop result from integrating out both particles in the loop at the same scale (see appendix) and subtracting the leading contributions that we found from matching contributions (20,21) and running without QCD (29). We refer to this procedure for obtaining the subleading terms in the rest of the present work.

Let us for the moment assume that $m_{\rm H^\pm} > m_{\rm W}$. When we integrate out the charged Higgs at $\mu = m_{\rm H^\pm}$, we obtain the subleading contributions from

(21,29,78):

$$
\begin{aligned}
\Delta'_{(H)} C_{O^1_{\rm LR}} &= x_{\rm tH} F_4(x_{\rm tH}) - \frac{1}{2} \\
\Delta'_{(H)} C_{O^2_{\rm LR}} &= \frac{Q_{\rm b}}{Q_{\rm t}} \Delta'_{(H)} C_{O^3_{\rm LR}} = \frac{x_{\rm tH}}{2}(F_3(x_{\rm tH}) + F_4(x_{\rm tH})) - \frac{1}{2} \\
\Delta'_{(H)} C_{P^{1,1}_{\rm L}} &= \Delta'_{(H)} C_{P^{1,3}_{\rm L}} \\
&= \left(\frac{x_{\rm tH}}{3}\left(2F_2(x_{\rm tH}) + F_3(x_{\rm tH}) + 2F_4(x_{\rm tH})\right) - \frac{11}{18}\right)\cot^2\beta \\
\Delta'_{(H)} C_{P^{1,2}_{\rm L}} &= \left(\frac{2x_{\rm tH}}{3}\left(F_2(x_{\rm tH}) - F_3(x_{\rm tH}) - 2F_4(x_{\rm tH})\right) + \frac{8}{9}\right)\cot^2\beta \\
\Delta'_{(H)} C_{P^{1,4}_{\rm L}} &= -Q_{\rm b} \Delta'_{(H)} C_{P^4_{\rm L}} = \left(x_{\rm tH} F_4(x_{\rm tH}) - \frac{1}{2}\right)\cot^2\beta \qquad (30) \\
\Delta'_{(H)} C_{P^2_{\rm L}} &= \frac{1}{Q_{\rm b}}\left(x_{\rm tH}\left(\frac{1}{2}F_3(x_{\rm tH}) + F_4(x_{\rm tH})\right) - \frac{3}{4} - \frac{\ln x_{\rm tH}}{6(x_{\rm tH}-1)}\right)\cot^2\beta\ .
\end{aligned}
$$

The functions $F_i(x)$ are given in appendix A. One may easily verify that the terms on the r.h.s. are of order $O(1/x_{\rm tH})$, indicating that they are truly subleading. Especially there is no (leading) logarithmic dependence of the matching contributions to $C_{P^2_{\rm L}}$ on the mass ratio $x_{\rm tH}$, since all such dependencies must come from the renormalization group.

As mentioned in section two, the choice of scale for the subleading contributions is ambiguous. This ambiguity can only be resolved by computing the power corrections, which fortunately differ from our treatment by a next-to-leading contribution. We shall define our procedure by assuming that setting the scale equal to the mass of the lightest particle in the loop is a suitable choice.

After scaling down from $\mu = m_{\rm H^\pm}$ and adding in the leading matching contributions at $\mu = m_{\rm W}$, we will also consider the subleading contributions. Analogous to the previous case we find:

$$
\begin{aligned}
\Delta'_{(\phi,W)} C_{O^1_{\rm LR}} &= -x_{\rm tW} F_4(x_{\rm tW}) + \frac{1}{2} \\
\Delta'_{(\phi,W)} C_{O^2_{\rm LR}} &= \frac{Q_{\rm b}}{Q_{\rm t}} \Delta'_{(\phi,W)} C_{O^3_{\rm LR}} = -\frac{x_{\rm tW}}{2}\left(F_3(x_{\rm tW}) + F_4(x_{\rm tW}) + \frac{1}{2}\right) \\
\Delta'_{(\phi,W)} C_{P^{1,1}_{\rm L}} &= \Delta'_{(\phi,W)} C_{P^{1,3}_{\rm L}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x_{\rm tW}+2}{3}\left(2F_2(x_{\rm tW})+F_3(x_{\rm tW})+2F_4(x_{\rm tW})\right)-\frac{11}{18}\\
\Delta'_{(\phi,W)}C_{P_{\rm L}^{1,2}} &= \frac{2(x_{\rm tW}+2)}{3}\left(F_2(x_{\rm tW})-F_3(x_{\rm tW})-2F_4(x_{\rm tW})\right)+\frac{8}{9}\\
\Delta'_{(\phi,W)}C_{P_{\rm L}^{1,4}} &= (x_{\rm tW}-2)F_4(x_{\rm tW})-\frac{1}{2}\\
\Delta'_{(\phi,W)}C_{P_{\rm L}^{2}} &= \frac{1}{Q_{\rm b}}\left((x_{\rm tW}+2)\left(\frac{1}{2}F_3(x_{\rm tW})+F_4(x_{\rm tW})\right)-\frac{\ln x_{\rm tW}}{2(x-1)}-\frac{3}{4}\right)\\
\Delta'_{(\phi,W)}C_{P_{\rm L}^{4}} &= \frac{1}{Q_{\rm b}}\left(\frac{7}{2}-2x_{\rm tH}F_3(x_{\rm tH})-5x_{\rm tH}F_4(x_{\rm tH})\right)\ . \qquad (31)
\end{aligned}
$$

3.1.4 Reduction by equations of motion

In order to use the results from previous calculations for the running between the W and the b scale, we have to match our operator basis to the operator basis employed there. To this end, we use the equations of motions, as in ref. [13]. For the effective Hamiltonian just below the W scale, one then finds:

$$
\begin{aligned}
H_{\rm eff} &= \frac{4G_{\rm F}}{\sqrt{2}}K_{\rm tb}K^*_{\rm ts}\sum_i C_i(m_{\rm W}^-)O_i(m_{\rm W}^-) \qquad (32)\\
&\stackrel{\rm EOM}{\longrightarrow} \frac{4G_{\rm F}}{\sqrt{2}}K_{\rm tb}K^*_{\rm ts}\Bigg[\left(-\frac{1}{2}C_{O^1_{\rm LR}}+C_{O^2_{\rm LR}}-\frac{1}{2}C_{P_{\rm L}^{1,1}}-\frac{1}{4}C_{P_{\rm L}^{1,2}}+\frac{1}{4}C_{P_{\rm L}^{1,4}}\right)O^2_{\rm LR}\\
&\quad+\left(-\frac{1}{2}C_{O^1_{\rm LR}}+C_{O^3_{\rm LR}}-\frac{1}{2}C_{P_{\rm L}^{1,1}}-\frac{1}{4}C_{P_{\rm L}^{1,2}}+\frac{1}{4}C_{P_{\rm L}^{1,4}}-\frac{1}{4}C_{P_{\rm L}^{4}}\right)O^3_{\rm LR}\Bigg]\\
&\quad+\frac{g_3^2}{16\pi^2}\,(\text{four-fermion operators})\ .
\end{aligned}
$$

Since we are only interested in the leading contributions from the QCD corrections caused by a large mass splitting, we may drop the contributions to the four-fermion operators in (32) as these are suppressed by a factor $g_3^2/16\pi^2$ and therefore nonleading.

The standard four-fermion operators $(\bar{b}_L\gamma_\mu s_L)(\bar{q}_L\gamma^\mu q_L)$, $q=u,c$ (with the appropriate CKM mixing coefficients) resulting from integrating out the W have to be added to this expression. The Wilson coefficients obtained this way at the W scale may then be used as input for the renormalization group running down to the b scale [6, 9, 12].

3.2 $m_{\mathrm{H}^\pm} > m_{\mathrm{t}}$

If the charged Higgs is heavier than the top quark, the picture becomes more involved. As we run down from large scales, we first encounter the threshold of the charged Higgs. Therefore, as a first step, we integrate out the charged Higgs. In the same way as in the previous case, we shall now be mainly concerned with the leading contributions in the limit $x_{\mathrm{tH}} \ll 1$.

In the range $m_{\mathrm{H}^\pm} > \mu > m_{\mathrm{t}}$, after integrating out the charged Higgs, we have to deal with four-fermion operators of dimension 6 that involve a b, an s, and a quark-anti-quark pair. Besides the operators (13), our operator basis contains:

$$\begin{aligned}
S_1 &= (\bar{s}_{\mathrm{L}}^{\alpha}\gamma_\mu b_{\mathrm{L}}^{\alpha})(\bar{t}_{\mathrm{R}}^{\beta}\gamma^\mu t_{\mathrm{R}}^{\beta}) \\
S_2 &= (\bar{s}_{\mathrm{L}}^{\alpha}\gamma_\mu b_{\mathrm{L}}^{\beta})(\bar{t}_{\mathrm{R}}^{\beta}\gamma^\mu t_{\mathrm{R}}^{\alpha}) \\
S_3 &= (\bar{s}_{\mathrm{L}}^{\alpha}\gamma_\mu b_{\mathrm{L}}^{\alpha})\sum_q \left(\bar{q}_{\mathrm{L}}^{\beta}\gamma^\mu q_{\mathrm{L}}^{\beta}\right) \\
S_4 &= (\bar{s}_{\mathrm{L}}^{\alpha}\gamma_\mu b_{\mathrm{L}}^{\beta})\sum_q \left(\bar{q}_{\mathrm{L}}^{\beta}\gamma^\mu q_{\mathrm{L}}^{\alpha}\right) \\
S_5 &= (\bar{s}_{\mathrm{L}}^{\alpha}\gamma_\mu b_{\mathrm{L}}^{\alpha})\sum_q \left(\bar{q}_{\mathrm{R}}^{\beta}\gamma^\mu q_{\mathrm{R}}^{\beta}\right) \\
S_6 &= (\bar{s}_{\mathrm{L}}^{\alpha}\gamma_\mu b_{\mathrm{L}}^{\beta})\sum_q \left(\bar{q}_{\mathrm{R}}^{\beta}\gamma^\mu q_{\mathrm{R}}^{\alpha}\right) \\
S_7 &= \mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\frac{m_{\mathrm{b}}}{m_{\mathrm{t}}}(\bar{s}_{\mathrm{L}}^{\alpha}t_{\mathrm{R}}^{\beta})(\bar{t}_{\mathrm{L}}^{\beta}b_{\mathrm{R}}^{\alpha}) \\
S_8 &= \mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\frac{m_{\mathrm{b}}}{m_{\mathrm{t}}}(\bar{s}_{\mathrm{L}}^{\alpha}t_{\mathrm{R}}^{\alpha})(\bar{t}_{\mathrm{L}}^{\beta}b_{\mathrm{R}}^{\beta}) \\
S_9 &= \frac{1}{4}\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\frac{m_{\mathrm{b}}}{m_{\mathrm{t}}}(\bar{s}_{\mathrm{L}}^{\alpha}\sigma_{\mu\nu}t_{\mathrm{R}}^{\beta})(\bar{t}_{\mathrm{L}}^{\beta}\sigma^{\mu\nu}b_{\mathrm{R}}^{\alpha}) \\
S_{10} &= \frac{1}{4}\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\frac{m_{\mathrm{b}}}{m_{\mathrm{t}}}(\bar{s}_{\mathrm{L}}^{\alpha}\sigma_{\mu\nu}t_{\mathrm{R}}^{\alpha})(\bar{t}_{\mathrm{L}}^{\beta}\sigma^{\mu\nu}b_{\mathrm{R}}^{\beta}) \; .
\end{aligned} \tag{33}$$

Here α and β are color indices of the quarks, and the sums run over all active flavors. Again, the inclusion of the additional factors g_3^2 is motivated by the Gilman-Wise trick [34], as are the factors $m_{\mathrm{b}}/m_{\mathrm{t}}$ to keep the anomalous dimension matrices mass independent. The different normalization of $S_1 \ldots S_6$ and $S_7 \ldots S_{10}$ will be explained below.

Integrating out the charged Higgs at $\mu = m_{\mathrm{H}^\pm}$, we find at leading order in $x_{\mathrm{tH}} \equiv (m_{\mathrm{t}}/m_{\mathrm{H}^\pm})^2$:

$$\begin{aligned}
C_{O^1_{\mathrm{LR}}} &= \frac{1}{2} x_{\mathrm{tH}} \\
C_{O^2_{\mathrm{LR}}} &= \frac{Q_{\mathrm{b}}}{Q_{\mathrm{t}}} C_{O^3_{\mathrm{LR}}} = -\frac{1}{2} x_{\mathrm{tH}} \\
C_{P^{1,1}_{\mathrm{L}}} &= C_{P^{1,3}_{\mathrm{L}}} = -\frac{1}{9} x_{\mathrm{tH}} \cdot \cot^2\beta \\
C_{P^{1,2}_{\mathrm{L}}} &= \frac{7}{18} x_{\mathrm{tH}} \cdot \cot^2\beta \\
C_{P^{1,4}_{\mathrm{L}}} &= -Q_{\mathrm{b}} C_{P^4_{\mathrm{L}}} = \frac{1}{2} x_{\mathrm{tH}} \cdot \cot^2\beta \\
C_{P^2_{\mathrm{L}}} &= \frac{1}{Q_{\mathrm{b}}} \left(-\frac{1}{4} x_{\mathrm{tH}}\right) \cot^2\beta \\
C_{S_2} &= -\frac{1}{2} x_{\mathrm{tH}} \cot^2\beta \\
C_{S_8} &= \frac{16\pi^2}{g_3^2} x_{\mathrm{tH}} \\
C_{S_i} &= 0, \qquad i = 1, 3 \ldots 7, 9, 10 \ .
\end{aligned} \tag{34}$$

Let us start again with the mixing back of the operators S into the operators O and P. Because of the chirality structure of the operators, we find two different situations at one loop. The operators $S_1, \ldots S_6$ appear to have a zeroth order mixing (g_3^0) at one loop into the operators P.

$$\gamma^{(0)} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} P^{1,1}_{\mathrm{L}} & P^{1,2}_{\mathrm{L}} & P^{1,3}_{\mathrm{L}} & P^{1,4}_{\mathrm{L}} & P^2_{\mathrm{L}} & P^4_{\mathrm{L}} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 2\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} & 0 \\
\frac{2}{3} & -\frac{4}{3} & \frac{2}{3} & 0 & \frac{2}{3Q_{\mathrm{b}}} & 0 \\
\frac{4}{3} & -\frac{8}{3} & \frac{4}{3} & 0 & \frac{2}{Q_{\mathrm{b}}}\sum_q Q_q & 0 \\
\frac{2n_f}{3} & -\frac{4n_f}{3} & \frac{2n_f}{3} & 0 & \frac{2}{3}\left(6 + \frac{1}{Q_{\mathrm{b}}}\sum_q Q_q - n_f\right) & 0 \\
0 & 0 & 0 & 0 & \frac{2}{Q_{\mathrm{b}}}\sum_q Q_q & 0 \\
\frac{2n_f}{3} & -\frac{4n_f}{3} & \frac{2n_f}{3} & 0 & \frac{2}{3}\left(\frac{1}{Q_{\mathrm{b}}}\sum_q Q_q - n_f\right) & 0
\end{pmatrix}
\end{array} . \tag{35}$$

However, by inspecting the equations of motion (32) one sees that the back mixing vanishes at this order; therefore we may simply drop this contribution.

As is well known, one has to consider this mixing at two-loop order. The anomalous dimension matrix can be derived from eq. (25) of ref. [9], and reads in our normalization

$$\hat{\gamma} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} \begin{array}{c} \begin{array}{cc} O^2_{\mathrm{LR}} & O^3_{\mathrm{LR}} \end{array} \\ \begin{pmatrix} -\frac{3}{2} & 0 \\ -\frac{119}{54} & \frac{224}{27} \\ \frac{70}{27}+\frac{3}{2}n_f & \frac{232}{27} \\ 3+\frac{35}{27}n_f & \frac{8}{27}n_f+4\bar{n}_f \\ -\frac{7}{3}-\frac{3}{2}n_f & -\frac{16}{3} \\ -2-\frac{119}{54}n_f & \frac{8}{27}n_f-4\bar{n}_f \end{pmatrix} \end{array} . \tag{36}$$

Here $n_f = n_u + n_d$ is the number of active flavors, and $\bar{n}_f = n_d + (Q_u/Q_d)n_u$.

On the other hand, the mixing of $S_7 \ldots S_{10}$ into the operators O does not vanish at one loop:

$$\hat{\gamma} = \begin{array}{c} \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{array} \begin{array}{c} \begin{array}{ccc} O^1_{\mathrm{LR}} & O^2_{\mathrm{LR}} & O^3_{\mathrm{LR}} \end{array} \\ \begin{pmatrix} 0 & 0 & -\frac{3}{2}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} \\ 0 & -\frac{1}{2} & -\frac{1}{2}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} \\ 0 & 0 & \frac{3}{2}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} \\ 0 & \frac{1}{2} & \frac{1}{2}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} \end{pmatrix} \end{array} . \tag{37}$$

Again one may verify that these entries of the ADM are consistent with the $\ln \mu$ dependence of the matching contributions (34).

Let us now turn to the mixing among the four-fermion operators. Since the considered operators are all of dimension $d+2$, and because the QCD interactions preserve chirality, the operators $S_{1,\ldots,6}$ and the operators $S_{7,\ldots,10}$ will mix only among themselves, respectively.

The one-loop mixing among the $S_1 \ldots S_6$ is well known [34]:

$$\hat{\gamma} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} \begin{array}{c} \begin{array}{cccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \end{array} \\ \begin{pmatrix} 1 & -3 & 0 & 0 & 0 & 0 \\ 0 & -8 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} \\ 0 & 0 & -\frac{11}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} \\ 0 & 0 & 3-\frac{n_f}{9} & \frac{n_f}{3}-1 & -\frac{n_f}{9} & \frac{n_f}{3} \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & -\frac{n_f}{9} & \frac{n_f}{3} & -\frac{n_f}{9} & \frac{n_f}{3}-8 \end{pmatrix} \end{array} . \tag{38}$$

For the mixing of $S_7 \dots S_{10}$ we find:

$$\hat{\gamma} = \begin{array}{c} \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{array} \begin{array}{c} \begin{array}{cccc} S_7 & S_8 & S_9 & S_{10} \end{array} \\ \begin{pmatrix} 1-2b & -3 & -\frac{7}{3} & -1 \\ 0 & -8-2b & -2 & \frac{2}{3} \\ -7 & -3 & -\frac{19}{3}-2b & 0 \\ -6 & 2 & 0 & \frac{8}{3}-2b \end{pmatrix} \end{array} . \tag{39}$$

Since the operators O_{LR} are dimension $d+1$, there is no mixing back into $S_{1,\dots,10}$.

Note that with our chosen normalization of the operators as given in (33) all relevant mixing occurs at order g_3^2, and all entries in the anomalous dimension matrix are dimensionless.

After running down to $\mu = m_{\mathrm{t}}$, we integrate out the t quark. The operators $S_1, S_2, S_{7,\dots,10}$ are removed, since they do not contribute to the matching; for the operators $S_{3,\dots,6}$ the t quark is excluded from the sum, because it is inactive for $\mu < m_{\mathrm{t}}$. Again we will take into account the subleading terms in x_{tH} according to the general prescription given in section 3.1.3. Then we will continue as in the case for the Standard Model with a heavy top, except that the coefficients $C_i(m_{\mathrm{t}}^+)$ are now nonvanishing.

4 Supersymmetric contributions

4.1 Flavor-changing chargino interactions

Let us denote by $\tilde{W}^{\pm}$, $\tilde{H}_1^-$ and $\tilde{H}_2^+$ (analogous to [14]) the superpartners of the W and the charged components of the Higgs fields, respectively. Define the two component spinors $\psi_j^{\pm}$ by

$$\psi_j^+ = \left(-i\tilde{W}^+, \tilde{H}_2^+\right), \quad \psi_j^- = \left(-i\tilde{W}^-, \tilde{H}_1^-\right), \qquad j = 1,2 \; . \tag{40}$$

The mass term for the W-inos and higgsinos then takes the following form:

$$L_{\mathcal{M}} = -\psi^- \mathcal{M} \psi^+ + \text{h.c.} \; , \tag{41}$$

where the mass matrix is given by

$$\mathcal{M} = \begin{pmatrix} M_2 & \sqrt{2} m_{\rm W} \sin\beta \\ \sqrt{2} m_{\rm W} \cos\beta & \mu_{\rm h} \end{pmatrix} \ , \tag{42}$$

where M_2 is the soft SUSY breaking mass term for the W-inos at the weak scale, and $\mu_{\rm h}$ is the renormalized Higgs mixing parameter.

This mass matrix may be diagonalized with the help of the two unitary matrices U, V such that

$$U^* \mathcal{M} V^\dagger = \mathcal{M}_\chi = {\rm diag}(\tilde{m}_1, \tilde{m}_2) \tag{43}$$

is a diagonal matrix with nonnegative entries. The corresponding charged mass-eigenstate 4-spinors are the charginos

$$\chi_i^+ = \begin{pmatrix} V_{ij} \psi_j^+ \\ U_{ij}^* \bar{\psi}_j^- \end{pmatrix} . \tag{44}$$

We shall find it more convenient to rewrite the interactions of the charginos by their charge conjugates

$$\chi_i^- \equiv \left(\chi_i^+\right)^c = C(\bar{\chi}_i^+)^T = \begin{pmatrix} U_{ij} \psi_j^- \\ V_{ij}^* \bar{\psi}_j^+ \end{pmatrix} , \tag{45}$$

so that when we refer to charginos below, we mean the χ_i^- given in (45).

Let us apply these definitions to the interactions of the charged gauginos and higgsinos and convert to 4-spinor notation. Ignoring for the moment the mixing of quarks and of squarks and concentrating on the terms involving b quarks, the relevant Lagrangian for chargino-quark-squark interactions reads:

$$L_{\chi b \tilde{t}} = -g_2 V_{i1}^* \tilde{t}_{\rm L}^\dagger (\bar{\chi}_i P_{\rm L} b) + g_2 \lambda_{\rm t} V_{i2}^* \tilde{t}_{\rm R}^\dagger (\bar{\chi}_i P_{\rm L} b) + g_2 \lambda_{\rm b} U_{i2} \tilde{t}_{\rm L}^\dagger (\bar{\chi}_i P_{\rm R} b) + {\rm h.c.} \ , \tag{46}$$

and the couplings λ_q are proportional to the Yukawa couplings:

$$\lambda_{\rm t} = \frac{m_{\rm t}}{\sqrt{2} m_{\rm W} \sin\beta} \quad , \quad \lambda_{\rm b} = \frac{m_{\rm b}}{\sqrt{2} m_{\rm W} \cos\beta} \ . \tag{47}$$

A similar expression is found for the interactions of the charginos with the quarks and squarks of the second family. In this case one can neglect the terms proportional to $\lambda_{\rm c}$, $\lambda_{\rm s}$, which originate in the coupling of the higgsino components of the charginos to the quark and squark fields.

Since the Yukawa couplings of the matter fields to the Higgs fields are not flavor diagonal in a weak interaction basis, we have to take into account the mixing among quarks and among squarks. Let us denote by $\tilde{q}_{l{\rm L,R}}$ as in ref. [17] the squark current eigenstates (where $q = u, d$, and $l = 1, 2, 3$ is the generation label), and $\tilde{q}_a$ $(a = 1, \ldots, 6)$ the corresponding mass eigenstates with masses $\tilde{m}_a$. We define the 6×3 squark mixing matrices $\Gamma_{Q{\rm L,R}}$ by

$$\tilde{q}_{\rm L,R} = \Gamma^{\dagger}_{Q{\rm L,R}} \tilde{q} \; . \tag{48}$$

The relevant chargino interactions involving down-type quarks may then be written as

$$L_{\chi d\tilde{u}} = -g_2 \sum_{j,a,l} \left[\tilde{u}^{\dagger}_a \, \bar{\chi}_j \left(G^{jal} P_{\rm L} - H^{jal} P_{\rm R} \right) d_l \right] + {\rm h.c.} \; , \tag{49}$$

where

$$\begin{aligned} G^{jal} &= V^*_{j1} \Gamma^{al}_{U{\rm L}} - V^*_{j2} (\Gamma_{U{\rm R}} \Lambda_U K)^{al} \\ H^{jal} &= U_{j2} (\Gamma_{U{\rm L}} \Lambda_D)^{al} \; . \end{aligned} \tag{50}$$

Here $\Lambda_U = M_U / (\sqrt{2} m_{\rm W} \sin\beta)$ and $\Lambda_D = M_D / (\sqrt{2} m_{\rm W} \cos\beta)$ are proportional to the Yukawa coupling matrices for up- and down-type quarks, respectively. Note that we neglect the masses of the light quarks, and therefore set the Yukawa couplings of the light quarks to zero.

Since we are interested in the $b \to s\gamma$ transition, we find it convenient to define

$$\mathcal{G}^{jal} = \frac{G^{jal}}{K_{{\rm t}l}} \quad \text{for } l = b, s; \qquad \mathcal{H}^{ja{\rm b}} = \frac{H^{ja{\rm b}}}{K_{\rm tb}} \; . \tag{51}$$

Unitarity of the mixing (48) implies that

$$\sum_{a=1}^{6} \Gamma^{ai}_{Q{\rm L,R}} \Gamma^{*ak}_{Q{\rm L,R}} = \delta_{ik} \quad , \quad \sum_{a=1}^{6} \Gamma^{ai}_{Q{\rm L,R}} \Gamma^{*ak}_{Q{\rm R,L}} = 0 \; , \tag{52}$$

and therefore

$$\sum_{a=1}^{6} \mathcal{G}^{*jas} \mathcal{G}^{ja{\rm b}} = \lambda_{\rm t}^2 |V_{j2}|^2 \quad , \quad \sum_{a=1}^{6} \mathcal{G}^{*jas} \mathcal{H}^{ja{\rm b}} = 0 \; . \tag{53}$$

After having described our conventions, let us now turn to the evaluation of the QCD correction. As the squarks and the charginos can have large mass splittings, the procedure of matching and running becomes more involved but still remains straightforward. We will give all ingredients, but the precise procedure will depend on the details of the spectrum.

4.2 Operators from heavy squarks

If we encounter the threshold of an up-type squark $\tilde{u}_a$ in the process of running down, we will integrate it out. This generates effective four-fermion operators made out of the quarks b, s, and the active charginos χ_j. We extend our operator basis by the following operators (no sum over j):

$$\begin{aligned}
W_{\mathrm{LR}}^{1,j} &= \mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\frac{m_{\mathrm{b}}}{\tilde{m}_j}(\bar{s}_{\mathrm{L}}b_{\mathrm{R}})(\bar{\chi}_{\mathrm{L}}^j\chi_{\mathrm{R}}^j) \\
W_{\mathrm{LR}}^{2,j} &= \frac{1}{4}\mu^{2\epsilon}\frac{g_3^2}{16\pi^2}\frac{m_{\mathrm{b}}}{\tilde{m}_j}(\bar{s}_{\mathrm{L}}\sigma^{\mu\nu}b_{\mathrm{R}})(\bar{\chi}_{\mathrm{L}}^j\sigma_{\mu\nu}\chi_{\mathrm{R}}^j) \\
W_{\mathrm{L}}^{j} &= \mu^{2\epsilon}\frac{g_3^2}{16\pi^2}(\bar{s}_{\mathrm{L}}\gamma^{\mu}b_{\mathrm{L}})(\bar{\chi}_{\mathrm{R}}^j\gamma_{\mu}\chi_{\mathrm{R}}^j)\ .
\end{aligned} \tag{54}$$

The matching contributions at $\mu = \tilde{m}_a$ for $\tilde{m}_a \gg \tilde{m}_j$ are:

$$\begin{aligned}
\Delta_{(\tilde{u}_a)}C_{O_{\mathrm{LR}}^1} &= \mathcal{G}^{*jas}\mathcal{H}^{ja\mathrm{b}}\cdot\frac{\tilde{m}_j}{m_{\mathrm{b}}}\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2\cdot(-1) \\
\Delta_{(\tilde{u}_a)}C_{O_{\mathrm{LR}}^2} &= 0 \\
\Delta_{(\tilde{u}_a)}C_{O_{\mathrm{LR}}^3} &= \mathcal{G}^{*jas}\mathcal{H}^{ja\mathrm{b}}\cdot\frac{\tilde{m}_j}{m_{\mathrm{b}}}\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2\cdot\frac{(-1)}{Q_{\mathrm{b}}} \\
\Delta_{(\tilde{u}_a)}C_{P_{\mathrm{L}}^{1,1}} &= \Delta_{(\tilde{u}_a)}C_{P_{\mathrm{L}}^{1,3}} = \mathcal{G}^{*jas}\mathcal{G}^{ja\mathrm{b}}\cdot\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2\cdot\left(\frac{5}{18}\right) \\
\Delta_{(\tilde{u}_a)}C_{P_{\mathrm{L}}^{1,2}} &= \mathcal{G}^{*jas}\mathcal{G}^{ja\mathrm{b}}\cdot\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2\cdot\left(-\frac{2}{9}\right) \\
\Delta_{(\tilde{u}_a)}C_{P_{\mathrm{L}}^{1,4}} &= 0 \\
\Delta_{(\tilde{u}_a)}C_{P_{\mathrm{L}}^{2}} &= \mathcal{G}^{*jas}\mathcal{G}^{ja\mathrm{b}}\cdot\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2\cdot\frac{1}{2Q_{\mathrm{b}}} \\
\Delta_{(\tilde{u}_a)}C_{P_{\mathrm{L}}^{4}} &= \mathcal{G}^{*jas}\mathcal{G}^{ja\mathrm{b}}\cdot\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2\cdot\frac{1}{Q_{\mathrm{b}}}
\end{aligned}$$

$$\begin{aligned}
\Delta_{(\tilde{u}_a)} C_{W^{1,j}_{\mathrm{LR}}} &= \Delta_{(\tilde{u}_a)} C_{W^{2,j}_{\mathrm{LR}}} = \mathcal{G}^{*jas}\mathcal{H}^{jab} \cdot \frac{16\pi^2}{g_3^2}\frac{\tilde{m}_j}{m_{\mathrm{b}}}\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2 \\
\Delta_{(\tilde{u}_a)} C_{W^{j}_{\mathrm{L}}} &= \mathcal{G}^{*jas}\mathcal{G}^{jab} \cdot \frac{16\pi^2}{g_3^2}\left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2 \cdot (-1)\ . \qquad (55)
\end{aligned}$$

The mixing back of these operators into the O's and P's is found to be:

$$\hat{\gamma} = \begin{array}{c} \\ W^{1,j}_{\mathrm{LR}} \\ W^{2,j}_{\mathrm{LR}} \\ W^{j}_{\mathrm{L}} \end{array} \begin{array}{c} \begin{array}{ccccc} O^{1,2}_{\mathrm{LR}} & O^{3}_{\mathrm{LR}} & P^{1,A}_{\mathrm{L}} & P^{2}_{\mathrm{L}} & P^{4}_{\mathrm{L}} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1/Q_{\mathrm{b}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2/3Q_{\mathrm{b}} & 0 \end{pmatrix} \end{array} . \qquad (56)$$

Since the charginos carry no color charge, the renormalization of these operators is particularly simple,

$$\hat{\gamma} = \begin{array}{c} \\ W^{1,j}_{\mathrm{LR}} \\ W^{2,j}_{\mathrm{LR}} \\ W^{j}_{\mathrm{L}} \end{array} \begin{array}{c} \begin{array}{ccc} W^{1,j}_{\mathrm{LR}} & W^{2,j}_{\mathrm{LR}} & W^{j}_{\mathrm{L}} \end{array} \\ \begin{pmatrix} -2b & 0 & 0 \\ 0 & \frac{16}{3} - 2b & 0 \\ 0 & 0 & -2b \end{pmatrix} \end{array} , \qquad (57)$$

and there is no mixing of the O and P operators back into these.

If we cross the threshold of chargino χ_j at $\mu = \tilde{m}_j$, the operators W^j will be removed; they do not give any matching contribution to leading order.

4.3 Operators from heavy charginos

Let us now consider the case that we encounter the threshold of chargino χ_j at $\mu = \tilde{m}_j$. If there are still active up-type squarks $\tilde{u}_a$, we have to extend our operator basis by the two-quark two-squark operators (no sum over a):

Dimension $d+1$:

$$\begin{aligned}
\tilde{Q}^{1,a}_{\mathrm{LR}} &= m_{\mathrm{b}}\, \tilde{u}^{\dagger\beta}_a \tilde{u}^{\alpha}_a\ \bar{s}^{\alpha}_{\mathrm{L}} b^{\beta}_{\mathrm{R}} \\
\tilde{Q}^{2,a}_{\mathrm{LR}} &= m_{\mathrm{b}}\, \tilde{u}^{\dagger\beta}_a \tilde{u}^{\beta}_a\ \bar{s}^{\alpha}_{\mathrm{L}} b^{\alpha}_{\mathrm{R}}\ .
\end{aligned}$$

Dimension $d+2$:

$$\begin{aligned}
\tilde{R}^{1,a}_{\mathrm{L}} &= i\tilde{u}^{\dagger\beta}_a \tilde{u}^{\alpha}_a\, (\bar{s}_{\mathrm{L}} \not{D} b_{\mathrm{L}})^{\alpha\beta} \\
\tilde{R}^{2,a}_{\mathrm{L}} &= i\tilde{u}^{\dagger\beta}_a (D^{\mu}\tilde{u}_a)^{\alpha}\ \bar{s}^{\alpha}_{\mathrm{L}} \gamma_{\mu} b^{\beta}_{\mathrm{L}}
\end{aligned}$$

$$
\begin{aligned}
\tilde{R}_{\rm L}^{3,a} &= i(D^\mu \tilde{u}_a)^{\dagger\beta}\tilde{u}_a^\alpha \; \bar{s}_{\rm L}^\alpha\gamma_\mu b_{\rm L}^\beta \\
\tilde{R}_{\rm L}^{4,a} &= i\tilde{u}_a^{\dagger\beta}\tilde{u}_a^\beta \; \mathrm{Tr}\;(\bar{s}_{\rm L}\not{D} b_{\rm L}) \\
\tilde{R}_{\rm L}^{5,a} &= i\tilde{u}_a^{\dagger\beta}(D^\mu \tilde{u}_a)^\beta \; \bar{s}_{\rm L}^\alpha\gamma_\mu b_{\rm L}^\alpha \\
\tilde{R}_{\rm L}^{6,a} &= i(D^\mu \tilde{u}_a)^{\dagger\beta}\tilde{u}_a^\beta \; \bar{s}_{\rm L}^\alpha\gamma_\mu b_{\rm L}^\alpha \; .
\end{aligned}
\tag{58}
$$

For a running over each active up-type squark we find the following leading matching contributions at $\mu = \tilde{m}_j$ for $\tilde{m}_j \gg \tilde{m}_a$:

$$
\begin{aligned}
\Delta_{(\chi_j)} C_{O^1_{\rm LR}} &= \mathcal{G}^{*jas}\mathcal{H}^{jab} \cdot \frac{\tilde{m}_j}{m_{\rm b}} \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot (-1) \\
\Delta_{(\chi_j)} C_{O^2_{\rm LR}} &= 0 \\
\Delta_{(\chi_j)} C_{O^3_{\rm LR}} &= \mathcal{G}^{*jas}\mathcal{H}^{jab} \cdot \frac{\tilde{m}_j}{m_{\rm b}} \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot \frac{1}{Q_{\rm b}} \\
\Delta_{(\chi_j)} C_{P^{1,1}_{\rm L}} &= \Delta_{(\chi_j)} C_{P^{1,3}_{\rm L}} = \mathcal{G}^{*jas}\mathcal{G}^{jab} \cdot \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot \left(-\frac{5}{18}\right) \\
\Delta_{(\chi_j)} C_{P^{1,2}_{\rm L}} &= \mathcal{G}^{*jas}\mathcal{G}^{jab} \cdot \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot \left(\frac{11}{9}\right) \\
\Delta_{(\chi_j)} C_{P^{1,4}_{\rm L}} &= 0 \\
\Delta_{(\chi_j)} C_{P^{2}_{\rm L}} &= \mathcal{G}^{*jas}\mathcal{G}^{jab} \cdot \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot \left(-\frac{3}{2}\right)\frac{1}{Q_{\rm b}} \\
\Delta_{(\chi_j)} C_{P^{4}_{\rm L}} &= \mathcal{G}^{*jas}\mathcal{G}^{jab} \cdot \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot \frac{1}{Q_{\rm b}} \\
\Delta_{(\chi_j)} C_{\tilde{Q}^{1,a}_{\rm LR}} &= \mathcal{G}^{*jas}\mathcal{H}^{jab} \cdot \frac{\tilde{m}_j}{m_{\rm b}} \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot (-2) \\
\Delta_{(\chi_j)} C_{\tilde{R}^{1,a}_{\rm L}} &= \Delta_{(\chi_j)} C_{\tilde{R}^{2,a}_{\rm L}} = \mathcal{G}^{*jas}\mathcal{G}^{jab} \cdot \left(\frac{m_{\rm W}}{\tilde{m}_j}\right)^2 \cdot 2 \\
\Delta_{(\chi_j)} C_{\tilde{Q}^{2,a}_{\rm LR}} &= \Delta_{(\chi_j)} C_{\tilde{R}^{n,a}_{\rm L}} = 0 \qquad n = 3,4,5,6 \; .
\end{aligned}
\tag{59}
$$

A straightforward calculation for the back-mixing at one-loop order (but order α_3^0 in our chosen normalization) gives

$$\gamma^{(0)} = \begin{array}{c} \\ \tilde{Q}^{1,a}_{\mathrm{LR}} \\ \tilde{Q}^{2,a}_{\mathrm{LR}} \\ \tilde{R}^{1,a}_{\mathrm{L}} \\ \tilde{R}^{2,a}_{\mathrm{L}} \\ \tilde{R}^{3,a}_{\mathrm{L}} \\ \tilde{R}^{4,a}_{\mathrm{L}} \\ \tilde{R}^{5,a}_{\mathrm{L}} \\ \tilde{R}^{6,a}_{\mathrm{L}} \end{array} \begin{array}{c} \begin{array}{ccccccc} O_{\mathrm{LR}} & P^{1,1}_{\mathrm{L}} & P^{1,2}_{\mathrm{L}} & P^{1,3}_{\mathrm{L}} & P^{1,4}_{\mathrm{L}} & P^{2}_{\mathrm{L}} & P^{4}_{\mathrm{L}} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6Q_{\mathrm{b}}} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 & -\frac{1}{6Q_{\mathrm{b}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} & 0 \end{pmatrix} \end{array} . \qquad (60)$$

Again one sees that, similarly to the case of the four-quark operators, the mixing into the magnetic moment operators vanishes after applying the equations of motion. Therefore we have to consider this mixing at two-loop order.

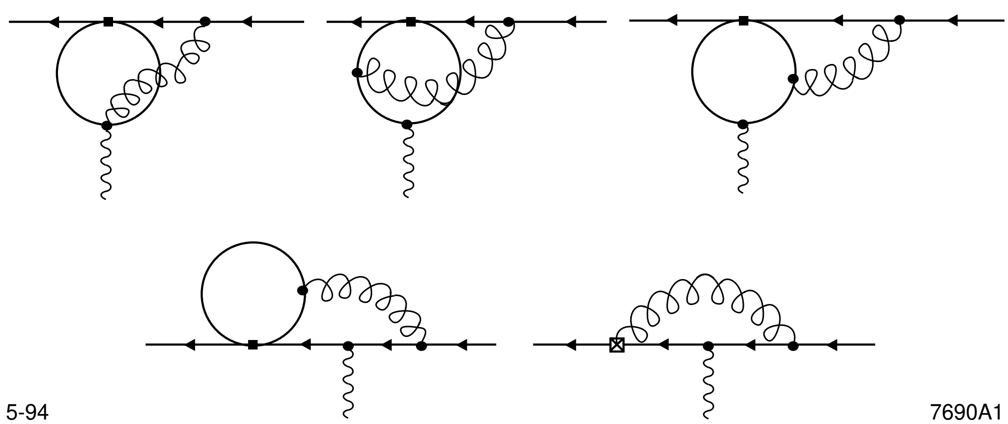

Figure 1: Feynman diagrams contributing to the mixing of the two-quark two-squark operators (58) into the operator O^3_{LR}. A full square denotes insertion of a two-quark two-squark operator, while an open square denotes a one-loop counterterm. Diagrams related by reflection to the ones above are not shown.

The actual two-loop calculation for mixing the two-quark two-squark operators (58) into the magnetic moment operators is performed analogously to the corresponding calculation for insertions of four-quark operators (see e.g. [4]). In figure 1 we show the relevant diagrams and one-loop counterterms contributing to mixing of the two-quark two-squark operators into the operator $O^3_{\rm LR}$. As we prefer to work off-shell, we have to consider only 1-PI diagrams. The main advantage is a simplification of the extraction of the divergent parts of interest by focussing on the coefficients of the tensor structures that are defined by our basis (13).

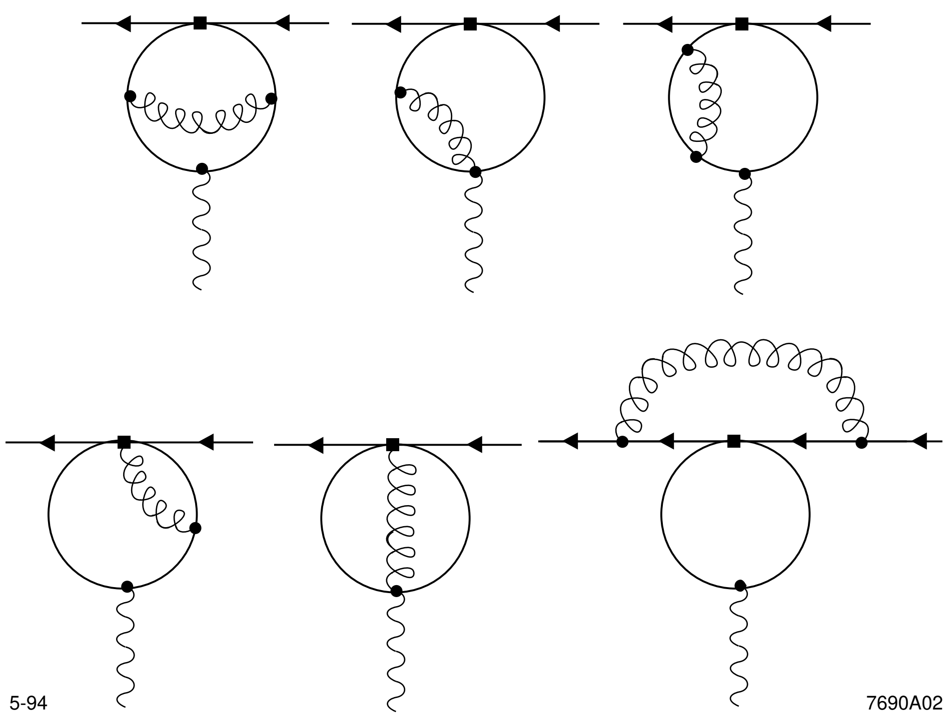

Figure 2: Feynman diagrams whose contribution to the mixing vanishes after application of the equations of motion.

Using the equations of motion (32) greatly reduces the computational effort, similar to the corresponding calculations with insertions of four-fermion operators. Figure 2 shows typical diagrams that do not contribute because their sum can be shown to be proportional to $(\gamma_\mu q^2 - q_\mu \not{q})$, and therefore need not be calculated.

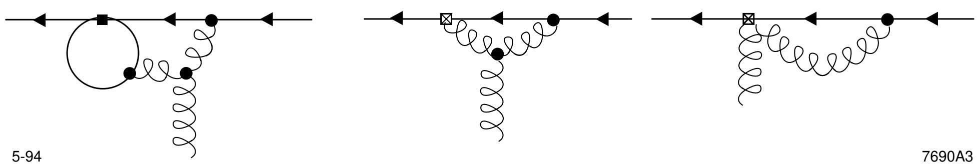

Figure 3: Additional Feynman diagrams that contribute to the mixing of the two-quark two-squark operators into the operator O^2_{LR}.

In the case of mixing into O^2_{LR}, we have to consider the additional diagrams and counterterms shown in figure 3 due to the non-Abelian interactions of the gluons.

We obtained the following mixing coefficients ($N = 3$):

$$\hat{\gamma} = \begin{array}{cc} & \begin{array}{cc} O^2_{\mathrm{LR}} & O^3_{\mathrm{LR}} \end{array} \\ \begin{array}{c} \tilde{Q}^{1,a}_{\mathrm{LR}} \\ \tilde{Q}^{2,a}_{\mathrm{LR}} \\ \tilde{R}^{1,a}_{\mathrm{L}} \\ \tilde{R}^{2,a}_{\mathrm{L}} \\ \tilde{R}^{3,a}_{\mathrm{L}} \\ \tilde{R}^{4,a}_{\mathrm{L}} \\ \tilde{R}^{5,a}_{\mathrm{L}} \\ \tilde{R}^{6,a}_{\mathrm{L}} \end{array} & \begin{pmatrix} \frac{N^2-2}{8N} & \frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}}\frac{N^2-1}{4N} \\ \frac{1}{4} & 0 \\ \frac{N^2-2}{8N} & \frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}}\frac{N^2-1}{4N} \\ -\frac{N^2-2}{16N} - \frac{N^2+2}{72N} & \left(-\frac{1}{4}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} + \frac{1}{18}\right)\frac{N^2-1}{2N} \\ -\frac{N^2-2}{16N} + \frac{N^2+2}{72N} & \left(-\frac{1}{4}\frac{Q_{\mathrm{t}}}{Q_{\mathrm{b}}} - \frac{1}{18}\right)\frac{N^2-1}{2N} \\ \frac{1}{4} & 0 \\ -\frac{1}{8} & 0 \\ -\frac{1}{8} & 0 \end{pmatrix} \end{array} . \tag{61}$$

In addition we need the mixing among the two-quark two-squark operators, where the squarks are of the same kind,

$$\hat{\gamma} = \begin{array}{cc} & \begin{array}{cccccccc} \tilde{Q}^{1,a}_{\mathrm{LR}} & \tilde{Q}^{2,a}_{\mathrm{LR}} & \tilde{R}^{1,a}_{\mathrm{L}} & \tilde{R}^{2,a}_{\mathrm{L}} & \tilde{R}^{3,a}_{\mathrm{L}} & \tilde{R}^{4,a}_{\mathrm{L}} & \tilde{R}^{5,a}_{\mathrm{L}} & \tilde{R}^{6,a}_{\mathrm{L}} \end{array} \\ \begin{array}{c} \tilde{Q}^{1,a}_{\mathrm{LR}} \\ \tilde{Q}^{2,a}_{\mathrm{LR}} \\ \tilde{R}^{1,a}_{\mathrm{L}} \\ \tilde{R}^{2,a}_{\mathrm{L}} \\ \tilde{R}^{3,a}_{\mathrm{L}} \\ \tilde{R}^{4,a}_{\mathrm{L}} \\ \tilde{R}^{5,a}_{\mathrm{L}} \\ \tilde{R}^{6,a}_{\mathrm{L}} \end{array} & \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{2} & -\frac{3}{2} & -4 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{25}{6} & \frac{1}{6} & 0 & \frac{13}{18} & -\frac{13}{18} \\ -4 & 0 & 0 & \frac{1}{6} & -\frac{25}{6} & 0 & -\frac{13}{18} & \frac{13}{18} \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -2 & -2 \\ -\frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -2 & -2 \end{pmatrix} \end{array} . \tag{62}$$

and for different types of squarks ($a \neq b$):

$$\hat{\gamma} = \begin{array}{c c} & \begin{array}{cccc} \tilde{R}_{\mathrm{L}}^{2,b} & \tilde{R}_{\mathrm{L}}^{3,b} & \tilde{R}_{\mathrm{L}}^{5,b} & \tilde{R}_{\mathrm{L}}^{6,b} \end{array} \\ \begin{array}{c} \tilde{R}_{\mathrm{L}}^{2,a} \\ \tilde{R}_{\mathrm{L}}^{3,a} \end{array} & \begin{pmatrix} \frac{1}{12} & -\frac{1}{12} & -\frac{1}{36} & \frac{1}{36} \\ -\frac{1}{12} & \frac{1}{12} & \frac{1}{36} & -\frac{1}{36} \end{pmatrix} \end{array} . \tag{63}$$

In addition we find a mixing of some of the two-quark two-squark operators into four-fermion operators:

$$\hat{\gamma} = \begin{array}{c c} & \begin{array}{cccc} S_3 & S_4 & S_5 & S_6 \end{array} \\ \begin{array}{c} \tilde{R}_{\mathrm{L}}^{2,a} \\ \tilde{R}_{\mathrm{L}}^{3,a} \end{array} & \begin{pmatrix} -\frac{1}{36} & \frac{1}{12} & -\frac{1}{36} & \frac{1}{12} \\ \frac{1}{36} & -\frac{1}{12} & \frac{1}{36} & -\frac{1}{12} \end{pmatrix} \end{array} . \tag{64}$$

If there are squarks lighter than the top quark, we also have to take into account the mixing of the four-fermion operators (33) into the operators $\tilde{R}$:

$$\hat{\gamma} = \begin{array}{c c} & \begin{array}{cccc} \tilde{R}_{\mathrm{L}}^{2,a} & \tilde{R}_{\mathrm{L}}^{3,a} & \tilde{R}_{\mathrm{L}}^{5,a} & \tilde{R}_{\mathrm{L}}^{6,a} \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{9} & \frac{1}{9} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{2}{9} & \frac{2}{9} \\ \frac{n_f}{3} & -\frac{n_f}{3} & -\frac{n_f}{9} & \frac{n_f}{9} \\ 0 & 0 & 0 & 0 \\ \frac{n_f}{3} & -\frac{n_f}{3} & -\frac{n_f}{9} & \frac{n_f}{9} \end{pmatrix} \end{array} . \tag{65}$$

In principle there is also a QCD-induced mixing into operators with two quarks and two down-type squarks, which we also would have to include if we were considering the contributions induced by gluinos and neutralinos. In most scenarios, the mass splitting of down-type squarks is much smaller than for up-type squarks. For the supersymmetric contributions to be numerically relevant the lightest squark (which is usually the lightest stop) must be significantly lighter than the other squarks. As has been argued in the introduction, contributions from these operators are strongly suppressed, and inclusion of these operators into the mixing would lead to only a minor effect compared to other neglected corrections. Furthermore, all Wilson coefficients that contribute to mixing via (63,64) are proportional to $\cot^2\beta$ and therefore suppressed in the large-$\tan\beta$ limit.

Again, if we cross the threshold $\mu = \tilde{m}_a$ of squark $\tilde{u}_a$, the matching contribution vanishes to leading order, and the operators $\tilde{O}^a$ and $\tilde{R}^a$ are simply removed. We will also add the corresponding subleading contributions each time a pair (a, j) of squarks and charginos has been integrated out, i.e. at $\mu = \min(\tilde{m}_a, \tilde{m}_j)$.

5 Results and Discussions

As the full anomalous dimension matrix is quite large and changes its structure every time we cross a threshold, it would be a major effort to diagonalize the anomalous dimension matrix in every step. It is much simpler to directly evaluate the solution (12) of the RGE numerically. Before we proceed, let us comment on some simplifications that result from the use of the equations of motion, since we are eventually only interested in the coefficient of the magnetic moment operators at the b scale.

First we note that the operators Q_{LR} and R_{L}^{1} (and in the case of $m_{\mathrm{H}^\pm} < m_{\mathrm{t}}$ their primed counterparts) which appear in intermediate stages of the calculations turn out to be superfluous, as they do not give any contribution in the process of matching, nor do they mix into any other operator. Second, although the coefficient of P_{L}^{2} does get matching contributions and many operators mix into it, it can be ignored, since it vanishes after applying the equations of motion. Third, the operators R_{L}^{2} and R_{L}^{3} mix only into P_{L}^{2}, which vanishes by equations of motion, and may therefore be omitted from the beginning. Extending this reasoning to $R_{\mathrm{L}}^{2\prime}$, $R_{\mathrm{L}}^{3\prime}$, $W_{\mathrm{LR}}^{1,j}$, W_{L}^{j} and $\tilde{R}_{\mathrm{L}}^{4,a}$ shows that they may also be disregarded.

Next, one may convince oneself that the apparent zeroth-order mixing of some operators [see eqs. (35), (60)] vanishes after application of the equations of motion, so all mixing occurs at order $(g_3^2/8\pi^2)$, as promised.

Let us first rediscuss the effect of the QCD corrections to the Standard Model contribution. For the contribution from the W-t loop there is a QCD enhancement of the coefficients $C^{\mathrm{eff}}_{O^2_{\mathrm{LR}}}(m_{\mathrm{W}})$ and $C^{\mathrm{eff}}_{O^3_{\mathrm{LR}}}(m_{\mathrm{W}})$ of the order of 10–18% and 15–22% for $m_{\mathrm{t}} = 130 \ldots 250\,\mathrm{GeV}$, respectively, which after scaling down to $\mu = m_{\mathrm{b}}$ and including the contribution from the four-fermion operators leads to an additional enhancement of the decay rate within the SM of the order of 12–23% [13], compared to the case when both t and W are integrated out at $\mu = m_{\mathrm{W}}$. This large correction, which seems to compare quite well with the naive estimate given in the introduction, is a confirmation that a full next-to-leading order calculation is quite important.

The magnitude of this effect may be understood by solving the renormalization group equation for the leading terms. After application of the equations of motion, their contribution turns out to be quite simple:

$$\begin{aligned} C^{\rm eff}_{O^2_{\rm LR}}(m_{\rm W})\Big|_{\rm SM} &= -\frac{5}{24}\left(\frac{\alpha_3(m_{\rm t})}{\alpha_3(m_{\rm W})}\right)^{14/23} + \frac{1}{3} + (\text{subleading}) \\ C^{\rm eff}_{O^3_{\rm LR}}(m_{\rm W})\Big|_{\rm SM} &= \frac{5}{3}\left(\frac{\alpha_3(m_{\rm t})}{\alpha_3(m_{\rm W})}\right)^{14/23} - \frac{8}{3} + (\text{subleading}) \end{aligned} \qquad (66)$$

Equation (66) leads to positive corrections essentially because the effective matching contributions at $\mu = m_{\rm t}$ (20) and at $\mu = m_{\rm W}$ (27) have opposite signs (which is a remnant of the GIM mechanism), and therefore lead to coefficients of opposite signs but comparable magnitude of the first two terms on the right-hand sides of (66). It has long been known [34] that the QCD corrections tend to soften the GIM-cancellations between different up-type quarks if they are nearly degenerate; but there remains a finite enhancement even for a heavy top quark (i.e., $m_{\rm t} \gg m_{\rm W}$), as can be explicitly seen from these expressions. Note that (66) gives only the leading terms, with the subleading terms being suppressed by only a factor of $(m_{\rm W}/m_{\rm t})^2$.

Next let us turn to the contribution from the loop with a charged Higgs. For this case we have solved the renormalization group equation numerically, using as input parameters:

$$\begin{aligned} m_{\rm b} &= 4.5\,\text{GeV}, \quad m_{\rm t} = 175\,\text{GeV}, \\ m_{\rm W} &= 80.22\,\text{GeV}, \quad \alpha_3(m_{\rm Z}) = 0.123\ . \end{aligned}$$

The resulting correction

$$\delta_{\rm H^\pm} = \left.\frac{C^{\rm eff}_{O^3_{\rm LR}}(m_{\rm b})}{C^{\rm eff,naive}_{O^3_{\rm LR}}(m_{\rm b})}\right|_{\rm H^\pm} - 1\ , \qquad (67)$$

to the naive result, obtained by integrating out t and $\rm H^\pm$ simultaneously at the W scale, is shown in figure 4 for $m_{\rm W} < m_{\rm H^\pm} < 750\,\text{GeV}$ and $\tan\beta =$ 1.5, 2, 3, and 10. At sufficiently large $\tan\beta$ (i.e. $\tan\beta > 3$), the correction

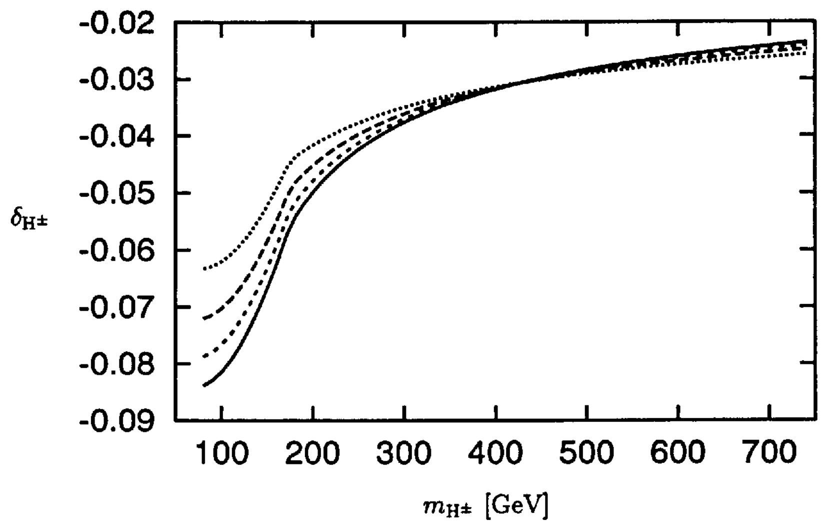

Figure 4: Correction (67) to the coefficient of $O^3_{\rm LR}(m_{\rm b})$ from a loop with t quark and charged Higgs with leading QCD corrections from large mass splitting to the case when both particles are integrated out at $\mu = m_{\rm W}$. The mass of the t quark is assumed to be 175 GeV. The dotted, long-dashed, dashed and solid line correspond to $\tan\beta = 1.5$, 2, 3, and 10, respectively.

turns out to be essentially independent of $\tan\beta$. This is quite understandable since the $\tan\beta$-dependent pieces are actually proportional to $\cot^2\beta$.

For a light charged Higgs, i.e., $m_{\rm H^\pm} < m_{\rm t}$, there appears to be a further reduction of this contribution compared to the naive result. Indeed, in the limit of large $\tan\beta$, and assuming there is no light squark or gluino with mass below $m_{\rm t}$, one finds the following simple analytical result for the charged Higgs contribution, valid for $m_{\rm b} < \mu < m_{\rm t}$:

$$\begin{aligned} C^{\rm eff}_{O^2_{\rm LR}}(\mu)\Big|_{\rm H^\pm} &= \frac{1}{4}\left(\frac{\alpha_3(m_{\rm t})}{\alpha_3(\mu)}\right)^{14/23} + ({\rm subl.}) + O(\cot^2\beta) \qquad (68)\\ C^{\rm eff}_{O^3_{\rm LR}}(\mu)\Big|_{\rm H^\pm} &= \frac{3}{4}\left(\frac{\alpha_3(m_{\rm t})}{\alpha_3(\mu)}\right)^{16/23} - 2\left(\frac{\alpha_3(m_{\rm t})}{\alpha_3(\mu)}\right)^{14/23} + ({\rm subl.}) + O(\cot^2\beta)\ . \end{aligned}$$

Hence no enhancement occurs as in the case of the SM contribution; on the contrary, the leading coefficients get suppressed as they are run down from the $m_{\rm t}$, compared to the subleading terms that (according to our discussion in section 3.1.3) get only suppressed by the evolution from $\mu = m_{\mathrm{H}^\pm}$ down to $\mu = m_{\mathrm{b}}$. The additional QCD corrections are then essentially due to the running from $\mu = m_{\mathrm{t}}$ to $\mu = m_{\mathrm{H}^\pm}$ for sufficiently small $m_{\mathrm{H}^\pm}$. Note that our corrections are counted relative to the case when both particles in the loop are integrated out at the common scale $\mu = m_{\mathrm{W}}$, which is obtained from (68) by substituting $\alpha_3(m_{\mathrm{t}}) \to \alpha_3(m_{\mathrm{W}})$. Thus, for $m_{\mathrm{H}^\pm} < m_{\mathrm{t}}$, integrating out t and $\mathrm{H}^\pm$ at the t scale appears to give a more accurate result than at $\mu = m_{\mathrm{H}^\pm}$ or $\mu = m_{\mathrm{W}}$.

On the other hand, for $m_{\mathrm{H}^\pm} > m_{\mathrm{t}}$ we found only a minor suppression of a few percents, which is essentially the result of a partial cancellation of the enhancement coming from the scaling between $m_{\mathrm{H}^\pm}$ and m_{t} (due to one negative eigenvalue of the submatrix (39) for the mixing of four-fermion operators), and of a reduction from the scaling between m_{t} to m_{W}. Unfortunately, we were unable to obtain a simple analytical solution for this case.

In the case of the chargino contribution, things are more complicated, since one has to consider in general the dependence of the amplitude as a function of several parameters, namely the mass spectrum and the mixing angles for the charginos and the up-type squarks. However, it turns out that the essential features may already be studied for the case of sufficiently large $\tan\beta$, which is in the center of recent interest [11, 18, 20, 23, 24, 25]. In this case, the parameter λ_{b} (47) may become of the same order of magnitude as the parameter λ_{t}. Assuming furthermore that mixing in the squark sector is essentially the same as in the quark sector, which is quite natural in supergravity models where the soft SUSY-breaking is characterized by a common scalar mass at some unification scale, the quantities $\mathcal{G}$ and $\mathcal{H}$, as defined in (51), are then necessarily of the same order of magnitude, the terms proportional to the ratio $\tilde{m}_j/m_{\mathrm{b}}$ will dominate the amplitude, and the corrections become $\tan\beta$-independent.

In this particular limit, one can find an analytical result for the leading terms. For the case of the chargino being much lighter than the squark, the coefficients read:

$$\begin{aligned} C^{\mathrm{eff}}_{O^2_{\mathrm{LR}}}(\mu)\Big|_\chi &= \tilde{C}\cdot\frac{1}{2}e^{14t/3} + (\mathrm{subl.}) + O(\cot^2\beta) \\ C^{\mathrm{eff}}_{O^3_{\mathrm{LR}}}(\mu)\Big|_\chi &= \tilde{C}\cdot\left[-4e^{14t/3} + e^{16t/3}\left(\frac{15}{2} + \frac{4\pi}{2bQ_{\mathrm{b}}}\left(\frac{1}{\alpha_3(\mu)} - \frac{1}{\alpha_3(\tilde{m}_a)}\right)\right)\right] \end{aligned}$$

$$+\ (\text{subl.}) + O(\cot^2\beta) \tag{69}$$

where

$$\tilde{C} \;=\; \mathcal{G}^{*jas}\mathcal{H}^{jab}\cdot\frac{\tilde{m}_j}{m_\mathrm{b}}\left(\frac{m_\mathrm{W}}{\tilde{m}_a}\right)^2 \,, \qquad t=\frac{1}{2b}\ln\left(\frac{\alpha_3(\mu)}{\alpha_3(\tilde{m}_a)}\right) \,,$$

while for the other case of a squark much lighter than a chargino, we get:

$$\begin{aligned}
C^{\mathrm{eff}}_{O^2_{\mathrm{LR}}}(\mu)\Big|_{\tilde{q}} &= \tilde{C}\cdot\left[\frac{99}{260}e^{14t/3}+\frac{1}{10}e^{t/2}+\frac{1}{52}e^{-4t}\right] + (\text{subl.}) + O(\cot^2\beta)\\
C^{\mathrm{eff}}_{O^3_{\mathrm{LR}}}(\mu)\Big|_{\tilde{q}} &= \tilde{C}\cdot\left[-\frac{198}{65}e^{14t/3}+\frac{495}{406}e^{16t/3}-\frac{96}{145}e^{t/2}-\frac{1}{91}e^{-4t}\right]\\
&\quad + (\text{subl.}) + O(\cot^2\beta)
\end{aligned} \tag{70}$$

where now

$$\tilde{C} \;=\; \mathcal{G}^{*jas}\mathcal{H}^{jab}\cdot\frac{m_\mathrm{W}^2}{\tilde{m}_j m_\mathrm{b}} \,, \qquad t=\frac{1}{2b}\ln\left(\frac{\alpha_3(\mu)}{\alpha_3(\tilde{m}_j)}\right) \,.$$

At first sight the terms proportional to $1/\alpha_3$ in (69) might be embarrassing, but a closer look shows that their difference is (to leading order) just some number times $\ln(\mu/\tilde{m}_a)$ and therefore finite in the limit $\alpha_3 \to 0$.

Unfortunately, the interpretation of these expressions is aggravated in both limiting cases since the number of free parameters in the general model is quite large, and due to eqs. (53) one has a supersymmetric version of the GIM mechanism, which leads to a partial cancellation of the leading terms under consideration. Therefore it is difficult to estimate the actual corrections due to the mass splitting between charginos and squarks by using (69) or (70).

Some features of these expressions may still be studied under the following assumptions:

- the squarks of the first two generations are degenerate with mass $\tilde{m}_u$,
- the mixing in the squark sector is the same as in the quark sector (i.e., the gluino-quark-squark couplings are flavor diagonal even in the mass eigenstate basis), and
- the mass matrix for the stop is given (in the $(t_\mathrm{L}, t_\mathrm{R})$ basis) by

$$M^2_{\tilde{t}} \;=\; \begin{pmatrix} \tilde{m}^2_{t_\mathrm{L}} & \tilde{m}^2_{t_\mathrm{LR}} \\ \tilde{m}^2_{t_\mathrm{LR}} & \tilde{m}^2_{t_\mathrm{R}} \end{pmatrix} \,. \tag{71}$$

This mass matrix is diagonalized by a unitary matrix T,

$$TM_{\tilde{t}}^2 T^{-1} = \begin{pmatrix} \tilde{m}_{t_1}^2 & 0 \\ 0 & \tilde{m}_{t_2}^2 \end{pmatrix} . \tag{72}$$

In this scenario, the quantities (51) take a particularly simple form:

$$\begin{aligned} \mathcal{G}^{jal} &\simeq V_{j1}T_{a1} - \lambda_{\rm t} V_{j2} T_{a2} \qquad \text{for } l = b, s;\ a = \tilde{t}_{1,2} \\ \mathcal{H}^{j a \rm b} &\simeq \lambda_{\rm b} U_{j2} T_{a1} \end{aligned} \tag{73}$$

while the sum over the squarks of the first two generations is determined by (53).

Let us for the moment neglect the mixing between $\tilde{t}_{\rm L}$ and $\tilde{t}_{\rm R}$, and consider the case $\tilde{m}_{t_{\rm LR}}^2 = 0$, $T = \mathbf{1}$. Evaluating the first line of (69) to lowest order, we find for the contribution of a light chargino and after summing over the different squarks:

$$\begin{aligned} C^{\rm eff}_{O^2_{\rm LR}}(\tilde{m}_j) &= \frac{1}{2}\lambda_{\rm b} U_{j2} V_{j1} \frac{\tilde{m}_j}{m_{\rm b}} \frac{m_{\rm W}^2}{\tilde{m}_u^2} \left(\frac{\tilde{m}_j^2}{\tilde{m}_u^2}\right)^{\frac{14}{3}\hat{\alpha}_3} \left[\left(\frac{\tilde{m}_u^2}{\tilde{m}_{t_{\rm L}}^2}\right)^{1+\frac{14}{3}\hat{\alpha}_3} - 1\right] \\ &\quad + O(\hat{\alpha}_3^2) + (\text{subl.}) + O(\cot^2\beta) \end{aligned} \tag{74}$$

with the abbreviation

$$\hat{\alpha}_3 = \frac{\alpha_3(\tilde{m}_j)}{4\pi} .$$

Similar, although rather lengthy expressions are obtained if the mixing between $\tilde{t}_{\rm L}$ and $\tilde{t}_{\rm R}$ is taken into account, and analogous results are found for the other coefficients in (69) and (70). As has already been pointed out in [23], the sign of the product $U_{j2}V_{j1}$ depends on the sign of $\mu_{\rm h}$, so that this leading contribution for large $\tan\beta$ can have either sign.

A closer look at (74) shows two counteracting effects: a reduction of the leading coefficient due to QCD running from $\tilde{m}_u$ down to $\tilde{m}_j$, while the term in square brackets shows an enhancement due to a QCD-softening of the GIM cancellation, independent on whether $\tilde{m}_{t_{\rm L}}$ is larger or smaller than $\tilde{m}_u$. The actual size of the corrections depends of course on the mass splitting between the squarks as well as on the splitting between the mass of the chargino and the squarks; since squarks can be an order of magnitude heavier than the lightest chargino, we estimate this coefficient to be of the order of

$$\frac{14}{3}\frac{\alpha_3(\tilde{m}_j)}{4\pi} \times \left(\ln\frac{\tilde{m}_u}{\tilde{m}_j},\ \ln\frac{\tilde{m}_u}{\tilde{m}_t}\right) \lesssim 15\% .$$

Similar results are found when analyzing the other expressions, so in general we will expect corrections up to $O(15\%)$ with either sign. An exceptional situation occurs when, due to these super-GIM cancellations, the lowest-order contribution to O^3_{LR} is accidentally lower than the contribution to O^2_{LR} by orders of magnitude, since the above reasoning did not take into account the mixing of O^2_{LR} into O^3_{LR} for scales below the heavy thresholds. In this case a sensible answer is obtained only when using the full expressions.

6 Conclusions

We have extended the calculation of the leading QCD corrections for the inclusive $b \to s\gamma$ decay to the MSSM in the framework of effective field theories. It was shown that proper treatment of the high-energy scale at which the particles in the loop are integrated out is important, as well as how to calculate the QCD corrections between if the masses of the particles in the loop are vastly different. To this end, we have calculated the leading order anomalous dimension matrices for the operators for the various scenarios that are relevant to this process in the MSSM.

We found that, while the SM contribution to the Wilson coefficients at the weak scale is enhanced in the limit of a heavy top quark by about 15–20%, the contribution from a loop with a charged Higgs is actually slightly reduced by a few percents. The result for the contribution from the chargino loops depends strongly on the mass spectrum of the squarks and the charginos as well as on the mixing angles. Typically, one expects corrections up to the order of 15% with either sign, which is less than the enhancement of the SM contribution.

Given a range of values for the inclusive decay, if one applies the above results to a parameter space analysis for a particular SUSY model, one will essentially find a relaxation of the bounds on the mass of the charged Higgs, especially in the region of large $\tan\beta$. The impact of the modification of the QCD corrections for the chargino loop contribution is not seen so easily, but we expect a smooth deformation of contours in analyses like [11, 25], with the strongest effect in those regions where the lowest order contribution to the coefficient $C_{O^3_{\mathrm{LR}}}$ is small although the chargino is relatively light.

Finally we would like to point out that for the inclusive decay rate, even after taking into account the real gluon emission and virtual corrections below the b scale [35], the leading order prediction remains uncertain by about 25% due to the residual scale dependence alone [10, 12]. Once a full next-to-leading order calculation is available for the SM, it may be combined with the above results to obtain predictions in the MSSM with comparable precision.

Acknowledgements

We would like to thank S. Brodsky, J. Hewett, and T. Rizzo for discussions and useful comments on the manuscript. The present work was stimulated by discussions with B. Grinstein during a stay at the former Superconducting Super Collider Laboratory. We benefitted also from conversations with F. Borzumati and J. Wells.

Finally, we would like to thank P. Cho for sending us the erratum to [13] prior to publication.

A Wilson coefficients at one loop

We quote here the results for the Wilson coefficients at one-loop order when both particles in the loop are integrated out at a common scale. These results will be used for the determination of subleading terms. They also provide an important cross-check for the leading terms obtained by the calculation in the effective theory, as well as for some of the entries in the anomalous dimension matrix.

It is convenient to use the following functions that appear in the evaluation of the coefficients of the basis operators:

$$\begin{aligned} F_1(x) &= \frac{x^2-5x-2}{12(x-1)^3} + \frac{x\ln x}{2(x-1)^4} \\ F_2(x) &= \frac{2x^2+5x-1}{12(x-1)^3} - \frac{x^2\ln x}{2(x-1)^4} \end{aligned}$$

$$
\begin{aligned}
F_3(x) &= \frac{x-3}{2(x-1)^2} + \frac{\ln x}{(x-1)^3} \\
F_4(x) &= \frac{x+1}{2(x-1)^2} - \frac{x \ln x}{(x-1)^3} \ .
\end{aligned} \tag{75}
$$

These functions are identical with those given in the appendix of ref. [17]. Some of their properties are:

$$
\begin{aligned}
&F_1\left(\frac{1}{x}\right) = xF_2(x), \quad F_2\left(\frac{1}{x}\right) = xF_1(x), \quad F_4\left(\frac{1}{x}\right) = xF_4(x) \\
&F_1(x) + F_2(x) = \frac{1}{2}F_4(x) = \frac{1}{4} - \frac{1}{2}xF_3(x) \\
&xF_1(x) + F_2(x) = \frac{1}{12} \\
&F_3\left(\frac{1}{x}\right) = -x\left(F_3(x) + 2F_4(x)\right) + \frac{x \ln x}{x-1} \ .
\end{aligned} \tag{76}
$$

A.1 Standard Model loop contributions

Integrating out the W, the charged would-be Goldstone bosons and an up-type quark simultaneously, we obtain the one-loop expression of the Wilson coefficients of the effective Hamiltonian (1):

$$
\begin{aligned}
C_{O^1_{\mathrm{LR}}} &= -xF_4(x) \\
C_{O^2_{\mathrm{LR}}} &= \frac{Q_\mathrm{b}}{Q_\mathrm{t}} C_{O^3_{\mathrm{LR}}} = -\frac{x}{2}\left(F_3(x) + F_4(x)\right) \\
C_{P^{1,1}_{\mathrm{L}}} &= C_{P^{1,3}_{\mathrm{L}}} = \frac{1}{3}(x+2)\left(2F_2(x) + F_3(x) + 2F_4(x)\right) \\
C_{P^{1,2}_{\mathrm{L}}} &= \frac{2}{3}(x+2)\left(F_2(x) - F_3(x) - 2F_4(x)\right) \\
C_{P^{1,4}_{\mathrm{L}}} &= (x-2)F_4(x) \\
C_{P^2_{\mathrm{L}}} &= \frac{1}{Q_\mathrm{b}}(x+2)\left(\frac{1}{2}F_3(x) + F_4(x) - \frac{\ln(x)}{6(x-1)}\right) \\
C_{P^4_{\mathrm{L}}} &= \frac{1}{Q_\mathrm{b}}\left(3 - 2xF_3(x) - 5xF_4(x)\right) \ .
\end{aligned} \tag{77}
$$

Here $x = (m_q/m_\mathrm{W})^2$. Note that for large x all coefficient functions are bounded, except for $C_{P^2_{\mathrm{L}}}$, which grows logarithmically with x. For small x,

$C_{P_{\rm L}^{1,1}}$, $C_{P_{\rm L}^{1,2}}$, $C_{P_{\rm L}^{1,3}}$ and $C_{P_{\rm L}^{2}}$ diverge logarithmically.

A.2 Charged Higgs loop contributions

Integrating out the charged Higgs and an up-type quark simultaneously, the corresponding expressions are ($y = (m_q/m_{\rm H^\pm})^2$):

$$\begin{aligned}
C_{O^1_{\rm LR}} &= yF_4(y) \\
C_{O^2_{\rm LR}} &= \frac{Q_{\rm b}}{Q_{\rm t}} C_{O^3_{\rm LR}} = \frac{y}{2}\left(F_3(y)+F_4(y)\right) \\
C_{P_{\rm L}^{1,1}} &= C_{P_{\rm L}^{1,3}} = \frac{1}{3} y\left(2F_2(y)+F_3(y)+2F_4(y)\right)\cot^2\beta \\
C_{P_{\rm L}^{1,2}} &= \frac{2}{3} y\left(F_2(y)-F_3(y)-2F_4(y)\right)\cot^2\beta \\
C_{P_{\rm L}^{1,4}} &= -Q_{\rm b} C_{P_{\rm L}^4} = yF_4(y)\cot^2\beta \\
C_{P_{\rm L}^2} &= \frac{1}{Q_{\rm b}} y\left(\frac{1}{2}F_3(y)+F_4(y)-\frac{\ln(y)}{6(y-1)}\right)\cot^2\beta\ .
\end{aligned} \tag{78}$$

A.3 Chargino loop contributions

Finally we give the expressions for integrating out a chargino and an up-type squark. Setting $z = (\tilde m_j/\tilde m_a)^2$, where m_j and m_a represent the mass of the chargino χ_j and of the up-type squark $\tilde u_a$ respectively, and using the couplings defined in eq. (51), one finds

$$\begin{aligned}
C_{O^1_{\rm LR}} &= \mathcal{G}^{*jas}\mathcal{H}^{jab}\cdot\frac{\tilde m_j}{m_{\rm b}}\left(\frac{m_{\rm W}}{\tilde m_a}\right)^2\cdot(-2)F_4(z) \\
C_{O^2_{\rm LR}} &= 0 \\
C_{O^3_{\rm LR}} &= \mathcal{G}^{*jas}\mathcal{H}^{jab}\cdot\frac{\tilde m_j}{m_{\rm b}}\left(\frac{m_{\rm W}}{\tilde m_a}\right)^2\cdot\frac{1}{Q_{\rm b}}\left(F_3(z)+F_4(z)\right) \\
C_{P_{\rm L}^{1,1}} &= C_{P_{\rm L}^{1,3}} = \mathcal{G}^{*jas}\mathcal{G}^{jab}\cdot\left(\frac{m_{\rm W}}{\tilde m_a}\right)^2\cdot\frac{1}{3}\left[4F_2(z)+\frac{1}{z}F_3\left(\frac{1}{z}\right)\right] \\
C_{P_{\rm L}^{1,2}} &= \mathcal{G}^{*jas}\mathcal{G}^{jab}\cdot\left(\frac{m_{\rm W}}{\tilde m_a}\right)^2\cdot\frac{2}{3}\left[2F_2(z)-\frac{1}{z}F_3\left(\frac{1}{z}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
C_{P_{\mathrm{L}}^{1,4}} &= 0 \qquad (79)\\
C_{P_{\mathrm{L}}^{2}} &= \mathcal{G}^{*jas}\mathcal{G}^{ja\mathrm{b}} \cdot \left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2 \cdot \left(-\frac{2}{Q_{\mathrm{b}}}\right)\left(\frac{1}{2}F_3(z) + F_4(z) - \frac{\ln(z)}{6(z-1)}\right)\\
C_{P_{\mathrm{L}}^{4}} &= \mathcal{G}^{*jas}\mathcal{G}^{ja\mathrm{b}} \cdot \left(\frac{m_{\mathrm{W}}}{\tilde{m}_a}\right)^2 \cdot \left(\frac{2}{Q_{\mathrm{b}}}\right) F_4(z)\ .
\end{aligned}
$$

After application of the equations of motion, these expressions are consistent with the corresponding expressions in [17].

References

[1] R. Ammar et al. (CLEO Collaboration), Phys. Rev. Lett. **71** (1993) 674.

[2] T. Inami, C.S. Lim, Prog. Theor. Phys. **65** (1981) 297.

[3] R. Grigjanis, P.J. O'Donnell, M. Sutherland, H. Navelet, Phys. Rep. **228** (1993) 93.

[4] B. Grinstein, R. Springer, M. Wise, Phys. Lett. **B202** (1988) 138; Nucl. Phys. **B339** (1990) 269.

[5] R. Grigjanis, P.J. O'Donnell, M. Sutherland, H. Navelet, Phys. Lett. **B213** (1988) 355; ibid. **B286** (1992) 413(E).

[6] G. Cella, G. Curci, G. Ricciardi, A. Viceré, Phys. Lett. **B248** (1990) 181; Phys. Lett. **B325** (1994) 227; preprint IFUP-TH 9/94, HUTP-94/A001, hep-ph/9406203, June 1994.

[7] M. Misiak, Phys. Lett. **B269** (1991) 161; Nucl. Phys. **B393** (1993) 23; Phys. Lett. **B321** (1994) 113.

[8] K. Adel, Y.P. Yao, Mod. Phys. Lett. **A8** (1993) 1679; preprint UM-TH-93-20, IP-ASTP-29-93, hep-ph/9308349.

[9] M. Ciuchini, E. Franco, G. Martinelli, L. Reina, L. Silvestrini, Phys. Lett. **B316** (1993) 127; M. Ciuchini, E. Franco, L. Reina, L. Silvestrini, preprint ROME-973-1993, hep-ph/9311357.

[10] A. Ali, C. Greub, Z. Phys. **C60** (1993) 433.

[11] F. Borzumati, preprint DESY 93-090, hep-ph 9310212, August 1993.

[12] A. Buras, M. Misiak, M. Münz, S. Pokorski, MPI-Ph/93-73, TUM-T31-50/93, hep-ph 9311345.

[13] P. Cho, B. Grinstein, Nucl. Phys. **B365** (1991) 279; Erratum: ibid., (in print).

[14] H. Haber, G. Kane, Phys. Rep. **117** (1985) 75.

[15] S. Bertolini, F. Borzumati, A. Masiero, Phys. Lett. **B192** (1987) 437; Nucl. Phys. **B294** (1987) 321.

[16] H.P. Nilles, Phys. Rep. **110** (1984) 1; A.B. Lahanas, D.V. Nanopoulos, Phys. Rep. **145** (1987) 1. For a recent introduction to supergravity phenomenology see e.g. R. Arnowitt, P. Nath, Lecture at Swieca Summer School, Campos do Jordao, Brasil, 1993, preprint CTP-TAMU-52/93.

[17] S. Bertolini, F. Borzumati, A. Masiero, G. Ridolfi, Nucl. Phys. **B353** (1991) 591.

[18] N. Oshimo, Nucl. Phys. **B404** (1993) 20.

[19] J.L. Lopez, D. Nanopoulos, G.T. Park, Phys. Rev. **D48** (1993) 974.

[20] M.A. Diaz, Phys. Lett. **B304** (1993) 278; Phys. Lett. **B322** (1994) 207.

[21] R. Barbieri, G.F. Giudice, Phys. Lett. **B309** (1993) 86.

[22] Y. Okada, Phys. Lett. **B315** (1993) 119.

[23] R. Garisto, J.N. Ng, Phys. Lett. **B315** (1993) 372.

[24] J.S. Hagelin, S. Kelley, T. Tanaka, Nucl. Phys. **B415** (1994) 293.

[25] S. Bertolini, F. Vissani, preprint SISSA 40/94/EP, hep-ph/9403397.

[26] G. Bhattacharyya, A. Raychaudhuri, preprint CERN-TH.7245/94, hep-ph/9405235, May 1994.

[27] E. Witten, Nucl. Phys. **B122** (1977) 109.

[28] S. Weinberg, Phys. Lett. **B91** (1979) 51.

[29] A. Cohen, H. Georgi, B. Grinstein, Nucl. Phys. **B232** (1984) 61.

[30] H. Georgi, Nucl. Phys. **B363** (1991) 301.

[31] H. Georgi, Ann. Rev. Nucl. Part. Sci. **43** (1993) 209.

[32] T. Appelquist, J. Carazzone, Phys. Rev. **D11** (1975) 2856.

[33] L. Abbott, Nucl. Phys. **B185** (1981) 189.

[34] F. Gilman, M. Wise, Phys. Rev. **D21** (1980) 3150.

[35] A. Ali, C. Greub, Phys. Lett. **B259** (1991) 182; Z. Phys. **C49** (1991) 431.