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Real Part of the Impedance for a Smooth Taper*

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Abstract

Real part of the transverse and longitudinal impedance of a pipe with slowly varying radius is calculated based on the energy radiated by the beam in the transition region.

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I. INTRODUCTION

In this paper we calculate the real part of the impedance for a smooth axisymmetric taper whose radius slowly varies along the axis. Previously Yokoya found a purely imaginary longitudinal and transverse impedances for such structures [1]. This, however, does not mean that the real part of the impedance vanishes identically; it only indicates that, in the domain of validity of Yokoya's result, it is much smaller than the imaginary part. For a small variation Δb of the pipe radius b, both $\operatorname{Im} Z_l(\omega)$ and $\operatorname{Re} Z_l(\omega)$ have been found in Ref. [2]. We generalize the result of Ref. [2] for arbitrary Δb, but limit our consideration to the real part of the impedance only.

The method used in this paper is based on the WKB theory of waveguide modes in smooth structures. General treatment of WKB approximation for Waveguides can be found in Ref. [3]. We calculate the amplitudes of waves generated by the beam in the taper and relate the radiated energy to the real part of the impedance. A similar approach was previously used for the calculation of the real part of the impedance of slots and holes in a vacuum chamber [4].

We present a detail derivation of the real part of the longitudinal impedance and discuss applicability limits of the theory. The derivation of the transverse impedance can be performed in the same fashion; in this paper we will give the final results omitting the lengthy derivation.

II. RADIATION CAUSED BY THE TANGENTIAL ELECTRIC FIELD AT THE WALL

We begin our consideration with a straight circular perfectly conducting pipe of constant radius b with a localized voltage $V\exp(-i\omega t)$ applied to the wall at the location $z = z_0$ and oscillating with the frequency ω. Mathematically, this situation corresponds to the following boundary condition,

$$E_z\big|_{r=b} = V\delta(z - z_0), \tag{1}$$

where z is the coordinate along the axis of the pipe, and we omit the factor $\exp(-i\omega t)$ throughout this paper. Equation (1) reads that the tangential electric field at the wall vanishes everywhere except the point $z = z_0$ where it has a singularity such that $\int E_z dz = V$.

The solution of the Maxwell equations inside the pipe with the boundary condition (1) can be found in terms of the axisymmetric TM modes:

$$E_z^{(m)} = \frac{j_{0m}^2}{b^2} J_0\left(j_{0m} \frac{r}{b} \right) \exp\left(\sigma i \varphi_{0m}(z)\right),$$

$$E_r^{(m)} = -\sigma \frac{i j_{0m} \kappa_{0m}}{b} J_1\left(j_{0m} \frac{r}{b} \right) \exp\left(\sigma i \varphi_{0m}(z)\right), \tag{2}$$

$$H_\theta^{(m)} = -\frac{i j_{0m} \omega}{bc} J_1\left(j_{0m} \frac{r}{b} \right) \exp\left(\sigma i \varphi_{0m}(z)\right),$$

where J_0 and J_1 are the Bessel functions of the zeroth and first order, j_{0m} is the *m*th root of J_0, b is the radius of the waveguide, $\varphi_m(z) = \kappa_{0m} z$, $\kappa_{0m} = \sqrt{\omega^2 - \omega_{0m}^2}/c$, where $\omega_{0m} = c j_{0m}/b$ is the cutoff frequency for the *m*th mode. The variable σ denotes the direction of propagation of the wave; $\sigma = +1$ corresponds to the modes propagating in the positive direction along the *z*-axis, and $\sigma = -1$ marks the waves traveling in the opposite direction. Below the cutoff frequency, $\omega < \omega_{0m}$, κ_{0m} is purely imaginary with a positive imaginary part $\operatorname{Im} \kappa_{0m} > 0$.

The electromagnetic field generated by the voltage V in Eq. (1) can be found using the results of Ref. [5]. It is given by the combination of the TM modes with the amplitudes a_m propagating in the positive and negative directions,

$$F = h(z - z_0) \sum_{m=1}^{\infty} a_m^{(+)} F^{(m)}(r, z, \sigma = 1) + h(-z + z_0) \sum_{m=1}^{\infty} a_m^{(-)} F^{(m)}(r, z, \sigma = -1), \tag{3}$$

where $h(z)$ is the step function, F denotes any of the components E_z, E_r, or H_θ, and $a_m^{(+)}$ and $a_m^{(-)}$ refer to the waves propagating in the positive and negative directions, respectively. For the amplitudes a_m one finds,

$$a_m^{(\sigma)}(z_0) = \frac{iV}{\kappa_{0m} j_{0m} J_1(j_{0m})} \exp\left(-\sigma i \kappa_{0m} z_0\right). \tag{4}$$

As an illustration, in Fig. 1 we plot the electric field E_z on the axis for two different values of the frequency ω.

It is clear that the solution given by Eqs. (3) and (4) plays a role of a Green function and allows us to find the fields for an arbitrary axisymmetric distribution of the tangential electric field at the wall.

III. A BEAM IN A PIPE WITH A VARYING RADIUS

We now consider a relativistic beam propagating in the pipe with a gradually varying radius $b(z)$, as shown in Fig. 3. We assume that the angle between the wall of the pipe and the axis is small, $b'(z) \ll 1$. The beam will be represented by an oscillating current traveling with the speed of light along the axis of the pipe,

$$I(z, t) = I_0 \exp(ikz), \tag{5}$$

where $k = \omega/c$.

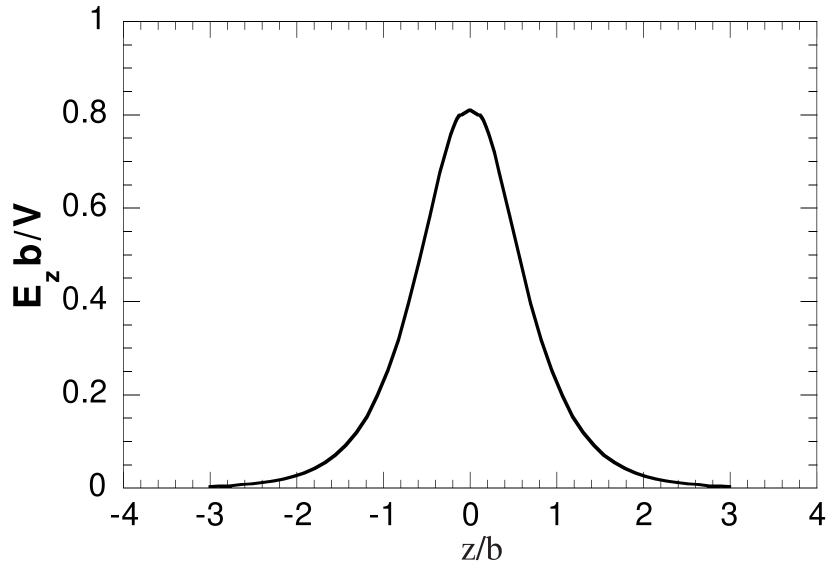

Fig. 1. Electric field E_z on the axis for $\omega b/c = 1$. The field exponentially decays away from the source because the frequency ω is below the cutoff frequency.

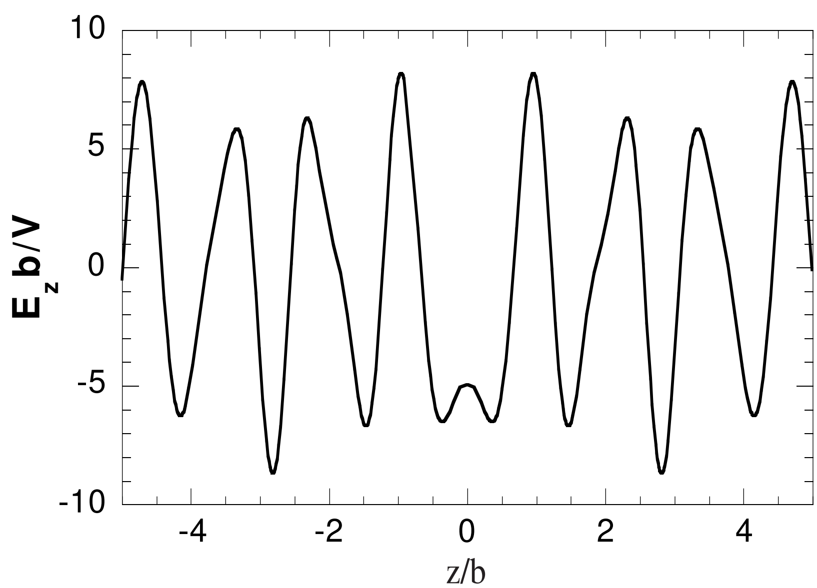

Fig. 2. Electric field E_z on the axis for $\omega b/c = 10$. Two waveguide modes, TM_{01} and TM_{02}, are excited and propagate away from the source in positive and negative directions.

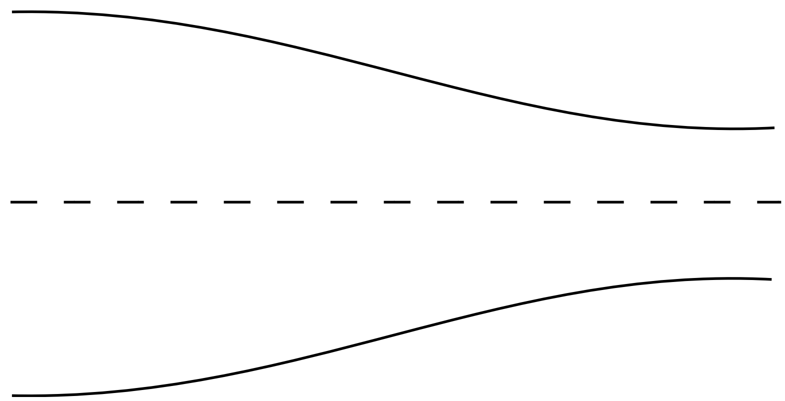

Fig. 3. A smooth taper.

The electromagnetic field generated by the beam in the pipe, $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$, can be represented as a sum of the field that the beam has in the vacuum, $\mathbf{E}^{vac}(\mathbf{r})$, plus the radiation field produced by the currents in the pipe wall, $\tilde{\mathbf{E}}(\mathbf{r})$,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{vac}(\mathbf{r}) + \tilde{\mathbf{E}}(\mathbf{r}), \tag{6}$$

(and a similar expression for the magnetic field). The electric field $\mathbf{E}^{vac}(\mathbf{r})$ has a radial component at the wall which magnitude is equal to $E_r^{(vac)} = \left[2I_0/b(z)c\right]\exp(ikz)$. Because the surface of the wall has an angle b' with the z-axis, this field gives rise to the tangential component at the wall,

$$E_t^{vac} = \frac{2I_0}{cb} b' \exp(ikz). \tag{7}$$

In order to satisfy the boundary condition at the perfect conductor, $E_t = 0$, the tangential component of the radiation field at the wall must be equal to $-E_t^{vac}$,

$$\tilde{E}_t = -E_t^{vac} = -\frac{2I_0}{cb} b' \exp(ikz). \tag{8}$$

The problem thus reduces to finding the solution of the Maxwell equations subject to the boundary condition given by Eq. (8). We will find this solution in the first approximation to the small parameter b' using the Green function derived in Section II.

IV. SHALLOW TAPER

Before addressing a general problem of arbitrary cross function $b(z)$, we consider first a shallow taper for which the change of the pipe radius is small,

$$b(z) = b_0 + \Delta b(z), \quad \Delta b \ll b. \tag{9}$$

Applying the boundary condition (8) to this case, in the first approximation, we can neglect the variation of the pipe radius and require that Eq. (8) be satisfied at the surface of the pipe with a radius equal to b_0. The solution can be easily found with the use of the Green function from Section I,

$$\tilde{\mathbf{E}}(\mathbf{r}) = \sum_m \left[c_m^{(+)}(z)\mathbf{E}^{(m)}(\mathbf{r},\sigma=1) + c_m^{(-)}(z)\mathbf{E}^{(m)}(\mathbf{r},\sigma=-1) \right], \tag{10}$$

where $\mathbf{E}^{(m)}(\sigma=\pm 1)$ is the electric field from Eq. (2), and

$$\begin{aligned} c_m^{(+)}(z) &= \frac{i}{\kappa_{0m} j_{0m} J_1(j_{0m})} \int_{-\infty}^{z} \tilde{E}_t(z')\exp(-i\kappa_{0m}z')dz', \\ c_m^{(-)}(z) &= \frac{i}{\kappa_{0m} j_{0m} J_1(j_{0m})} \int_{z}^{\infty} \tilde{E}_t(z')\exp(i\kappa_{0m}z')dz'. \end{aligned} \tag{11}$$

Having found the electric field in the pipe, one can try to find the longitudinal impedance using the definition

$$Z_l = -\frac{1}{I_0}\int_{-\infty}^{\infty} dz E_z(z, r=0)\exp(-ikz). \tag{12}$$

This will however give a trivial result

$$Z_l = \frac{Z_0}{2\pi b_0}\left[\Delta b(\infty) - \Delta b(-\infty)\right], \tag{13}$$

which is due to the change of the cross section of the pipe and vanishes in the case of a collimator for which $\Delta b(\infty) = \Delta b(-\infty)$. It turns out that a nontrivial part of the impedance is proportional to Δb^2, and a straightforward way to find it would be to develop a second order of the perturbation theory for the electromagnetic field, and then to use Eq. (12). We, however, choose a simpler approach that, though does not allow to find the imaginary part of the impedance, will give us $\operatorname{Re} Z_l$ without going to the second order theory (cf. Ref. [4]). The approach is based on the exact relation between the power P radiated into the waveguide due to the presence of the taper and the real part of the impedance,

$$P = \frac{1}{2} I_0^2 \operatorname{Re} Z(\omega). \tag{14}$$

In our case, the radiated power is

$$P = \sum_m P_{0m}^{(TM)}\left(\left| c_m^{(+)}(z=\infty) \right|^2 + \left| c_m^{(-)}(z=-\infty) \right|^2 \right), \tag{15}$$

where $P_{0m}^{(TM)} = \frac{1}{8}\omega\kappa_{0m} j_{0m}^2 J_1^2(j_{0m})$ is the energy flow in the TM mode of unit amplitude, and the sum carries over the propagating modes only. The amplitudes of outgoing waves at $z \to \pm\infty$ are

$$c_m^{(\pm)}(z=\pm\infty) = -\frac{2iI_0}{cb\kappa_{0m} j_{0m} J_1(j_{0m})}\hat{b}'(k \mp \kappa_{0m}), \tag{16}$$

where the cap denotes the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} dz f(z) \exp(ikz). \tag{17}$$

This gives the following result for the impedance,

$$\operatorname{Re} Z_l(\omega) = \frac{Z_0 k}{4\pi b_0^2} \sum_m \frac{1}{\kappa_{0m}} \left(\left| \hat{b}'(k - \kappa_{0m}) \right|^2 + \left| \hat{b}'(k + \kappa_{0m}) \right|^2 \right). \tag{18}$$

For any given frequency, the contribution to $\operatorname{Re} Z_l$ comes only from the finite number of modes whose cutoff frequencies are $\omega_{0m} < \omega$.

The impedance (18) agrees completely with the Warnock result [2].

V. WKB APPROXIMATION IN A WAVEGUIDE WITH A SLOWLY VARYING RADIUS

To lift the assumption of the shallowness of the taper we need to consider the propagation of the eigenmodes in a waveguide with a varying radius. In general, for an arbitrary function $b(z)$, the electromagnetic waves given by Eq. (2) do not provide a correct representation of the eigenmodes in this case. A TM mode incident on the taper from a straight pipe will experience a transformation to other modes and a possible reflection if the cutoff frequency at the smallest cross section of the taper is higher than the frequency of the mode. However, if the transition is smooth and the conversion into other modes can be neglected, the parameters of the propagating mode adiabatically adjust to the local value of the radius and the mode keeps its identity. The propagation of the mode in this case can be described within the framework of the WKB approximation which is valid if the inverse wavenumber of the mode, $|\kappa_{0m}|^{-1}$, is much smaller than the characteristic length l of the taper. Typically, $|\kappa_{0m}|^{-1} < b$, and the condition $|\kappa_{0m}|^{-1} \ll l$ is automatically satisfied for $b \ll l$. Only for modes that are close to the cutoff frequency, corresponding to $\kappa_{0m} \to 0$, the WKB approximation fails and a more accurate treatment is needed.

In the WKB approximation, the longitudinal wavenumber of the mode, κ_{0m}, is a function of position z,

$$\kappa_{0m}(z) = \sqrt{k^2 - \frac{j_{0m}^2}{b^2(z)}}, \tag{19}$$

and the corresponding phase φ_{0m} is given by

$$\varphi_{0m}(z) = \int^{z} \kappa_{0m}(z') dz', \tag{20}$$

where the lower limit can be chosen arbitrarily. The cutoff frequency of the mode is also a function of the position, $\omega_{0m}(z) = c j_{0m} / b(z)$. The structure of the mode is obtained from Eq. (2) by substituting $b(z)$ for b, and adding an amplitude factor $a(z)$ which is also a function of z (for the sake of brevity, we write here only the equation for the z-component of the electric field),

$$E_z^{(m)} = a(z)\frac{j_{0m}^2}{b^2(z)}J_0\left(j_{0m}\frac{r}{b(z)}\right)\exp\left(i\sigma\varphi_{0m}(z)\right). \tag{21}$$

The amplitude $a(z)$ turns out to be inversely proportional to the square root of the longitudinal wavenumber κ_{0m} [3],

$$a(z) = \frac{\text{const}}{\sqrt{\kappa_{0m}(z)}}. \tag{22}$$

This dependence can be easily understood from the fact that the power flow in the wave, $P_{0m} \propto |a_{0m}|^2 \kappa_{0m}$, must be constant at each cross section of the pipe.

The above equations are applicable in the region where $\omega > \omega_{0m}(z)$. When a wave approaches a turning point z_r at which $\omega = \omega_{0m}(z_r)$, it will be reflected in the opposite direction. As a result of the reflection, the wave gets a phase advance $-\pi/2$ [3]. We can write down the correspondence between the incident and reflected waves in the following form

$$\begin{aligned} &a(z)\frac{j_{0m}^2}{b^2(z)}J_0\left(j_{0m}\frac{r}{b(z)}\right)\exp\left[i\sigma\left(\varphi_{0m}(z) - \varphi_{0m}(z_r)\right)\right] \to \\ &a(z)\frac{j_{0m}^2}{b^2(z)}J_0\left(j_{0m}\frac{r}{b(z)}\right)\exp\left[-i\sigma\left(\varphi_{0m}(z) - \varphi_{0m}(z_r)\right) - i\frac{\pi}{2}\right]. \end{aligned} \tag{23}$$

The above consideration assumes that the wave propagating in the waveguide with varying cross section changes its amplitude but does not transform into the other modes. This impose an important requirement on the smoothness of the taper. Namely, in order to be able to neglect the transformation effects, the length of the transition region should satisfy the following condition [3]

$$kb^2 \ll l, \tag{24}$$

(we assume here that variation of the radius Δb is of the order of b). For a given taper, Eq. (24) will determine the frequency range for which the results of this paper will be applicable.

VI. GREEN FUNCTION IN WKB APPROXIMATION AND LONGITUDINAL IMPEDANCE

We are now in a position to derive a Green function corresponding to the solution of the Maxwell equations for a long taper with the boundary condition given by Eq. (1). The result can be obtained from the following argument. Because of the smoothness of the pipe, the amplitude of the TM_{0m} modes radiated by the source (1) in its vicinity (that is at the distances of several local radii, but much smaller than the length of the taper) is given by Eq. (4) in which $\kappa_{0m} = \kappa_{0m}(z_0)$ is the local value of the wave number. Traveling from the point z_0 to the point z, the wave will change its phase and amplitude in accordance with Eqs. (21) and (22). This results in the following expression for the amplitude of the wave $a_m(z, z_0)$ at the position z,

$$a_m(z,z_0)=\frac{iV}{\sqrt{\kappa_{0m}(z)\kappa_{0m}(z_0)}\,j_{0m}J_1(j_{0m})}\exp\left[-\sigma i\left(\varphi_{0m}(z)-\varphi_{0m}(z_0)\right)\right]. \tag{25}$$

We can now repeat the arguments of Section III and find that the radiation of the beam (5) is given by the same Eq. (10) with the amplitudes

$$c_m^{(\pm)}(z)=-\frac{2iI_0}{cb\sqrt{\kappa_{0m}(z)}\,j_{0m}J_1(j_{0m})}\int d\zeta\,\frac{b'(\zeta)}{b(\zeta)\sqrt{\kappa_{0m}(\zeta)}}\exp\left[ik\zeta\mp i\left(\varphi_{0m}(\zeta)-\varphi_{0m}(z_0)\right)\right] \tag{26}$$

(we remind that in Eq. (2) one now has to use the phase $\varphi_{0m}(z)$ from Eq. (20) rather than $\kappa_{0m}z$). The integration in Eq. (26) runs from $-\infty$ to z for $c_m^{(+)}$, and from z to ∞ for $c_m^{(-)}$.

Using Eq. (26) we can find the energy radiated by the beam in the transition region, Eq. (15), and the real part of the impedance, Eq. (14). The result is

$$\begin{aligned}\operatorname{Re} Z_l^{prop}(\omega)=\frac{Z_0k}{4\pi}\sum_m\Bigg(&\left|\int_{-\infty}^{\infty}d\zeta\,\frac{b'(\zeta)}{b(\zeta)\sqrt{\kappa_{0m}(\zeta)}}\exp\left[ik\zeta-i\varphi_{0m}(\zeta)\right]\right|^2\\ &+\left|\int_{-\infty}^{\infty}d\zeta\,\frac{b'(\zeta)}{b(\zeta)\sqrt{\kappa_{0m}(\zeta)}}\exp\left[ik\zeta+i\varphi_{0m}(\zeta)\right]\right|^2\Bigg).\end{aligned} \tag{27}$$

The superscript "prop" indicates that the impedance is due to the modes that propagate trough the taper, that is $\min(b(z))>j_{0m}/k$. There may also be modes that are reflected by the taper, for which $\min[b(z)]<j_{om}/k<\max[b(z)]$. Contribution of those modes is given by the following expression,

$$\operatorname{Re} Z_l^{refl}(\omega)=\frac{Z_0k}{\pi}\sum_m\left|\int_{-\infty}^{z_r}d\zeta\,\frac{b'(\zeta)}{b(\zeta)\sqrt{\kappa_{0m}(\zeta)}}\exp(ik\zeta)\cos\left(\varphi_{0m}(\zeta)-\varphi_{0m}(z_r)+\frac{\pi}{4}\right)\right|^2. \tag{28}$$

The total impedance $\operatorname{Re} Z_l$ is the sum of Eqs. (27) and (28),

$$\operatorname{Re} Z_l(\omega)=\operatorname{Re} Z_l^{prop}(\omega)+\operatorname{Re} Z_l^{refl}(\omega). \tag{29}$$

VII. REAL PART OF THE TRANSVERSE IMPEDANCE

Calculation of the real part of the transverse impedance can be performed similar to the longitudinal one. One has to find the energy of the dipole TE and TM modes radiated by the beam into the pipe while passing through the taper. Here we write down the expressions for $\operatorname{Re} Z_t$ without derivation.

First, we need to define a local wavenumber for TM_{1m} and TE_{1m} modes propagating in a pipe with varying radius,

$$k_{1m}^{(TM)}(z)=\sqrt{k^2-j_{1m}^2/b(z)^2},\ k_{1m}^{(TE)}(z)=\sqrt{k^2-j_{1m}'^2/b(z)^2}, \tag{30}$$

where j_{1m} and j'_{1m} are the mth roots of the Bessel function J_1 and its derivative J'_1 respectively. Introducing the phase of a mode as an integral of the wavenumber,

$$\varphi_{1m}^{(TM)}(z) = \int_0^z k_{1m}^{(TM)}(\zeta)d\zeta\,, \tag{31}$$

and a similar expression for $\varphi_{1m}^{(TE)}(z)$, we can write the real part of the impedance in the following form,

$$\operatorname{Re} Z_t(\omega) = Z_t^{prop}(\omega) + Z_t^{refl}(\omega), \tag{32}$$

where

$$Z_t^{prop}(\omega) = Z_{TE+}^{prop}(\omega) + Z_{TE-}^{prop}(\omega) + Z_{TM+}^{prop}(\omega) + Z_{TM-}^{prop}(\omega), \tag{33}$$

and

$$Z_t^{refl}(\omega) = Z_{TE}^{refl}(\omega) + Z_{TM}^{refl}(\omega). \tag{34}$$

Each term in Eqs. (33) and (34) is associated with the energy radiated into TE or TM modes propagating in the positive (+) or negative (-) directions. The superscripts "prop" and "refl" refer to the modes that can either propagate through the taper or will be reflected back due to the increase of the cutoff frequency in the narrow part of the taper. The expressions for each term in Eqs. (33) and (34) are

$$Z_{TE\pm}^{prop}(\omega) = \frac{Z_0}{2\pi}\sum_m \left| \int_{-\infty}^{\infty} \frac{b'(z)dz}{b^2(z)\sqrt{k_{1m}^{(TE)}(z)}} e^{ikz \pm i\varphi_{1m}^{(TE)}(z)} \right|^2, \tag{35}$$

$$Z_{TM\pm}^{prop}(\omega) = \frac{Z_0}{2\pi}\sum_m \frac{1}{j_{1m}'^2 - 1} \left| \int_{-\infty}^{\infty} \frac{b'(z)dz}{b^2(z)\sqrt{k_{1m}^{(TM)}(z)}} e^{ikz \pm i\varphi_{1m}^{(TM)}(z)} \right|^2, \tag{36}$$

$$Z_{TE}^{refl}(\omega) = \frac{2Z_0}{\pi}\sum_m \left| \int_{z_r}^{\pm\infty} \frac{b'(z)dz}{b^2(z)\sqrt{k_{1m}^{(TE)}(z)}} e^{ikz} \cos\left(\varphi_{1m}^{(TE)}(z) - \varphi_{1m}^{(TE)}\left(z_r^{(TE)}\right) - \frac{\pi}{4} \right) \right|^2, \tag{37}$$

$$Z_{TM}^{refl}(\omega) = \frac{2Z_0}{\pi}\sum_m \frac{1}{j_{1m}'^2 - 1} \left| \int_{z_r'}^{\pm\infty} \frac{b'(z)dz}{b^2(z)\sqrt{k_{1m}^{(TM)}(z)}} e^{ikz} \cos\left(\varphi_{1m}^{(TM)}(z) - \varphi_{1m}^{(TM)}\left(z_r^{(TM)}\right) - \frac{\pi}{4} \right) \right|^2 . \tag{38}$$

where $z_r^{(TE)}$ and $z_r^{(TM)}$ are the points at which $k_{1m}^{(TM)} = 0$ and $k_{1m}^{(TE)} = 0$, respectively. The summation in Eqs. (35) and (36) carries over the modes for which $\min(b(z)) > j_{1m}/k$, and $\min(b(z)) > j'_{1m}/k$, respectively. Similarly, the summation in Eqs. (37) and (38) is performed over the modes for which $\min[b(z)] < j_{1m}/k < \max[b(z)]$, and $\min[b(z)] < j'_{1m}/k < \max[b(z)]$, respectively. The evanescent modes, such that j_{1m}/k, $j'_{1m}/k > \max[b(z)]$ do not contribute to the real part of the impedance.

In the limit of a shallow taper considered in Section IV, $\operatorname{Re} Z_t$ can be simplified. First, the contribution of the reflected modes can be neglected. Second, $\varphi_{1m}^{(TE)}(z)$ and

$\varphi_{1m}^{(TM)}(z)$ can be substituted by $k_{1m}^{(TE)}z$ and $k_{1m}^{(TM)}z$, respectively, where the $k_{1m}^{(TE)}$ and $k_{1m}^{(TM)}$ can be considered as not depending on *z*. Finally, $b(z)$ can replaced by b_0. The result

$$\operatorname{Re} Z_t(\omega) = \frac{Z_0}{2\pi b_0^4}\left\{\sum_m \frac{1}{k_{1m}^{(TM)}\left(j_{1m}'^2 - 1\right)}\left[\left|\int_{-\infty}^{\infty} dz b'(z) e^{ikz + ik_{1m}^{(TM)}z}\right|^2 + \left|\int_{-\infty}^{\infty} dz b'(z) e^{ikz - ik_{1m}^{(TM)}z}\right|^2\right] + \right.$$
$$\left. \sum_m \frac{1}{k_{1m}^{(TE)}}\left[\left|\int_{-\infty}^{\infty} dz b'(z) e^{ikz + ik_{1m}^{(TE)}z}\right|^2 + \left|\int_{-\infty}^{\infty} dz b'(z) e^{ikz - ik_{1m}^{(TE)}z}\right|^2\right]\right\}, \tag{39}$$

agrees with [6].

VIII. ACKNOWLEDGMENT

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