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Null Surfaces, Initial Values, and Evolution Operators for Spinor Fields*

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ABSTRACT

We analyze the initial value problem for spinor fields obeying the Dirac equation, with particular attention to the characteristic surfaces. The standard Cauchy initial value problem for first-order differential equations is to construct a solution function in a neighborhood of space and time from the values of the function on a selected initial value surface. On the characteristic surfaces the solution function may be discontinuous, so the standard Cauchy construction breaks down. For the Dirac equation, the characteristic surfaces are null surfaces. An alternative version of the initial value problem may be formulated using null surfaces; the initial value data needed differs from that of the standard Cauchy problem. For the case discussed here, the values of separate components of the spinor function on an intersecting pair of null surfaces comprise the necessary initial value data. We present an expression for the construction of a solution from null surface data; two analogues of the quantum mechanical Hamiltonian operator determine the evolution of the system.

I. Introduction

The Cauchy problem, initial values, and characteristics of the Klein-Gordon equation for scalar fields was discussed in Ref. [1]. We extend the discussion here to include spinor fields obeying the free particle Dirac equation. Although the approach is the same, and frequent reference is made to Ref. [1], we have attempted to make this discussion of spinor fields self-contained.

The standard Cauchy problem for a first-order partial differential equation is to use the values of the solution function on an initial value surface to determine the values of the function in a neighborhood of the surface [2]. Solution of the problem proceeds by showing how the initial value data and the differential equation determine all the time derivatives of the function on the initial surface, and thus allow a power series development of the function for future times.

If the solution function has a discontinuity across some special surface, then the standard Cauchy problem cannot be solved using that surface for the initial value data. Such surfaces are called characteristic surfaces, or simply characteristics. The characteristics of the Dirac equation for free particles in flat space are easily shown to be null surfaces. Thus the discontinuities of the solution functions propagate at the velocity of light, independently of the mass parameter in the equations. The characteristics of Maxwell's equations, the Klein-Gordon equation, and the vacuum Einstein equations of general relativity are also all null surfaces [1,3].

We briefly review the standard Cauchy problem for the Dirac equation. We then obtain the characteristics and solve the analogous problem for the planar null surfaces. Initial data for this situation consists of the values of disparate components of the spinor function on the pair of null surfaces. We give a simple evolution operator expression in terms of null coordinates, which is a direct

analogue of the usual expression of quantum theory, except that two analogues of the Hamiltonian operator determine the evolution of the solution; that is, are generators of displacements in the null coordinates [4–8]. The result is remarkably similar to that obtained for the scalar field [1].

II. Development Of a Spinor Function from a Constant Time Hypersurface

Since the Dirac equation may be expressed in Hamiltonian form, the development of a solution from initial value data given on a constant time hypersurface is one of the most familiar problems of physics [9]. We express the Dirac equation in Hamiltonian form as [10]

$$i\psi_{|t}(\vec{x},t) = [-i\vec{\alpha}.\bigtriangledown + \beta m]\psi(\vec{x},t) = H\psi(\vec{x},t)\ . \tag{1}$$

We use units in which $\hbar = c = 1$. The slash notation indicates differentiation with respect to the indicated variable, in this case time t. We write a Taylor series expansion as

$$\psi(\vec{x},t) = \psi(\vec{x},t)_{t=0} + t\psi_{|t}(\vec{x},t)_{t=0} + \frac{t^2}{2!}\ \psi_{|t|t}(\vec{x},t)_{t=0} + \dots\ . \tag{2}$$

All of the time derivatives at $t = 0$ may be readily obtained from the Dirac equation and the value of the spinor function at $t = 0$, which we call $h(\vec{x})$. When substituted in the series, these give the familiar result

$$\psi(\vec{x},t) = \psi(\vec{x},t)_{t=0} + (-iHt)\psi(\vec{x},t)_{t=0} + \frac{(-iHt)^2}{2!}\ \psi(\vec{x},t)_{t=0} + \ldots = e^{-iHt}h(\vec{x})\ . \tag{3}$$

The indicated exponential of the Hamiltonian operator is thereby seen as the finite time displacement operator.

III. Characteristics of the Dirac Equation

The characteristics of the Dirac equation are found in the same way as for the Klein-Gordon equation in [1], except that since the Dirac equation is first-order in time, the problem is even easier. We suppose that the initial value of the solution is given on a hypersurface S, $t = T(\vec{x})$, by a four-component spinor.

$$\psi(\vec{x}, T(\vec{x})) = h(\vec{x}) \ , \tag{4}$$

and that the solution is given by a Taylor series expansion

$$\psi(\vec{x}, t) = \psi(\vec{x}, t)_{t=T} + [t - T(\vec{x})]\psi_{|t}(\vec{x}, t)_{t=T} + \frac{[t - T(\vec{x})]^2}{2!} \ \psi_{|t|t}(\vec{x}, t)_{t=T} + \dots \ . \tag{5}$$

To find the first time derivative, we first express the gradient of the initial value h in terms of $t = T(\vec{x})$,

$$\bigtriangledown h(\vec{x}) = \bigtriangledown \psi(\vec{x}, t)_{t=T} + \psi_{|t}(\vec{x}, t)_{t=T} \bigtriangledown T(\vec{x}) \ . \tag{6}$$

From this and the Dirac equation (1), we write an equation for the time derivative of ψ on S,

$$i[I - \vec{\alpha}. \bigtriangledown T(\vec{x})] \, \psi_{|t}(\vec{x}, t)_{t=T} - -i\vec{\alpha}. \bigtriangledown h(\vec{x}) + \beta m h(\vec{x}) \ . \tag{7}$$

This may be solved for the time derivative if the matrix in square brackets has an inverse. The determinant of that matrix is easily found to be

$$|I - \vec{\alpha}.\nabla T(\vec{x})| = (1 - \nabla T(\vec{x})^2)^2 \ . \tag{8}$$

Thus, the condition that S be a characteristic is that the above quantity be zero, or

$$\nabla T(\vec{x})^2 = 1 \ . \tag{9}$$

This is the same as the characteristic equation for the Klein-Gordon equation; thus the characteristics are null surfaces, and are independent of the mass parameter in

the Dirac equation. The particular null surfaces we study in the remainder of this paper are

$$\begin{aligned} u &= t - x = 0 \quad u \quad \text{characteristic or null surface} , \\ v &= t + x = 0 \quad v \quad \text{characteristic or null surface} . \end{aligned} \tag{10}$$

IV. Construction of a Solution Function in Terms of Null Coordinates

The covariant form of the Dirac equation is [10]

$$\gamma^{\alpha} i \, \frac{\partial}{\partial x^{\alpha}} \, \psi = m\psi \ . \tag{11}$$

We will study this using the null coordinates $u = t - x$ and $v = t + x$, and suppress dependence on y and z in all that follows [11]. The gamma matrices are then $\gamma^u = \gamma^0 - \gamma^1$ and $\gamma^v = \gamma^0 + \gamma^1$. The Dirac equation and the gamma matrix algebra in terms of the null coordinates are

$$\left[\gamma^u i \frac{\partial}{\partial u} + \gamma^v i \frac{\partial}{\partial v} \right] \ \psi(u,v) = m\psi(u,v) \ , \tag{12}$$

$$\frac{\gamma^u \gamma^v}{4} + \frac{\gamma^v \gamma^u}{4} = 1 \ , \qquad (\gamma^u)^2 = (\gamma^v)^2 = 0 \ . \tag{13}$$

We define a pair of projection operators as

$$\Lambda_u = \frac{\gamma^u \gamma^v}{4} \ , \qquad \Lambda_v = \frac{\gamma^v \gamma^u}{4} \ . \tag{14}$$

The usual projection operator properties follow from (13); that is,

$$\Lambda_u + \Lambda_v = 1 \ , \qquad \Lambda_u \Lambda_u = \Lambda_u \ , \qquad \Lambda_v \Lambda_v = \Lambda_v \ , \qquad \Lambda_u \Lambda_v = \Lambda_v \Lambda_u = 0 \ . \tag{15}$$

To analyze the Dirac equation on and near the null surfaces $u = 0$ and $v = 0$, we make an expansion in the mass m, since the solutions of the zero mass Dirac equation are functions of only u or only v. Thus, we write

$$\psi(u,v) = \psi^{(0)}(u,v) + m\psi^{(1)}(u,v) + m^2\psi^{(2)}(u,v) + \dots \ . \tag{16}$$

Substitution of this into the Dirac equation gives the following set of iterative equations:

$$\begin{aligned} \left[\gamma^u i\frac{\partial}{\partial u} + \gamma^v i\frac{\partial}{\partial v}\right]\ \psi^{(0)}(u,v) \ &= \ 0 \ , \\ \left[\gamma^u i\frac{\partial}{\partial u} + \gamma^v i\frac{\partial}{\partial v}\right]\ \psi^{(n)}(u,v) \ &= \ m\psi^{(n-1)}(u,v) \ . \end{aligned} \tag{17}$$

Solution of the zero-order equation is easy by inspection,

$$\psi^{(0)}(u,v) = \Lambda_u f(u) + \Lambda_v g(v) \ . \tag{18}$$

Here, the functions f and g are any differentiable 4-tuple functions of$u$ and v; we will call these the generating functions. It is instructive to consider a representation of the gamma matrices in which the projection operators L are diagonal; then the Λ_u contains ones at entries (1,1) and (2,2), and zeros everywhere else; similarly, Λ_v has ones at entries (3,3) and (4,4) and zeros everywhere else.

As for the zero order solution, they are (in transpose notation):

$$\psi^{(0)\dagger}(u,v) = (f_1(u), f_2(u), g_1(v), g_2(v)) \ ; \tag{19}$$

thus, only two components of f and two components of g enter the solution. The first order equation (17) is

$$\left[\gamma^u i\frac{\partial}{\partial u} + \gamma^v i\frac{\partial}{\partial v}\right]\ \psi^{(1)}(u,v) = \Lambda_u f(u) + \Lambda_v g(v) \ . \tag{20}$$

Only a particular solution to this is needed since the homogeneous solution may be absorbed into the zero-order solution. We break the equation into u and v parts by premultiplying by Γ_u and using (12) to get

$$i\frac{\partial}{\partial v}\Lambda_u\psi^{(1)}(u,v) = \frac{1}{4}\gamma^u\Lambda_v g(v) \ . \tag{21}$$

The right-hand side of this is a function of v only, so the left-hand side must be also. Thus, we may write a solution by integration,

$$\Lambda_u\psi^{(1)}(u,v) = \frac{-i}{4}\gamma^u\Lambda_v\int_{v_0}^{v} dv' g(v') \ . \tag{22}$$

Here, v_0 is an arbitrary parameter. In the same way, we may obtain a similar expression for the v projection of the solution,

$$\Lambda_v\psi^{(1)}(u,v) = \frac{-i}{4}\gamma^v\Lambda_u\int_{u_0}^{u} du' f(u') \ . \tag{23}$$

The sum of (21) and (22) gives the complete first-order solution

$$\psi^{(1)}(u,v) = \frac{-i}{4}\left[\gamma^u\Lambda_v\int_{v_0}^{v} dv' g(v') + \gamma^v\Lambda_u\int_{u_0}^{u} du' f(u')\right] \ . \tag{24}$$

The second-order equation (16) may now be written

$$i\frac{\partial}{\partial v}\ \Lambda_u\psi^{(2)}(u,v) = \frac{-i}{4}\left[\gamma^u\Lambda_v\int_{v_0}^{v} dv' g(v') + \gamma^v\Lambda_u\int_{u_0}^{u} du' f(u')\right] \ . \tag{25}$$

As before, we premultiply by γ^u to obtain, with use of (14) and (15),

$$\frac{\partial}{\partial v}\Lambda_u\psi^{(2)}(u,v) = \frac{-1}{4}\Lambda_u\int_{u_0}^{u} du' f(u') \tag{26}$$

Since the right-hand side of this is a function of only u, we may write a particular solution by inspection:

$$\Lambda_u \psi^{(2)}(u,v) = \frac{-1}{4}\ (v-v_0)\Lambda_u \int_{u_0}^{u} du' f(u')\ . \tag{27}$$

In the same way, we may obtain the v projection of the solution,

$$\Lambda_v \psi^{(2)}(u,v) = \frac{-1}{4}\ (u-u_0)\Lambda_v \int_{v_0}^{v} dv' g(v')\ . \tag{28}$$

Summing (27) and (28), we have the complete second order solution:

$$\psi^{(2)}(u,v) = \frac{-1}{4}\ (v-v_0)\left[\Lambda_u \int_{u_0}^{u} du' f(u') + (u-u_0)\Lambda_v \int_{v_0}^{v} dv' g(v')\right]\ . \tag{29}$$

The procedure is now clear; we may continue to all orders—the even orders are similar in form, and the odd orders are similar in form. The complete series solution is:

$$\begin{aligned}\psi(u,v)\ &=\ \sum_{n=0}^{\infty} \frac{[-\frac{m^2}{4}(v-v_0)\Gamma_u]^n}{n!}\left(1-\frac{im}{4}\ \gamma^v\Gamma_u\right)\ \Lambda_u f(u) \\ &+\ \sum_{j=0}^{\infty} \frac{[-\frac{m^2}{4}(u-u_0)\Gamma_v]^j}{j!}\left(1-\frac{im}{4}\ \gamma^u\Gamma_v\right)\ \Lambda_v g(v)\ ,\end{aligned} \tag{30}$$

where the multiple integral operator Γ_u^n is defined as

$$\Gamma_u^n f(u) = \int_{u_0}^{u}\int_{u_0}^{u'}\cdots\int_{u_0}^{u^{n-1}} du^n f(u^n)\ . \tag{31}$$

The series (29) may be readily summed to give a concise expression for the solution

$$\begin{aligned}\psi(u,v)\ &=\ \exp\left\{-\frac{m^2}{4}\ (v-v_0)\Gamma_u\right\}\left(1-\frac{im}{4}\ \gamma^v\Gamma_u\right)\ \Lambda_u f(u) \\ &+\ \exp\left\{-\frac{m^2}{4}\ (u-u_0)\Gamma_v\right\}\left(1-\frac{im}{4}\ \gamma^u\Gamma_v\right)\ \Lambda_v g(v)\ .\end{aligned} \tag{32}$$

This gives a solution of the free particle Dirac equation for any pair of generating functions f and g; the generating functions together have only four independent components, however, so this initial value data contains the same amount of information as that in the standard Cauchy problem.

V. Initial Values on Null Surfaces

The generating functions f and g are arbitrary functions, and are simply related to the initial values of the solution on the null surfaces, as we will now discuss. In this and the following section, we set $u_0 = v_0 = 0$ without loss of generality.

To get a relation between the initial values of the function ψ on the null surfaces and the generating functions f and g, we set $u = 0$ and then $v = 0$ in (32), and easily find

$$\psi_0(u,0) = \left(1 - \frac{\mathrm{im}}{4}\,\gamma^v\,\Gamma_u\right)\,\Lambda_u f(u) + \Gamma_v\Lambda_v g(0)\ , \tag{33}$$

$$\psi_0(0,v) = \left(1 - \frac{\mathrm{im}}{4}\,\gamma^u\,\Gamma_v\right)\,\Lambda_v g(v) + \Gamma_u\Lambda_u f(0)\ . \tag{34}$$

We wish to make a convenient choice for the values of $f(0)$ and $g(0)$; taking $u = v = 0$ in (33),(34), we see that

$$\psi_0(0,0) = \Lambda_u f(0) + \Lambda_v g(0)\ . \tag{35}$$

Accordingly, we choose $f(0) = g(0) = \psi_0(0,0)$; then the quantities that appear in the evolution expression (32) are

$$\left(1 - \frac{\mathrm{im}}{4}\,\gamma^v\,\Gamma_u\right)\,\Lambda_u f(u) = \psi_0(0,0) - \Lambda_v\psi_0(0,0)\ , \tag{36}$$

$$\left(1 - \frac{\mathrm{im}}{4}\,\gamma^u\,\Gamma_v\right)\;\Lambda_v g(v) = \psi_0(0,0) - \Lambda_u\psi_0(0,0)\;. \tag{37}$$

Thus, the spinorial expressions in parentheses turn the arbitrary 4-tuple functions into the indicated initial data on the null surfaces. These expressions may be inverted to obtain f and g .

$$\Lambda_u f(u) = \left(1 + \frac{\mathrm{im}}{4}\,\gamma^v\,\Gamma_u\right)\;\left[\psi_0(0,0) - \Lambda_v\psi_0(0,0)\right]\;, \tag{38}$$

$$\Lambda_v g(v) = \left(1 + \frac{\mathrm{im}}{4}\,\gamma^u\,\Gamma_v\right)\;\left[\psi_0(0,0) - \Lambda_u\psi_0(0,0)\right]\;. \tag{39}$$

In terms of the initial values of the solution, expression (31) now reads

$$\begin{aligned}\psi(u,v) \;&=\; \exp\left\{-\frac{m^2}{4}\;(v-v_0)\Gamma_u\right\}\;\left[\psi_0(0,0) - \Lambda_v\psi_0(0.0)\right]\;,\\ &+\exp\left\{-\frac{m^2}{4}\;(u-u_0)\Gamma_v\right\}\;\left[\psi_0(0,0) - \Lambda_u\psi_0(0,0)\right]\;.\end{aligned} \tag{40}$$

This is a complete expression for the evolution of the solution from its values on the pair of null surfaces $u = 0$ and $v = 0$. Notice that only the square of the mass appears in the evolution operators.

The form of the solution in (40) is very similar to that obtained in [1] for scalar fields obeying the Klein–Gordon equation. The two exponential evolution operators are identical, and only the forms of the initial data expressions are slightly different, since the present one contains spin information. Thus, in terms of the null coordinates the dynamics of scalar and spinor fields is remarkably similar—much more so than in terms of the usual Cartesian coordinates.

VI. Plane Waves

We wish to verify the consistency of (40) for plane-wave solutions of the Dirac equation. We will show that if the appropriate initial data functions are put into (40) then the solution function generated is the appropriate plane wave. We write a plane wave solution in Cartesian and null coordinates as

$$\psi(u,v) = \exp\{-iEt + ikx\}\ \ w(E,k) = \exp\{-i\lambda u - i\tau v\}\ \ w(\lambda,\tau)\ . \tag{41}$$

Here w is the usual Dirac 4-tuple spin function. The null momenta are given by

$$\lambda = \frac{E+k}{2}\ , \qquad \tau = \frac{E-k}{2}\ , \qquad \lambda\tau = \frac{E^2-k^2}{4} = \frac{m^2}{4} = \mu^2\ . \tag{42}$$

Thus, the appropriate initial value quantities that enter the evolution expression (40) are

$$\psi_0(u,0) - \Lambda_v\psi_0(0,0) = (\exp\{-i\lambda u\} - \Lambda_v)\ \ w(\lambda,\tau)\ ,$$

$$\psi_0(0,v) - \Lambda_u\psi_0(0,0) = (\exp\{-i\tau v\} - \Lambda_u)\ \ w(\lambda,\tau)\ . \tag{43}$$

We substitute these expressions into (40), abbreviating for convenience $B = -i\lambda$, $C = -i\tau$, and $-m^2/4 = BC$, and find

$$\begin{aligned}
\psi(u,v)\ \ &=\ \ \exp\{BCv\Gamma_u\}\ (\exp\{Bu\} - \Lambda_v)\ w(\lambda,\tau) \\
&\quad + \exp\{BCu\Gamma_v\}\ \ (\exp\{Cv\} - \Lambda_u)\ w(\lambda,\tau) \\
&=\ \ \exp\{BCv\Gamma_u\}\ \exp\{Bu\} \\
&\quad + \exp\{BCu\Gamma_v\}\ \exp\{Cv\}]\ w(\lambda,\tau) - F_c(uv)\ w(\lambda,\tau)\ .
\end{aligned} \tag{44}$$

Here, F_c denotes the function related to a constant generator, defined and evaluated as

$$F_c(uv) = \sum_{n=0}^{\infty} \frac{(-\mu^2 uv)^2}{(n!)^2} \;=\; \exp\left\{-\frac{m^2}{4}\right\}\; v\Gamma_u 1 \;=\; \exp\left\{-\frac{m^2}{4}\right\}\; u\Gamma_v 1\;. \quad (45)$$

Notice that this function is symmetric in u and v. The quantity in parentheses in (44) is straightforward to evaluate; we expand the exponentials in double series as

$$\exp\{BCv\Gamma_u\}\;\exp\{Bu\} \;+\; \exp\{BCu\Gamma_v\}\;\exp\{Cv\}$$

$$= \sum_{n=0,j=0}^{\infty} \frac{(BCv\Gamma_u)^n}{n!}\;\frac{(Bu)^j}{j!} \;+\; \sum_{m=0,k=0}^{\infty} \frac{(BCu\Gamma_v)^m}{m!}\;\frac{(Cv)^k}{k!}\;. \quad (46)$$

It is easy to obtain the operation of Γ^n on powers of u,

$$\Gamma_u^n u^j \;=\; \frac{u^{j+n}}{(j+1)(j+2)\,\ldots\,(j+n)}\;. \quad (47)$$

We substitute this into (46), and rearrange summation indices to find

$$\left[\exp\{BCv\Gamma_u\}\,\exp\{Bu\} \;+\; \exp\{BCu\Gamma_v\}\,\exp\{Cv\}\right]$$

$$= \sum_{n\le j}^{\infty} \frac{(Cv)^n}{n!}\;\frac{(Bu)^j}{j!} \;+\; \sum_{m\ge k}^{\infty} \frac{(Cv)^m}{m!}\;\frac{(Bu)^k}{k!}$$

$$= \sum_{n=0,j=0}^{\infty} \frac{(Cv)^n}{n!}\;\frac{(Bu)^j}{j!} \;+\; \sum_{m=0,k=0}^{\infty} \frac{(Cv)^m}{m!}\;\frac{(Bu)^k}{k!} \;+\; \sum_{n=0}^{\infty} \frac{(-\mu^2 uv)^2}{(n!)^2}$$

$$= \exp\{Bu + Cv\} + F_c(uv)\;. \quad (48)$$

We now combine (40) and (44) to get the complete solution

$$\psi(u,v) = \exp\{Bu + Cv\}\;\; w(\lambda,\tau) = \exp\{-i\lambda u - i\tau v\}\;\; w(\lambda,\tau)\;, \quad (49)$$

which is the correct plane wave solution. We have thus checked the consistency of a known solution with our formalism. It follows, moreover, that the evolution

operator equation (40) will produce the correct solution in any case that may be expanded as a superposition of plane waves.

We notice that the constant terms in the initial data expressions (36),(37) play a very important role in the formalism, and cannot be ignored.

VII. Relation to the Hamiltonian Viewpoint

The evolution operators in (40) are close symbolic analogues of the usual evolution operator in expression (3), as we will discuss in some detail.

A Taylor series expansion in t of a function may be expressed symbolically as an exponential, as follows:

$$\psi(\vec{x},t) \;=\; \exp\left\{(t-t_0)\;\frac{\partial}{\partial t}\right\}\;\psi(\vec{x},t_0) \;\equiv\; \sum_{n=0}^{\infty}\;\frac{(t-t_0)^n}{n!}\;\psi_{|nt}\,(\vec{x},t)_{t=t_0}\;; \qquad (50)$$

Notice that the quantity in the exponent generates a time shift. In standard quantum mechanics, the time derivative is related to the energy operator by $\partial/\partial t = -iE$. To define the dynamics, the energy operator in terms of t is identified with the Hamiltonian operator H in terms of x; It is postulated that this is a Schroedinger equation, with an operator equivalence

$$\frac{\partial}{\partial t} = -iH\;. \qquad (51)$$

Then, for a wave function, the Taylor series expression reads

$$\psi(\vec{x},t) = \exp\left\{-i(t-t_0)H\right\}\;\;\psi(\vec{x},t_0)\;, \qquad (52)$$

which is the standard form for the time evolution operator, as in (3).

We can use the same procedure to obtain our result (40), but only symbolically, and up to a normalization constant and constant terms. We begin with an expression for a double Taylor series expansion of a function of u and v in analogy with (50).

$$\psi(u,v) \;=\; \frac{1}{2}\left[\exp\left\{(v-v_0)\;\frac{\partial}{\partial v}\;\;\psi(u,v_0)\;\;+\;\;\exp\left\{(u-u_0)\;\frac{\partial}{\partial u}\;\;\psi(u_0,v)\right\}\;\;.\;\;(53)\right.\right.$$

In analogy with the Schroedinger equation, we define the dynamics with the Klein Gordon equation.In terms of u and v, this is

$$\frac{\partial^2}{\partial u \partial v}\;\psi(u,v) \;=\; -\frac{m^2}{4}\;\psi(u,v)\;, \tag{54}$$

which solutions of the Dirac equation must obey. In analogy with (51). this allows us to identify the v derivative operator with the following operator in terms of u,

$$\frac{\partial}{\partial v} \;=\; -\frac{m^2}{4}\;(\frac{\partial}{\partial u})^{-1} \;\equiv\; -\frac{m^2}{4}\;\Gamma_u\;. \tag{55}$$

Substituting this into (53), we obtain (40)— if we do not concern ourselves about the normalization factor and the constant terms, which the symbolic derivation does not seem to explain.

In conclusion, our main result (40) presents the evolution of a spinor field in terms of two analogues of the Hamiltonian operator, and the dynamics of the spinor field are remarkably similar to that of the scalar field.

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