Particle Trapping with an RF Cavity

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It is in principle possible to trap charged particles in an rf cavity. For simplicity, the cavity is considered to be a closed rectangular box with three dimensions ℓ_x, ℓ_y and ℓ_z, and the cavity mode is supposed to provide

$$\begin{aligned}
E_x &= E_{x0} \sin\frac{\pi x}{\ell_x} \cos\frac{\pi y}{\ell_y} \cos\frac{\pi z}{\ell_z} e^{-i\omega t} \\
E_y &= E_{y0} \cos\frac{\pi x}{\ell_x} \sin\frac{\pi y}{\ell_y} \cos\frac{\pi z}{\ell_z} e^{-i\omega t} \\
E_z &= E_{z0} \cos\frac{\pi x}{\ell_x} \cos\frac{\pi y}{\ell_y} \sin\frac{\pi z}{\ell_z} e^{-i\omega t} \\
B_x &= \frac{ic\pi}{\omega}(\frac{E_{z0}}{\ell_y} - \frac{E_{y0}}{\ell_z}) \cos\frac{\pi x}{\ell_x} \sin\frac{\pi y}{\ell_y} \sin\frac{\pi z}{\ell_z} e^{-i\omega t} \\
B_y &= \frac{ic\pi}{\omega}(\frac{E_{x0}}{\ell_z} - \frac{E_{z0}}{\ell_x}) \sin\frac{\pi x}{\ell_x} \cos\frac{\pi y}{\ell_y} \sin\frac{\pi z}{\ell_z} e^{-i\omega t} \\
B_z &= \frac{ic\pi}{\omega}(\frac{E_{y0}}{\ell_x} - \frac{E_{x0}}{\ell_y}) \sin\frac{\pi x}{\ell_x} \sin\frac{\pi y}{\ell_y} \cos\frac{\pi z}{\ell_z} e^{-i\omega t}
\end{aligned} \tag{1}$$

where ω is the mode frequency given by

$$\omega = \pi c \sqrt{\frac{1}{\ell_x^2} + \frac{1}{\ell_y^2} + \frac{1}{\ell_z^2}} \tag{2}$$

In this note we calculate the trapping mechanism of the field pattern (1). The trapping occurs for either sign of the charge of the particles.

Maxwell equations also requires for this rf mode to obey

$$\frac{E_{x0}}{\ell_x} + \frac{E_{y0}}{\ell_y} + \frac{E_{z0}}{\ell_z} = 0 \tag{3}$$

The mode pattern (1) traps particles around the node point $(0, 0, 0)$. Such a trapping mechanism is of course considered favorable for a particle trap. On the other hand, it may also happen that undesirable stray particles are trapped at nodes of an rf device, which then perturbs the normal operation of an accelerator. Such trappings are then undesirable and are to be avoided.

Consider the motion of a cold particle near the center of the cavity $x \ll \ell_x, y \ll \ell_y, z \ll \ell_z$. The field seen by this particle is approximately

$$\begin{aligned}
\vec{E} &\approx \pi e^{-i\omega t} \left[\frac{E_{x0}}{\ell_x} x\hat{x} + \frac{E_{y0}}{\ell_y} y\hat{y} + \frac{E_{z0}}{\ell_z} z\hat{z} \right] \\
\vec{B} &\approx 0
\end{aligned} \tag{4}$$

The magnetic field is negligible not only because it is second order in x, y, z, but also because the magnetic force is weak for nonrelativistic particles. For what follows, we replace $e^{-i\omega t}$ by $\cos\omega t$. In this work, we do not consider how the particles are injected into the cavity, or how the particles can be cooled by some means.

The equation of motion of a particle with rest mass M and charge Q is

$$M\ddot{x} = QE_x \approx Q\frac{E_{x0}}{\ell_x}x\cos\omega t \tag{5}$$

and similarly for the y- and z-motions. The three dimensions are decoupled, and by a change of variable from t to $v = \frac{1}{2}\omega t$, they read

$$\frac{d^2u}{dv^2} + (\beta\cos 2v)u = 0 \tag{6}$$

For the x-motion, for example, we have

$$\beta = -\frac{4QE_{x0}}{M\ell_x\omega^2} \tag{7}$$

Particle trapping is then based on the alternating focusing nature of Eq.(6). Indeed, the alternating sign of $\cos 2v$ provides an opportunity for the u-motion to be focussed regardless of the sign of β. Although the field can not focus all three dimensions at a given instant of time,¶The values of β for the three dimensions can not all have the same sign, as shown by an inspection of Eq.(3). the alternating rf field may still provide net focusings to all three dimensions, thus trap the particles.

Equation (6) is a special case of the Mathieu's equation. In order for the u-motion to be stable, β and ω must satisfy some condition. The stability boundary is determined by the existence of periodic solutions of $u(v)$ of the types

$$u(v) = \sum_{m=0}^{\infty} A_m \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (2m+1)v \tag{8}$$

Following the standard Mathieu's equation analysis,[1] it can be shown that for the periodic solutions (8) to exist, the value of β must satisfy

$$2 \pm \beta = \cfrac{\beta^2}{2\cdot 3^2 - \cfrac{\beta^2}{2\cdot 5^2 - \cfrac{\beta^2}{2\cdot 7^2 - \cdots}}} \tag{9}$$

where $+$ sign is for the existence of the sin solution, while the $-$ sign is for the cos solution. A numerical calculation gives the results $\beta = \mp 1.816$. We thus find that the stability condition is

$$|\beta| < 1.816 \tag{10}$$

$a*$

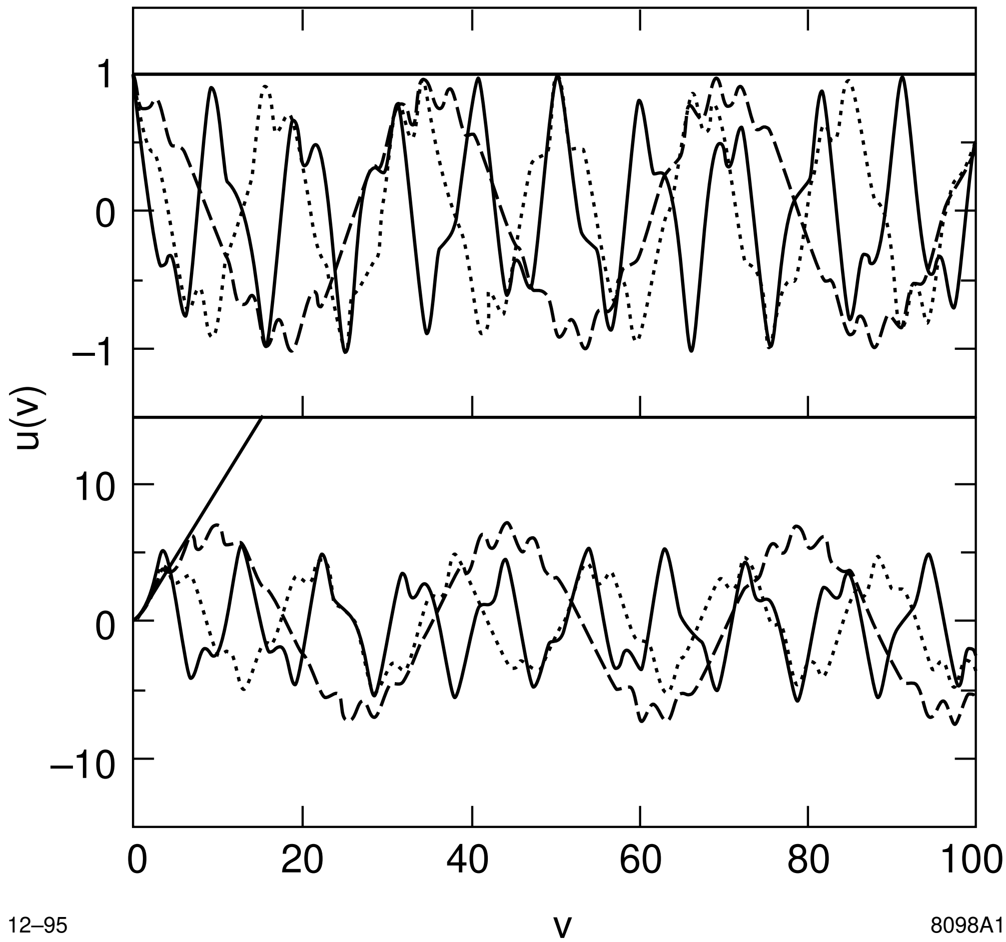

Fig. 1. Particle motion in an rf trap.

Condition (10) can be written as

$$QE_0\ell \leq 4.3Mc^2 \tag{11}$$

where E_0 is the cavity field strength, ℓ is the size of the cavity. This condition is equivalent to requiring the energy gained from the cavity electric field over the distance of the cavity dimension must not render the particle relativistic. Such a condition is easily satisfied in practice.

We can track the motion of a particle using Eq.(6). Figure 1(a) shows the results for tracking a particle with initial conditions $u(0) = 1$ and $\frac{du}{dv}(0) = 0$ for 4 values of $\beta = 0$ (solid curve), 0.5 (dashed curve), 1 (dotted curve), and 1.5 (solid curve), for the range of v from 0 to 100. The motion is stable for these values of β. Figure 1(b) is the same with the initial conditions $u(0) = 0$ and $\frac{du}{dv}(0) = 1$. A strong focusing occurs when $\beta \sim 1$, away from 0 or the threshold value of 1.816.

Figures 2(a) and 2(b) show the particle motion near the threshold $\beta = 1.816$. The two values of β are 1.814 (solid curve) and 1.818 (dotted curve). The transition from

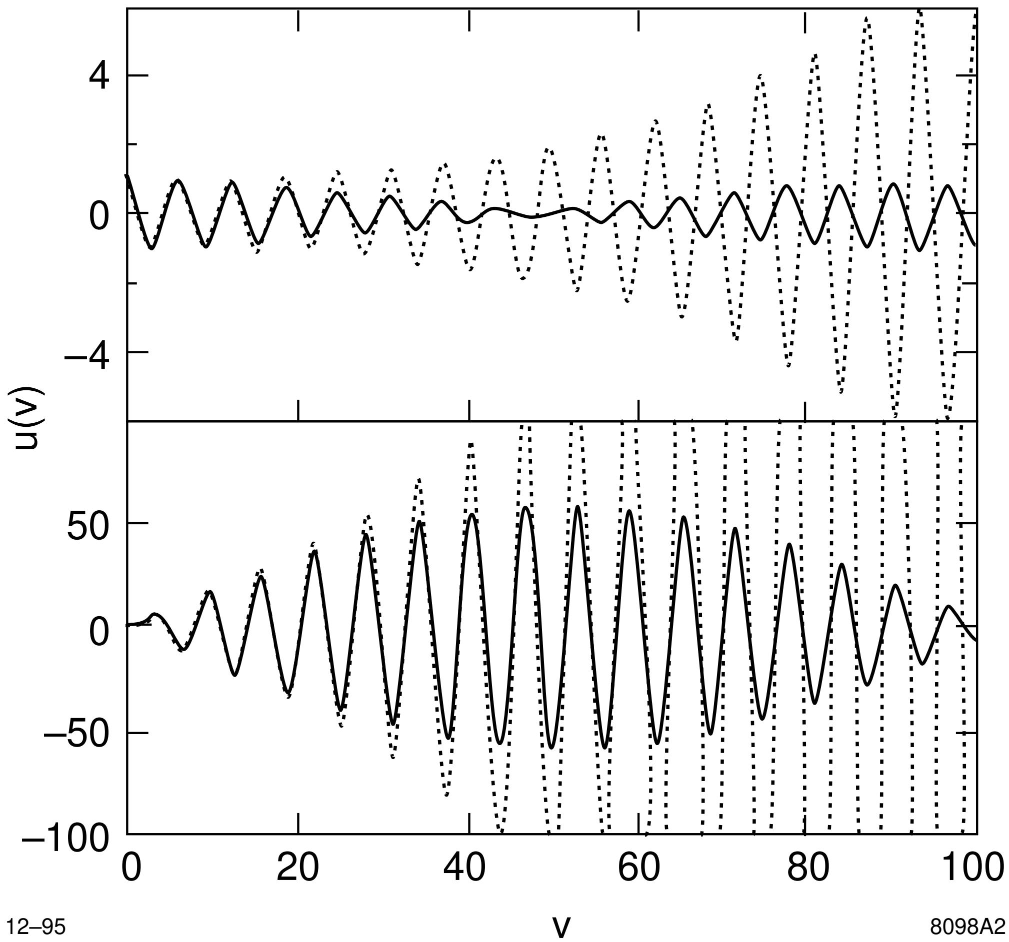

Fig. 2. Particle motion in an rf trap near the instability threshold.

stable to unstable motions is rather sudden.

It may be instructive to examine the qualitative behavior of the particle motion using a simplified model instead of the Mathieu's equaton analysis. To do so, consider the replacement

$$\beta\cos 2v \to \begin{cases} |\beta| & \text{when } \beta\cos 2v > 0 \\ -|\beta| & \text{when } \beta\cos 2v < 0 \end{cases} \tag{12}$$

In other words, replace the $\beta\cos 2v$ focusing by a piecewise constant alternating focusing. We expect to obtain a stability condition similar to Eq.(10) but with somewhat different numerical coefficients.

With the replacement (12), the particle motion over one period of time $v_{\text{final}} - v_{\text{initial}} = \pi$ is described by a matrix map

$$\begin{bmatrix} u \\ \frac{1}{\sqrt{|\beta|}}\frac{du}{dv} \end{bmatrix}_{\text{final}} = T \begin{bmatrix} u \\ \frac{1}{\sqrt{|\beta|}}\frac{du}{dv} \end{bmatrix}_{\text{initial}} \tag{13}$$

where

$$T = \begin{bmatrix} \cos(\frac{\pi}{2}\sqrt{|\beta|}) & \sin(\frac{\pi}{2}\sqrt{|\beta|}) \\ -\sin(\frac{\pi}{2}\sqrt{|\beta|}) & \cos(\frac{\pi}{2}\sqrt{|\beta|}) \end{bmatrix} \begin{bmatrix} \cosh(\frac{\pi}{2}\sqrt{|\beta|}) & \sinh(\frac{\pi}{2}\sqrt{|\beta|}) \\ \sinh(\frac{\pi}{2}\sqrt{|\beta|}) & \cosh(\frac{\pi}{2}\sqrt{|\beta|}) \end{bmatrix} \tag{14}$$

Stability of motion is determined by the condition $|\mathrm{trace}(T)| \leq 2$, or

$$|\cos(\frac{\pi}{2}\sqrt{|\beta|})\cosh(\frac{\pi}{2}\sqrt{|\beta|})| \leq 1 \tag{15}$$

Like the Mathieu's equation, there are infinite number of stable bands of values of β. The stable band near $\beta = 0$ has the largest width, and it spans the region

$$|\beta| < 1.425 \tag{16}$$

Equation (16) is to be compared with the exact result, Eq.(10).

The model (12) can also be used to estimate the oscillation frequencies of the particle motion in the rf cavity trap. In the stable region, the oscillation frequency is determined by

$$f = \frac{\omega}{2\pi}\frac{1}{2\pi}\cos^{-1}[\cos(\frac{\pi}{2}\sqrt{|\beta|})\cosh(\frac{\pi}{2}\sqrt{|\beta|})] \tag{17}$$

If $|\beta| \leq 1$, we have

$$f \approx \frac{\omega|\beta|}{16\sqrt{3}} \tag{18}$$

The three oscillation frequencies are therefore

$$f_x \approx \frac{QE_{x0}}{4\sqrt{3}M\ell_x\omega}, \qquad \text{similarly for } f_y, f_z \tag{19}$$

The period V of oscillation can be read off from Figs. 1 and 2. This gives $V \approx 35$, 17, 10, 104 for $\beta = 0.5$, 1, 1.5 and 1.814, respectively. These periods can be compared with the approximate expression (19), which gives $V \approx \frac{8\sqrt{3}}{\beta}$. For $\beta = 0.5$, 1, 1.5 and 1.814, this gives $V = 28$, 14, 9, and 7.6. The result for $\beta = 1.814$ is way off, but the other values are reasonable.

Take $\ell_x = \ell_y = \ell_z = 30$ cm, Eq.(2) gives $\omega = 0.54 \times 10^{10}$ s^{-1} (0.86 GHz).[2] Consider antiprotons, we have $Q = e$, $Mc^2 = 938$ MeV. Let the cavity field be such that $E_{x0} = E_{y0} = 100$ KV/cm and, from Eq.(3), $E_{z0} = -200$ KV/cm. We have $|\beta_x| = |\beta_y| = \frac{1}{2}|\beta_z| = 0.28 \times 10^{-4}$, easily satisfying the stability condition (10). Using Eq.(20), we can also estimate the oscillation frequencies to be $f_x = f_y = \frac{1}{2}f_z = 85$ KHz.

The rf trap mentioned in the above example can in principle be used to trap cold electrons or positrons. We then have $|\beta_x| = |\beta_y| = \frac{1}{2}|\beta_z| = 0.051$, and the particles are trapped with $f_x = f_y = \frac{1}{2}f_z = 160$ MHz.

The map (14) has the Courant-Snyder invariant

$$J = (1 - \cos^2 \cosh^2)u^2 + [u \sin \sinh + \frac{du}{dv}(\cos \sinh + \sin \cosh)]^2 \tag{20}$$

where the arguments of sin, cos, sinh, and cosh functions are $\frac{\pi}{2}\sqrt{|\beta|}$. When $|\beta| \ll 1$, we have the invariant[3]

$$\frac{J}{\pi^2|\beta|} \approx (\frac{du}{dv})^2 + \frac{\pi}{2}\sqrt{|\beta|}u\frac{du}{dv} + \frac{\pi^2}{12}|\beta|u^2 + \mathcal{O}(|\beta|^2) \tag{21}$$

The invariant (21) allows an estimate of the particle oscillation amplitude given the initial conditions $u(0)$ and $\frac{du}{dv}(0)$. For example, when $u(0) = 0$, Eq.(21) gives a maximum oscillation amplitude

$$u_{\max} \approx \frac{4}{\pi}\sqrt{\frac{3}{|\beta|}}\frac{du}{dv}(0) \tag{22}$$

Equation (22) can be compared with the tracking results shown in Fig.1(b).

If a cloud of particles are trapped, and if the rms kinetic energy is K, the rms size of the particle cloud is

$$\sigma_x = \frac{4}{\pi}\sqrt{\frac{K\ell_x}{QE_{0x}}}, \qquad \text{and similarly for } \sigma_y, \sigma_z \tag{23}$$

Note the result (23) does not depend explicitly on the mass of the particle. Take $K = 1$ KeV, then the previous numerical examples (for both antiprotons and positrons) give $\sigma_x = \sigma_y = \sqrt{2}\sigma_z = 7$ mm. Space charge effects however have been ignored in Eq.(23).

I would like to thank Sandro Ruggiero and Paul Channel for their helpful comments.

1. References

1. Morse and Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, 1978.

2. One may want to avoid this highly degenerate mode by having $\ell_{x,y,z}$ slightly different from one another, but here we just want a crude numerical example.

3. Carried out to higher orders, we have $\mathcal{O}(|\beta|^2) = -\frac{\pi^4}{240}|\beta|^2(\frac{du}{dv})^2 - \frac{\pi^5}{720}|\beta|^{5/2}u\frac{du}{dv} - \frac{\pi^6}{5040}|\beta|^3u^2 + \mathcal{O}(|\beta|^{7/2})$.