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Renormalization Scale Setting for Evolution Equation of Non-Singlet Structure Functions and Their Moments

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Abstract

We use the BLM procedure to eliminate the renormalization scale ambiguity in the evolution equation for the non-singlet deep-inelastic structure function $F_2^{NS}(x,Q)$. The scale of the QCD coupling in the $\overline{\mathrm{MS}}$ scheme has the form $Q^*(x) = Q(1-x)^{1/2}/x\ f(x)$, where x is the Bjorken variable and $f(x)$ is a smoothly varying function bounded between 0.30 to 0.45. Equivalently, the evolution of the nth moment of the structure function should contain an effective Λ_{QCD} pattern, with $\Lambda_n \sim n^{1/2}$. This variation of Λ_n agrees with experimental data.

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I. INTRODUCTION

One of the most serious problems preventing precise empirical tests of QCD is the ambiguity of the renormalization scale μ of perturbative predictions. Formally, any physical quantity should be μ independent. However, in practice, spurious μ dependence appears as one can only have finite-order predictions in a perturbative theory. Unless one specifies the argument of the coupling $\alpha_s(\mu)$ in the truncated predictions, the range of theoretical uncertainty can be much larger than the experimental error. Although the uncertainty of the prediction contributed from the renormalization scale ambiguity is expected to reduce as the order of the prediction gets higher, we still need a rational way of choosing a value for μ such that the truncated prediction will approximate the true prediction as close as possible.

A conventional way of setting the renormalization scale in perturbation theory is setting μ to be the momentum transfer or energy scale Q of the system (provided there is only one energy scale); this eliminates large logarithmic terms $\ln(Q/\mu)$ and so gives a more convergent series. However, often an order-1 variation of μ/Q leads to a significant uncertainty in the prediction of perturbative QCD.

In multi-scale processes, the problem of scale setting becomes compounded, since the choice for μ can depend on any combination of the available physical scales. An example of this is the proper scale for the running coupling constant that appears in the DGLAP evolution equation for the deep-inelastic structure function. In addition to the momentum transfer Q of the lepton, the physical scale can also depend on the Bjorken ratio $x = Q^2/(2p \cdot q)$. Equivalently, the scale controlling the evolution of each moment $M_n(Q)$ of the structure function can depend on both Q and n.

Collins [1], Neubert [2], and Lepage and Mackenzie [3] have emphasized that the renormalization scale should not be fixed by an *ad hoc* procedure but rather, should be determined systematically as the mean virtuality of the underlying physical subprocess. From this point of view, the choice of the renormalization scale μ for a particular prediction to a certain order thus depends on the specific experimental measure and the truncation order.

In this paper, we discuss how to obtain the optimal scale for the structure function evolution using the BLM (Brodsky-Lepage-Mackenzie) [4] scale-fixing procedure. In this procedure, the vacuum polarization diagrams that contribute to the non-zero QCD beta function are resummed into the running coupling. More technically, we absorb into the running coupling all factors of the number of flavors n_f that appear in the coefficients of the perturbative expansion. This criterion automatically sets the scale to a value that reflects the average gluon virtuality of the subprocess.

On the one hand, we can calculate the evolution of the moment by fixing the scale as a function of Q and x before integrating over x to obtain the moments; on the other hand, we can set the scale from the moment-evolution equation for each n. We give the appropriate scale for the moment-evolution equation for each n and show that the two procedures give the same result to the order that we consider.

II. THE BLM SCALE-FIXING PROCEDURE

The crucial idea of BLM procedure [4,5] is that, in the α_V scheme [α_V is defined through the heavy quark potential $V(Q^2)$ by $V(Q^2) = -4\pi C_F \alpha_V(Q)/Q^2$], the scale must be chosen to absorb all the vacuum polarization (non-zero beta function) contributions into the running coupling constant. Thus one has to choose the argument of the coupling constant in each order of perturbation theory so that there is no n_f dependence in the coefficients of the coupling constants (those light-by-light contributions that are not associated with renormalization are not resummed). This is a good choice of scale since all the vacuum-polarization contributions are then automatically summed to all orders for any finite-order prediction. For a next-to-leading order (NLO) perturbative prediction in *any* renormalization scheme, the procedure simply translates to choosing the scale such that there is no n_f dependence in the NLO coefficient of the coupling constant, by the transitivity of the procedure to this order [5].

To show explicitly how to set the scale for an NLO perturbative QCD prediction, consider a prediction in any scheme of the following form,

$$\rho = \rho_0 \alpha(\mu) \left\{ 1 + [A(\mu) n_f + B(\mu)] \frac{\alpha(\mu)}{\pi} \right\} + O(\alpha^3), \tag{1}$$

where μ is the renormalization scale. The parameters ρ, ρ_0, A and B are all in general dependent on one or more physical scales such as center-of-mass energy, momentum transfer and Bjorken parameter x. The μ dependence of the coefficients $A(\mu)$ and $B(\mu)$, fixed by the renormalization group

$$\frac{d\rho}{d\mu} = 0 + O(\alpha^3) \tag{2}$$

are of the form

$$\begin{aligned} A(\mu) &= A' - \frac{1}{3} \ln(\mu) \\ B(\mu) &= B' + \frac{11}{6} C_A \ln(\mu) \ , \end{aligned} \tag{3}$$

where A' and B' are μ independent.

We apply the BLM procedure to eliminate the n_f and β_0 dependence in Eq. (1). We choose the renormalization scale μ to be

$$Q^* = \mu \exp(3A(\mu)), \tag{4}$$

so that

$$A(Q^*) = 0 \tag{5}$$

[note that the right-hand side of Eq. (4) is μ independent by Eq. (3)].

Using the BLM scale, the perturbative QCD prediction becomes

$$\rho = \rho_0 \alpha(Q^*) \left[1 + B(Q^*) \frac{\alpha(Q^*)}{\pi} \right] + O(\alpha^3). \tag{6}$$

Rewrite it in terms of the original coefficients $A(\mu)$ and $B(\mu)$ using Eq. (3),

$$\rho = \rho_0 \alpha(Q^*) \left\{ 1 + \left[\frac{11}{2} C_A A(\mu) + B(\mu) \right] \frac{\alpha(Q^*)}{\pi} \right\} + O(\alpha^3). \tag{7}$$

This is the scale-fixed perturbative QCD prediction. With the chosen scale Q^*, all vacuum polarization is resummed into the running coupling constant, with the coefficient of α^2 independent of n_f.

In short, by comparing Eq. (1) and Eq. (7), setting the scale for an observable is simply equivalent to replacing n_f by $\frac{11}{2}C_A$, and using Q^* as the argument of the coupling constant (μ dependences in $A(\mu)$ and $B(\mu)$ cancel).

For a prediction with μ preset to any energy scale Q, as is usually given in the literature, the procedure for resetting the scale appropriately is identical to the one presented above except with Q replacing μ.

As we have seen, to obtain the renormalization scale only the n_f term of the NLO result is needed. Therefore, one can improve a leading-order result by setting an appropriate scale without calculating the full NLO prediction. Extensions of the method to higher orders are given in [5], [6], and [7].

III. SCALE SETTING OF THE STRUCTURE-FUNCTION EVOLUTION EQUATION

The non-singlet, structure-function evolution equation is [8]

$$Q^2 \frac{\partial \mathcal{F}_2^{\rm NS}}{\partial Q^2}(x,Q) = \int_x^1 \frac{dy}{y}\left[P_{qq}\left(\frac{x}{y},\alpha(Q)\right) \pm P_{\bar{q}q}\left(\frac{x}{y},\alpha(Q)\right)\right]\mathcal{F}_2^{\rm NS}(y,Q), \tag{8}$$

where, in terms of experimental quantities,

$$\mathcal{F}_2^{\rm NS} = \begin{cases} (F_2^{\rm eP} - F_2^{\rm eN})/x \ \ \text{for+ }, \\ (F_2^{\bar{\nu}\rm P} - F_2^{\nu\rm P})/x \ \ \text{for− }, \end{cases} \tag{9}$$

and $P_{qq} \pm P_{\bar{q}q}$ are the non-singlet evolution kernels, for which the NLO perturbative predictions in $\overline{\rm MS}$ scheme with the renormalization scale preset to momentum transfer Q are [9], [10], [12]:

$$P_{qq}(x,\alpha_{\overline{\rm MS}}(Q)) = \frac{\alpha_{\overline{\rm MS}}(Q)}{2\pi}C_F\left(\frac{1+x^2}{1-x}\right)_+ + \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{2\pi}\right)^2\left\{C_F^2\left[-2\left(\frac{1+x^2}{1-x}\right)\ln x\ln(1-x)\right.\right.$$

$$
\begin{aligned}
&-5(1-x)-\left(\frac{3}{1-x}+2x\right)\ln x-\frac{1}{2}(1+x)\ln^2 x\Big] \\
&+\frac{1}{2}C_F C_A\left[\left(\frac{1+x^2}{1-x}\right)\left(\ln^2 x-\frac{11}{3}\ln\frac{1-x}{x^2}+\frac{367}{16}-\frac{\pi^2}{3}\right)\right. \\
&\left.+2(1+x)\ln x+\frac{61}{12}-\frac{215}{12}x\right] \\
&\left.+\frac{2}{3}C_F T\left[\left(\frac{1+x^2}{1-x}\right)\left(\ln\frac{1-x}{x^2}-\frac{29}{12}\right)+\frac{1}{4}+\frac{13}{4}x\right]\right\}_+ \\
&+\delta(1-x)\int_0^1 dx P_{\bar{q}q}(x,\alpha_{\overline{\rm MS}}(Q))+O(\alpha_s^3).
\end{aligned} \tag{10a}
$$

and

$$
\begin{aligned}
P_{\bar{q}q}(x,\alpha_{\overline{\rm MS}}(Q)) = &\left(\frac{\alpha_{\overline{\rm MS}}(Q)}{2\pi}\right)^2 (C_F-\frac{1}{2}C_A)C_F\left[2(1+x)\ln x+4(1-x)\right. \\
&\left.+\frac{1+x^2}{1+x}\left(\ln^2 x-4\ln x\ln(1+x)-4{\rm Li}_2(-x)-\frac{\pi^2}{3}\right)\right],
\end{aligned} \tag{10b}
$$

with

$$
\begin{aligned}
T &= n_f/2 \\
C_A &= 3 \\
C_F &= 4/3 \\
{\rm Li}_2(x) &= -\int_0^x \frac{dy}{y}\ln(1-y).
\end{aligned} \tag{11}
$$

The + distribution notation serves as a regulator defined as

$$
\int_z^1 dx f(x)\left[\frac{g(x)}{1-x}\right]_+ = \int_z^1 dx \frac{(f(x)-f(1))g(x)}{1-x} - f(1)\int_0^z dx\frac{g(x)}{1-x}. \tag{12}
$$

Since all the coupling constants reside in the evolution kernels, we set the scale for the structure-function evolution equation by setting the scale for the kernels. When the kernels $P_{qq} \pm P_{\bar{q}q}$ have been recast in the form of Eq. (1), the coefficient of n_f can be obtained as

$$
A(x) = \frac{1}{6}\left[\ln\left(\frac{1-x}{x^2}\right)-\frac{29}{12}+\left(\frac{1-x}{1+x^2}\right)\left(\frac{1}{4}+\frac{13}{4}x\right)\right]. \tag{13}
$$

Using Eq. (4), BLM scale-fixing procedure gives for the BLM scale

$$
Q^*(x) = Q\exp(3A(x)) = Q\frac{(1-x)^{1/2}}{x}\ f(x), \tag{14}
$$

where

$$f(x) = \exp\left[\frac{1}{2}\left(\frac{1-x}{1+x^2}\right)\left(\frac{1}{4}+\frac{13}{4}x\right)-\frac{29}{24}\right] \tag{15}$$

is a smoothly varying function bounded between 0.30 and 0.45 as shown in Fig. 1.

Note that the above manipulation is done inside the + distribution notation. It is because after setting the scale to be dependent on x, the coupling constants must also be included in the + distribution notation in order to preserve the Adler sum rule.

Finally, the scale-fixed evolution kernels can be obtained by rewriting Eq. (10) in the form of Eq. (7), using the BLM scale given by Eq. (14):

$$\begin{aligned}
\tilde{P}_{qq}(x,\alpha_{\overline{\rm MS}}(Q^*(x))) = & \left\{\frac{\alpha_{\overline{\rm MS}}(Q^*(x))}{2\pi}C_F\left(\frac{1+x^2}{1-x}\right)+\left(\frac{\alpha_{\overline{\rm MS}}(Q^*(x))}{2\pi}\right)^2\right. \\
& \times\left\{C_F^2\left[-2\left(\frac{1+x^2}{1-x}\right)\ln x\ln(1-x)\right.\right. \\
& \left. -5(1-x)-\left(\frac{3}{1-x}+2x\right)\ln x-\frac{1}{2}(1+x)\ln^2 x\right] \\
& +\frac{1}{2}C_F C_A\left[\left(\frac{1+x^2}{1-x}\right)\left(\ln^2 x+\frac{2027}{144}-\frac{\pi^2}{3}\right)\right. \\
& \left.\left.\left. +\ 2(1+x)\ln x-6x+6\right]\right\}\right\}_+ \\
& +\delta(1-x)\int_0^1 dx\,\tilde{P}_{\bar{q}q}(x,\alpha_{\overline{\rm MS}}(Q^*(x)))+O(\alpha_s^3),
\end{aligned} \tag{16a}$$

and

$$\begin{aligned}
\tilde{P}_{\bar{q}q}(x,\alpha_{\overline{\rm MS}}(Q^*(x))) = & \left(\frac{\alpha_{\overline{\rm MS}}(Q^*(x))}{2\pi}\right)^2(C_F-\frac{1}{2}C_A)C_F\left[2(1+x)\ln x+4(1-x)\right. \\
& \left.+\frac{1+x^2}{1+x}\left(\ln^2 x-4\ln x\ln(1+x)-4{\rm Li}_2(-x)-\frac{\pi^2}{3}\right)\right]\ .
\end{aligned} \tag{16b}$$

The scale fixed-structure function evolution equation is then given by

$$Q^2\frac{\partial\mathcal{F}_2^{NS}}{\partial Q^2}(x,Q)=\int_x^1\frac{dy}{y}\left[\tilde{P}_{qq}\left(\frac{x}{y},\alpha_{\overline{\rm MS}}(Q^*(\frac{x}{y}))\right)\pm\tilde{P}_{\bar{q}q}\left(\frac{x}{y},\alpha_{\overline{\rm MS}}(Q^*(\frac{x}{y}))\right)\right]\mathcal{F}_2^{\rm NS}(y,Q). \tag{17}$$

This equation can also be written in terms of α_V using the relation [4]

$$\alpha_{\overline{\rm MS}}(\mu)=\alpha_V(\mu\exp(5/6))(1+2\alpha_V/\pi+\ldots). \tag{18}$$

IV. VERIFICATION OF THE SCALE

A good way to see the validity of the scale set for the structure-function evolution equation is to apply it to moment evolution and compare the results with experimental data. The moment analysis of the Fermilab muon deep-inelastic scattering data [11] showed an interesting variation of the effective $\Lambda_{\rm QCD}$ with n, both at leading and next-to-leading order. Here we would like to demonstrate that this variation can be readily explained by the choice of the renormalization scale even at leading order.

The nth ordinary moment of the non-singlet structure function is defined as:

$$M_n(Q^2) = \int_0^1 dx x^{n-1} \mathcal{F}_2^{\rm NS}(x, Q^2). \tag{19}$$

The Nachtman moments will give the same result up to higher twist terms.

Taking the nth moment of the structure function evolution equation (Eq. (17)), interchanging the order of integration on the right-hand side and rearranging, one can obtain the equation of evolution for the moment:

$$\frac{\partial \ln M_n(Q^2)}{\partial \ln Q^2} = \int_0^1 dx x^{n-1} \left(\tilde{P}_{qq}(x, \alpha(Q^*(x))) \pm \tilde{P}_{\bar{q}q}(x, \alpha(Q^*(x))\right), \tag{20}$$

which is just the nth moment of the evolution kernels.

Let us consider only the leading order prediction of the kernel function in Eq. (16a), then Eq. (20) becomes

$$\frac{\partial \ln M_n(Q^2)}{\partial \ln Q^2} = \int_0^1 \left(\frac{\alpha(Q^*(x))}{2\pi} P_{qq}^0(x) \right)_+ x^{n-1} dx, \tag{21}$$

here $Q^*(x)$ is given by Eq. (14), and

$$P_{qq}^0(x) = C_F \frac{1+x^2}{1-x} \tag{22}$$

the leading order coefficient.

The crucial difference between the conventional calculation, that is, using the momentum transfer Q as the renormalization scale, and the calculation here, namely using the BLM scale, is that the argument of the coupling constant in Eq. (21) is now x dependent.

To see the effect of the scale setting on the leading-order analysis, expand $\alpha(Q^*)$ in powers of $\beta_0\alpha(Q)$ (similar results can be obtained without the expansion), that is,

$$\alpha(Q^*(x)) = \alpha(Q)\left\{1 + \beta_0 \ln\left(\frac{Q}{Q^*(x)}\right)\frac{\alpha(Q)}{2\pi} + \dots\right\}, \tag{23}$$

then Eq. (21) becomes

$$\frac{\partial \ln M_n(Q^2)}{\partial \ln Q^2} = \frac{\alpha(Q)}{2\pi} A_n \left[1 + \beta_0 B_n \frac{\alpha(Q)}{2\pi} + \dots\right], \tag{24}$$

where

$$A_n = \int_0^1 dx x^{n-1} P^0_{qq}(x)_+, \text{ and} \tag{25}$$

$$B_n = \frac{1}{A_n}\int_0^1 dx x^{n-1}\left[P^0_{qq}(x)\ln\left(\frac{Q}{Q^*(x)}\right)\right]_+. \tag{26}$$

Now Eq. (24) can be integrated with respect to $\ln Q^2$ by using, to LO (to compare with LO experimental analysis),

$$\alpha(Q) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)}, \tag{27}$$

and the solution for $M_n(Q)$ is then

$$M_n(Q)^{-1/d_n} = C_n \ln(Q^2/\Lambda^2)\exp\left[-\frac{2B_n}{\ln(Q^2/\Lambda^2)}\right], \tag{28}$$

where $d_n = -2A_n/\beta_0$ and C_n are constants.

This is our prediction for the evolution of the moments.

For experimental data taken at large Q, the exponential term can be expanded and we have an equation linear in $\ln(Q^2/\Lambda^2)$:

$$M_n(Q)^{-1/d_n} = C_n\left[\ln(Q^2/\Lambda^2) - 2B_n\right]. \tag{29}$$

Had we set the scale to Q, we would have had

$$M_n(Q)^{-1/d_n} = C_n\left[\ln(Q^2/{\Lambda_n}^2)\right], \tag{30}$$

where Λ_n is an "effective" Λ at LO and will depend on n as predicted by Eq. (29) as

$$\Lambda_n = \Lambda / \exp(-B_n). \tag{31}$$

Values of $\exp(-B_n)$ for n from 2 to 10 are plotted in Fig. 2; these are well-described by the form

$$\exp(-B_n) = 0.656/n^{1/2}. \tag{32}$$

Therefore, we expect the dependence of Λ_n on n to be

$$\Lambda_n = \frac{n^{1/2}}{0.656}\Lambda. \tag{33}$$

In fact, the data of Gordon *et al* does show the predicted variation. In Fig. 3 we show the results extracted for Λ_n against n. A fit to Eq. (33) yields $\Lambda_{\overline{\rm MS}} = 170 \pm 120$ MeV, where the error is estimated from higher-order terms in Eq. (23).

V. SCALE SETTING FOR THE MOMENT-EVOLUTION EQUATION

The BLM scale fixing procedure can also be applied to the moment-evolution equation *after* the moment integration has been carried out, with evolution kernels given by Eq. (10). When the resulting equation is written in the form of Eq. (1), the coefficient of n_f is given by

$$A = \frac{\int_0^1 dx x^{n-1} \left(P^0_{qq} A(x)\right)_+}{A_n}, \tag{34}$$

where $A(x)$ is given by Eq. (13). This is related to our previously defined B_n as

$$A = -\frac{B_n}{3}. \tag{35}$$

Again, using Eq. (4), the BLM scale for the moment-evolution equation is

$$Q^*_n = Q \exp(-B_n). \tag{36}$$

Now the scale is x independent but n dependent.

The scale for the moment-evolution equation can also be read from Fig. 2 and has a fit of $Q^*_n/Q = 0.656/n^{1/2}$. The $n^{1/2}$ behavior for large n was also predicted in [4].

Keeping only the leading-order term of P_{qq}, the moment-evolution equation can be written analogous to Eq. (24) as

$$\frac{\partial \ln M_n(Q^2)}{\partial \ln Q^2} = \frac{\alpha(Q_n^*)}{2\pi} A_n. \tag{37}$$

Again, it can be solved by rewriting the coupling constant using Eq. (27) to be

$$\begin{aligned} M_n(Q)^{-1/d_n} &= C_n \ln(Q_n^{*\,2}/\Lambda^2) \\ &= C_n \left[\ln(Q^2/\Lambda^2) - 2B_n\right], \end{aligned} \tag{38}$$

which is simply Eq. (29). Actually, the commutativity of scale setting and moment integration works at any higher order, because a choice of the renormalization scale can only affect the result by an order higher than that of the calculation.

VI. CONCLUSION

We have applied the BLM procedure to set the scale in the evolution equations for the non-singlet structure functions and their moments in deep-inelastic scattering. The variation of the effective Λ obtained from the nth-order moment-evolution data is explained by this method, and an unambiguous value of $\Lambda_{\overline{\rm MS}}$ is obtained.

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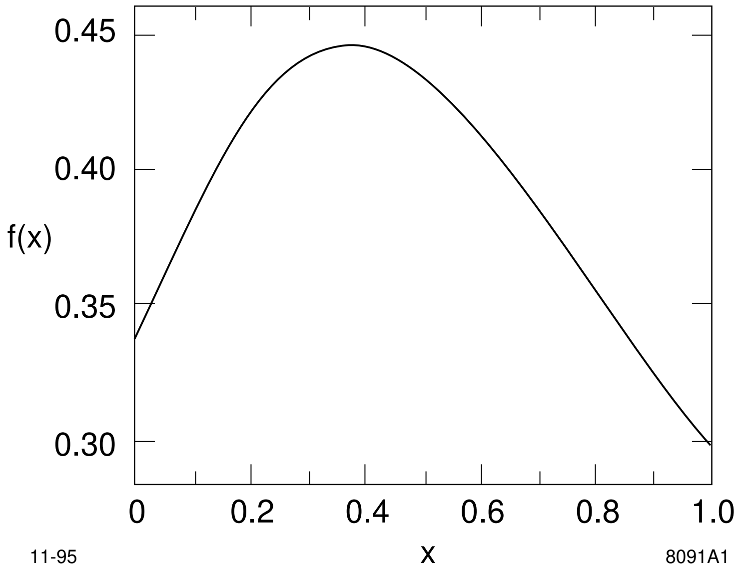

Fig. 1

FIG. 1. Plot of $f(x)$ against x.

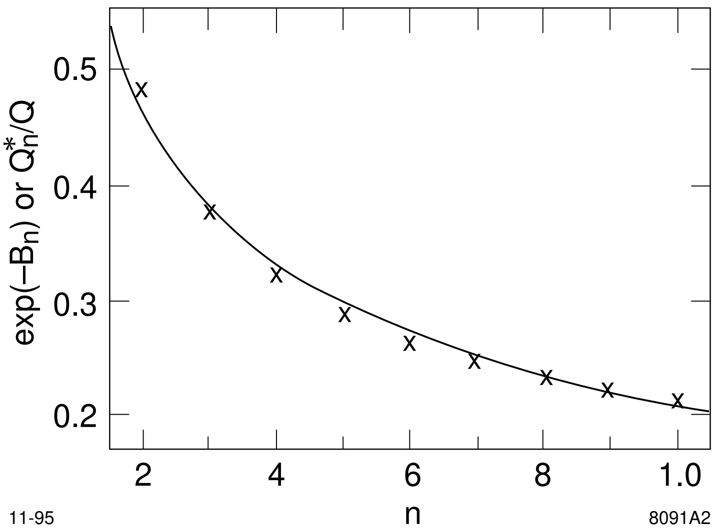

Fig. 2

FIG. 2. Plot of $\exp(-B_n)$ (or Q_n^*/Q) against n. The fitted curve has the form $\exp(-B_n) = 0.656/n^{1/2}$.

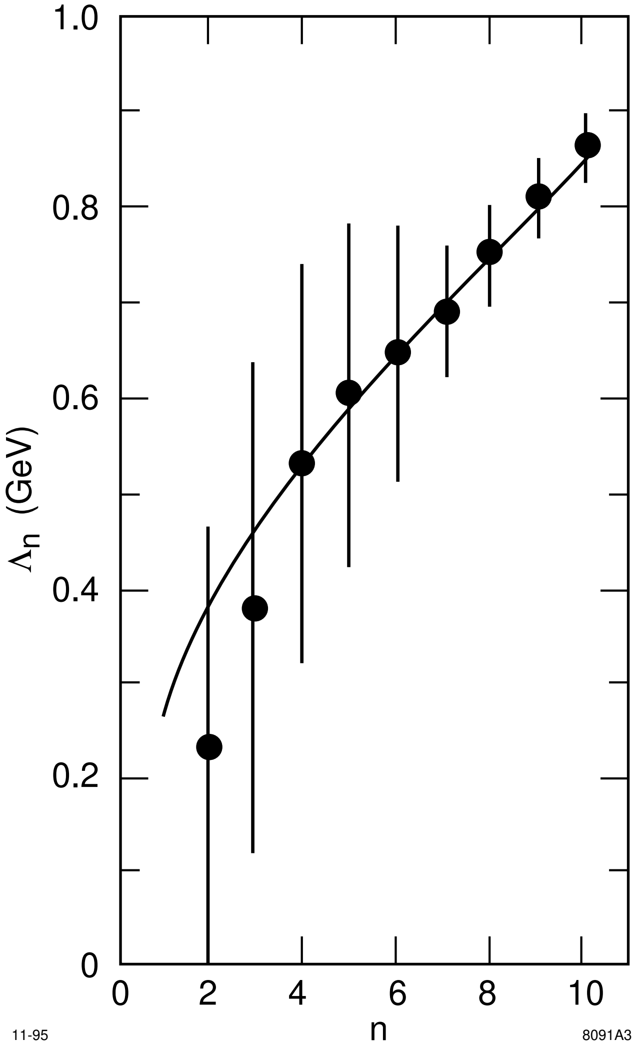

Fig. 3

FIG. 3. Plot of Λ_n against n, after B. A. Gordon et al.