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OPTIMAL RENORMALIZATION SCALES AND COMMENSURATE SCALE RELATIONS *

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Abstract

Commensurate scale relations relate observables to observables and thus are independent of theoretical conventions, such as the choice of intermediate renormalization scheme. The physical quantities are related at commensurate scales which satisfy a transitivity rule which ensures that predictions are independent of the choice of an intermediate renormalization scheme. QCD can thus be tested in a new and precise way by checking that the observables track both in their relative normalization and in their commensurate scale dependence. For example, the radiative corrections to the Bjorken sum rule at a given momentum transfer Q can be predicted from measurements of the e^+e^- annihilation cross section at a corresponding commensurate energy scale $\sqrt{s} \propto Q$, thus generalizing Crewther's relation to non-conformal QCD. The coefficients that appear in this perturbative expansion take the form of a simple geometric series and thus have no renormalon divergent behavior. We also discuss scale-fixed relations between the threshold corrections to the heavy quark production cross section in e^+e^- annihilation and the heavy quark coupling α_V which is measurable in lattice gauge theory.

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1. Introduction

The leading power-law prediction in perturbative QCD for an observable such as the e^+e^- annihilation cross section is generally of the form $\Sigma_{n=1}^{N} r_n \alpha_s^n(\mu)$. The predictions for a physical quantity are formally invariant under a change of the renormalization scale μ; however, since the series is only known for a finite number of terms N, there is an unavoidable dependence on the choice of μ. The choice of renormalization scheme used to define and normalize the coupling α_s is also apparently a matter of convention. In principle, one can carry out perturbative calculations in any renormalization scheme such as modified minimal subtraction $\alpha_{\overline{\rm MS}}$ or by expanding in effective charges defined from any perturbatively-calculable observable.

Thus a critical problem in making reliable predictions in quantum chromodynamics is how to deal with the dependence of the truncated perturbative series on the choice of renormalization scale and scheme. Because there is no *a priori* range of allowed values of the renormalization scale and the parameters of the renormalization scheme, it is even difficult to quantify the uncertainties due to the renormalization conventions. For processes where only the leading and next-to-leading order predictions are known, the theoretical uncertainties from the choice of renormalization scale and scheme are evidently much larger than the experimental uncertainties. The problems of convention dependence are compounded by the fact that the infinite series in PQCD is generally divergent due to "renormalon" contributions growing as $n!\alpha_s^n(\mu)$. "Renormalons" are singularities in the Borel transform of asymptotic series coming from particular subset of Feynman diagrams [1] which dictate the divergent behavior of large-order expansion coefficients.

The uncertainties introduced by the conventions in the renormalization procedure are amplified in processes involving more than one physical scale such as jet observables and semi-inclusive reactions. In the case of jet production at e^+e^- colliders, the the jet fractions depend both on the total center of mass energy s and the jet resolution parameter y (which gives an upperbound ys to the invariant mass squared of each individual jet). Kramer and Lamper [2] have shown that different scale-setting strategies can lead to very different behaviors for the renormalization scale in the small y region. The experimental fits indicate that in general one must choose a scale $\mu^2 \ll s$ for 4-jet production rates [3]. In the case of QCD predictions for exclusive processes such as the decay of heavy hadrons to specific channels and baryon form factors at large momentum transfer, the scale ambiguities for the underlying quark-gluon subprocesses are even more acute since the coupling constant $\alpha_s(\mu)$ enters at a high power. Furthermore, since the external momenta entering an exclusive reaction are partitioned among the many propagators of the underlying hard-scattering amplitude, the physical scales that control these processes are inevitably much softer than the overall momentum transfer.

Scheme and scale ambiguities are also an obstacle for testing the Standard Model to high precision. The situation is complicated by the fact that computations in different sectors of the Standard Model are carried out using different renormalization schemes. For example, in quantum electrodynamics, higher order radiative corrections are computed in the traditional "on- shell" scheme using Pauli-Villars regularization. The QED coupling $\alpha_{\rm QED}$ is defined from the Coulomb scattering of heavy test charges at zero momentum transfer. The scale k^2 in the running QED coupling $\alpha_{\rm QED}(k^2)$ is then set by the virtuality of the photon propagator in order to sum all vacuum polarization corrections. However, in the non-Abelian sectors of the Standard Model, higher order computations are usually carried out using the $\overline{\rm MS}$ dimensional regularization scheme. The renormalization scale μ that appears in perturbative expansions in the QCD coupling $\alpha_{\overline{\rm MS}}(\mu^2)$ is usually treated as an arbitrary parameter. These ambiguities and disparities in choices of scales and schemes lead to uncertainties in establishing the accuracy and range of validity of perturbative QCD predictions and in testing the hypothesis of grand unification [4].

The fact that physical quantities cannot depend on theoretical conventions does not preclude the possibility that we can choose an optimal renormalization scale for the truncated series. This is analogous to gauge invariance: physical results cannot depend on the choice of gauge; however, there are often special gauges, such as the radiation gauge, light-cone gauge, Landau gauge, Fried-Yennie gauge, which allow the entire physical result to be calculated and presented in a simple way. Similarly, it is possible that the renormalization scale ambiguity problem can be resolved if one can optimize the choices of scale according to some sensible criteria. As we shall see in the case of the generalized Crewther relation, an appropriate choice of scales also makes the physical interpretation more transparent.

In the BLM procedure [5], one first selects a renormalization scheme. The renormalization scales are then fixed by the requirement that all contributions to the $\beta-$function such as quark and gluon loop contributions are re-summed into the running couplings. The coefficients of the perturbative series are thus identical to the perturbative coefficients of the corresponding conformally-invariant theory with $\beta = 0$. The BLM method has the important advantage of "pre-summing" the large and strongly divergent terms in the PQCD series which grow as $n!(\beta_0\alpha_s)^n$, *i.e.*, the infrared renormalons associated with coupling- constant renormalization [1, 6, 7, 8]. The renormalization scales Q^* in the BLM method are physical in the sense that they reflect the mean virtuality of the gluon propagators [5, 9, 10, 11].

In should be emphasized, that the BLM renormalization scales Q^* and the series coefficients r_n necessarily depend on the choice of renormalization scheme since the scheme sets the units of measure – just as the number of inches of a given length differ from the number of centimeters for the same length. Nevertheless, the actual predictions for physical quantities are independent of the scheme if one uses BLM scale-setting.

It is interesting to compare Padé resummation predictions for single-scale perturbative QCD series in which the initial renormalization scale choice is taken as the characteristic scale $\mu = Q$ as well as the BLM scale $\mu = Q^*$. One finds [12] that the Padé predictions for the summed series are in each case independent of the initial scale choice, an indication that the Padé results are thus characteristic of the actual QCD prediction. However, the BLM scale generally produces a faster convergence to the complete sum than the conventional scale choice. This can be understood by the fact that the BLM scale choice immediately sums into the coupling all repeated vacuum polarization insertions to all orders, thus eliminating the large $(\beta_0 \alpha_s)^n$ terms in the series as well as the $n!$ growth characteristic of the infrared renormalon structure of PQCD [1, 6].

A basic principle of renormalization theory is the requirement that relations between physical observables must be independent of renormalization scale and scheme conventions to any fixed order of perturbation theory [13]. In this talk, we shall discuss high precision perturbative predictions which have no scale or scheme ambiguities. These predictions, called "Commensurate Scale Relations,"[14] are valid for any renormalizable quantum field theory, and thus may provide a uniform perturbative analysis of the electroweak and strong sectors of the Standard Model.

Commensurate scale relations relate observables to observables, and thus are independent of theoretical conventions, such as choice of intermediate renormalization scheme. The scales of the effective charges that appear in commensurate scale relations are fixed by the requirement that the couplings sum all of the effects of the non-zero β function, as in the BLM method [5]. The coefficients in the perturbative expansions in the commensurate scale relations are thus identical to those of a corresponding conformally-invariant theory with $\beta = 0$.

A helpful tool and notation for relating physical quantities is the effective charge. Any perturbatively calculable physical quantity can be used to define an effective charge [15, 16, 17] by incorporating the entire radiative correction into its definition. An important result is that all effective charges $\alpha_A(Q)$ satisfy the Gell-Mann-Low renormalization group equation with the same β_0 and β_1; different schemes or effective charges only differ through the third and higher coefficients of the β function. Thus, any effective charge can be used as a reference running coupling constant in QCD to define the renormalization procedure. More generally, each effective charge or renormalization scheme, including $\overline{\mathrm{MS}}$, is a special case of the universal coupling function $\alpha(Q, \beta_n)$ [13, 18]. Peterman and Stückelberg have shown [13] that all effective charges are related to each other through a set of evolution equations in the scheme parameters β_n.

For example, consider the entire radiative corrections to the annihilation cross section expressed as the "effective charge" $\alpha_R(Q)$ where $Q = \sqrt{s}$:

$$R(Q) \equiv 3 \sum_f Q_f^2 \left[1 + \frac{\alpha_R(Q)}{\pi} \right] . \tag{1}$$

Similarly, we can define the entire radiative correction to the Bjorken sum rule as the effective charge $\alpha_{g_1}(Q)$ where Q is the lepton momentum transfer:

$$\int_0^1 dx \left[g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2) \right] \equiv \frac{1}{3} \left| \frac{g_A}{g_V} \right| \left[1 - \frac{\alpha_{g_1}(Q)}{\pi} \right] . \tag{2}$$

The commensurate scale relations connecting the effective charges for observables A and B have the form $\alpha_A(Q_A) = \alpha_B(Q_B) \left(1 + r_{A/B} \frac{\alpha_B}{\pi} + \cdots \right)$, where the coefficient $r_{A/B}$ is independent of the number of flavors n_F contributing to coupling constant renormalization. The ratio of scales $\lambda_{A/B} = Q_A/Q_B$ is unique at leading order and guarantees that the observables A and B pass through new quark thresholds at the same physical scale. One also can show that the commensurate scales satisfy the transitivity rule $\lambda_{A/B} = \lambda_{A/C}\lambda_{C/B}$, which is the renormalization group property which ensures that predictions in PQCD are independent of the choice of an intermediate renormalization scheme C. In particular, scale-fixed predictions can be made without reference to theoretically-constructed renormalization schemes such as $\overline{\rm MS}$. QCD can thus be tested in a new and precise way by checking that the observables track both in their relative normalization and in their commensurate scale dependence.

A scale-fixed relation between any two physical observables A and B can be derived by applying BLM scale-fixing to their respective perturbative predictions in, say, the $\overline{\rm MS}$ scheme, and then algebraically eliminating $\alpha_{\overline{\rm MS}}$. The choice of the BLM scale ensures that the resulting commensurate scale relation between A and B is independent of the choice of the intermediate renormalization scheme [14]. Thus, using this formalism, one can relate any perturbatively calculable observable, such as the annihilation ratio $R_{e^+e^-}$, the heavy quark potential, and the radiative corrections to structure function sum rules to each other without any renormalization scale or scheme ambiguity [14]. Commensurate scale relations can also be applied in grand unified theories to make scale and scheme invariant predictions which relate physical observables in different sectors of the theory.

The scales that appear in commensurate scale relations are physical since they reflect the mean virtuality of the gluons in the underlying hard subprocess [5, 10]. As emphasized by Mueller [6], commensurate scale relations isolate the effect of infrared renormalons associated with the non-zero β function. The usual factorial growth of the coefficients in perturbation theory due to quark and gluon vacuum polarization insertions is eliminated since such effects are resummed into the running couplings. The perturbative series is thus much more convergent.

In the next section we discuss an elegant example: a surprisingly simple connection between the radiative corrections to the Bjorken sum rule at a given momentum transfer Q is predicted from measurements of the e^+e^- annihilation cross section at a corresponding commensurate energy scale $\sqrt{s} \propto Q$[14, 19]. The coefficients that appear in the perturbative expansion are a simple geometric series, and thus have no divergent renormalon behavior $n!\alpha_s^n(\mu)$ in the coefficients. This relation generalizes Crewther's relation to non-conformal QCD. Another useful example is the connection between the moments of structure functions and other observables [5, 20].

The heavy-quark potential $V(Q^2)$ can be identified as the two-particle-irreducible scattering amplitude of test charges; *i.e.* the scattering of two infinitely-heavy quark and antiquark at momentum transfer $t = -Q^2$. The relation $V(Q^2) = -4\pi C_F \alpha_V(Q^2)/Q^2$ with C_F given by $C_F = (N_C^2 - 1)/2N_C = 4/3$ then defines the "effective charge" $\alpha_V(Q)$. This coupling provides a physically-based alternative to the usual $\overline{\mathrm{MS}}$ scheme.

As in the corresponding case of Abelian QED, the scale Q of the coupling $\alpha_V(Q)$ is identified with the exchanged momentum. All vacuum polarization corrections due to fermion pairs are incorporated in terms of the usual vacuum polarization kernels defined in terms of physical mass thresholds. The first two terms $\beta_0 = 11 - \frac{2}{3}n_f$ and $\beta_1 = 102 - \frac{38}{3}n_f$ in the expansion of the β-function defined from the logarithmic derivative of $\alpha_V(Q)$ are universal, *i.e.*, identical for all effective charges.

The scale-fixed relation between α_V and the conventional $\overline{\mathrm{MS}}$ coupling is

$$\alpha_{\overline{\mathrm{MS}}}(Q) = \alpha_V(e^{5/6}Q)(1 + \frac{2\alpha_V}{\pi} + ...), \qquad (3)$$

The factor $e^{5/6} \simeq 0.4346$ is the ratio of commensurate scales between the two schemes to this order. It arises because of the convention used in defining the modified minimal subtraction scheme. The scale in the $\overline{\mathrm{MS}}$ scheme is thus a factor ~ 0.4 smaller than the physical scale. The coefficient 2 in the NLO coefficient is a feature of the non Abelian couplings of QCD; the same coefficient occurs even if the theory were conformally invariant with $\beta_0 = 0$. The commensurate scale relation between α_V, as defined from the heavy quark potential, and $\alpha_{\overline{\rm MS}}$ provides an analytic extension of the $\overline{\rm MS}$ scheme in which flavor thresholds are automatically. taken into account at their proper respective scales [21].

The use of α_V as the expansion parameter with BLM scale-fixing has also been found to be valuable in lattice gauge theory, greatly increasing the convergence of perturbative expansions relative to those using the bare lattice coupling [9]. In fact the new lattice calculations of the Υ- spectrum [22] has been used to determine the normalization of the static heavy quark potential and its effective charge using as input a line splitting of the quarkonium spectrum:

$$\alpha_V^{(3)}(8.2\mathrm{GeV}) = 0.196(3). \tag{4}$$

where the effective number of light flavors is $n_f = 3$. The corresponding modified minimal subtraction coupling evolved to the Z mass is given by

$$\alpha_{\overline{\rm MS}}^{(5)}(M_Z) = 0.115(2). \tag{5}$$

One can also apply commensurate scale relations to the domain of exclusive processes at large momentum transfer and exclusive weak decays of heavy hadrons in QCD [23]. In our new work with Chueng-Ryong Ji and Alex Pang, we use the BLM method to fix the renormalization scale of the QCD coupling in exclusive hadronic amplitudes such as the pion form factor and the photon-to-pion transition form factor at large momentum transfer. Renormalization-scheme-independent commensurate scale relations can then be established which connect the hard-scattering subprocess amplitudes which control exclusive processes to other QCD observables such as the heavy quark potential and the electron-positron annihilation cross section. The coupling α_V is particularly useful for analyzing exclusive amplitudes. Each gluon propagator with four-momentum k^μ in the hard-scattering quark-gluon scattering amplitude is associated with the coupling $\alpha_V(k^2)$ [24, 5].

A direct measurement of α_V would in principle require the scattering of heavy quarks. In fact, as we shall discuss in section 3, the threshold corrections to heavy quark production in e^+e^- annihilation depend directly on α_V at specific scales Q^*. Two distinctly different scales arise as arguments of α_V near threshold: the relative momentum of the quarks governing the soft gluon exchange responsible for the Coulomb potential, and a large momentum scale approximately equal to twice the quark mass for the corrections induced by transverse gluons. One thus can obtain a direct determination of α_V the coupling in the heavy quark potential, which can be compared with lattice gauge theory predictions. The corresponding QED results for τ pair production allow for a measurement of the magnetic moment of the τ and could be tested at a future τ-charm factory.

2. Commensurate Scale Relations and The Generalized Crewther Relation in Quantum Chromodynamics

In 1972 Crewther [25] derived a remarkable consequence of the operator product expansion for conformally-invariant gauge theory. Crewther's relation has the form

$$3S = KR' \tag{6}$$

where S is the value of the anomaly controlling $\pi^0 \to \gamma\gamma$ decay, K is the value of the Bjorken sum rule in polarized deep inelastic scattering, and R' is the isovector part of the annihilation cross section ratio $\sigma(e^+e^- \to\text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. Since S is unaffected by QCD radiative corrections [26], Crewther's relation requires that the QCD radiative corrections to $R_{e^+e^-}$ exactly cancel the radiative corrections to the Bjorken sum rule order by order in perturbation theory.

However, Crewther's relation is only valid in the case of conformally-invariant gauge theory, *i.e.* when the coupling α_s is scale invariant. However, in reality the radiative corrections to the Bjorken sum rule and the annihilation ratio are in general functions of different physical scales. Thus Crewther's relation cannot be tested directly in QCD unless the effects of the nonzero β function for the QCD running coupling are accounted for, and the energy scale $\sqrt{s}$ in the annihilation cross section is related to the momentum transfer Q in the deep inelastic sum rules. Recently Broadhurst and Kataev [27] have explicitly calculated the radiative corrections to the Crewther relation and have demonstrated explicitly that the corrections are proportional to the QCD β function.

We can use the known expressions to three loops [28, 29, 30] in $\overline{\rm MS}$ scheme and choose the leading-order and next-to-leading scales Q^* and Q^{**} to re-sum all quark and gluon vacuum polarization corrections into the running couplings. The values of these scales are the physical values of the energies or momentum transfers which ensure that the radiative corrections to each observable passes through the heavy quark thresholds at their respective commensurate physical scales. The final result connecting the effective charges (see section 1) is remarkably simple:

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_R(Q^*)}{\pi} - \left(\frac{\alpha_R(Q^{**})}{\pi}\right)^2 + \left(\frac{\alpha_R(Q^{***})}{\pi}\right)^3 + \cdots . \qquad (7)$$

The coefficients in the series (aside for a factor of C_F, which can be absorbed in the definition of α_s) are actually independent of color and are the same in Abelian, non-Abelian, and conformal gauge theory. The non-Abelian structure of the theory is reflected in the scales Q^* and Q^{**}. Note that the a calculational device; it simply serves as an intermediary between observables and does not appear in the final relation (7). This is equivalent to the group property defined by Peterman and Stückelberg [13] which ensures that predictions in PQCD are independent of the choice of an intermediate renormalization scheme. (The renormalization group method was developed by Gell-Mann and Low [31] and by Bogoliubov and Shirkov [32]).

The connection between the effective charges of observables such as Eq. (7) is referred to as a "commensurate scale relation" (CSR). A fundamental test of QCD will be to verify empirically that the observables track in both normalization and shape as given by the CSR. The commensurate scale relations thus provide fundamental tests of QCD which can be made increasingly precise and independent of the choice of renormalization scheme or other theoretical convention. More generally, the CSR between sets of physical observables automatically satisfy the transitivity and symmetry properties [33] of the scale transformations of the renormalization "group" as originally defined by Peterman and Stückelberg [13]. The predicted relation between observables must be independent of the order one makes substitutions; *i.e.* the algebraic path one takes to relate the observables.

The relation between scales in the CSR is consistent with the BLM scale-fixing procedure [5] in which the scale is chosen such that all terms arising from the QCD $\beta-$function are resummed into the coupling. Note that this also implies that the coefficients in the perturbation CSR expansions are independent of the number of quark flavors f renormalizing the gluon propagators. This prescription ensures that, as in quantum electrodynamics, vacuum polarization contributions due to fermion pairs are all incorporated into the coupling $\alpha(\mu)$ rather than the coefficients. The coefficients in the perturbative expansion using BLM scale-fixing are the same as those of the corresponding conformally invariant theory with $\beta = 0$. In practice, the conformal limit is defined by $\beta_0, \beta_1 \to 0$, and can be reached, for instance, by adding enough spin-half and scalar quarks as in $N = 4$ supersymmetric QCD. Since all the running coupling effects have been absorbed into the renormalization scales, the BLM scale-setting method correctly reproduces the perturbation theory coefficients of the conformally invariant theory in the $\beta \to 0$ limit.

Let us now discuss in more detail the derivation of Eq. (7). The perturbative series of $\alpha_{g_1}(Q)/\pi$ using dimensional regularization and the $\overline{\rm MS}$ scheme with the renormalization scale fixed at $\mu = Q$ has been computed [28] through three loops in perturbation theory. The effective charge for the annihilation cross section has also been computed [29, 30] to the same order in the $\overline{\rm MS}$ scheme with the renormalization scale fixed at $\mu = Q = \sqrt{s}$. The two effective charges can be related to each other by eliminating $\alpha_{\overline{\rm MS}}$. The scales Q^* and Q^{**} are set by resumming all dependence on $\beta = 0$ and β_1 into the effective charge. The application of the NLO BLM formulas then leads to

$$\begin{aligned}
\frac{\alpha_{g_1}(Q)}{\pi} &= \frac{\alpha_R(Q^*)}{\pi} - \frac{3}{4}C_F\left(\frac{\alpha_R(Q^{**})}{\pi}\right)^2 \\
&\quad + \left[\frac{9}{16}C_F^2 - \left(\frac{11}{144} - \frac{1}{6}\zeta_3\right)\frac{d^{abc}d^{abc}}{C_F N}\frac{\left(\sum_f Q_f\right)^2}{\sum_f Q_f^2}\right]\left(\frac{\alpha_R(Q^{***})}{\pi}\right)^3, \qquad (8)\\
Q^* &= Q\exp\left[\frac{7}{4} - 2\zeta_3 + \left(\frac{11}{96} + \frac{7}{3}\zeta_3 - 2\zeta_3^2 - \frac{\pi^2}{24}\right)\left(\frac{11}{3}C_A - \frac{2}{3}f\right)\frac{\alpha_R(Q)}{\pi}\right], (9)\\
Q^{**} &= Q\exp\left[\frac{523}{216} + \frac{28}{9}\zeta_3 - \frac{20}{3}\zeta_5 + \left(-\frac{13}{54} + \frac{2}{9}\zeta_3\right)\frac{C_A}{C_F}\right]. \qquad (10)
\end{aligned}$$

In practice, the scale Q^{***} in the above expression can be chosen to be Q^{**}. Notice that aside from the light-by-light contributions, all the ζ_3, ζ_5 and π^2 dependencies have been absorbed into the renormalization scales Q^* and Q^{**}. Understandably, the π^2 term should be absorbed into renormalization scale since it comes from the analytical continuation of $R(Q)$ to the Euclidean region.

For the three-flavor case, where the light-by-light contribution vanishes, the series remarkably simplifies to the CSR of Eq. (7). The form suggests that for the general $SU(N)$ group the natural expansion parameter is $\widehat{\alpha} = (3C_F/4\pi)\,\alpha$. The use of $\widehat{\alpha}$ also makes it explicit that the same formula is valid for QCD and QED. That is, in the limit $N_C \to 0$ the perturbative coefficients in QCD coincide with the perturbative coefficients of an Abelian analog of QCD.

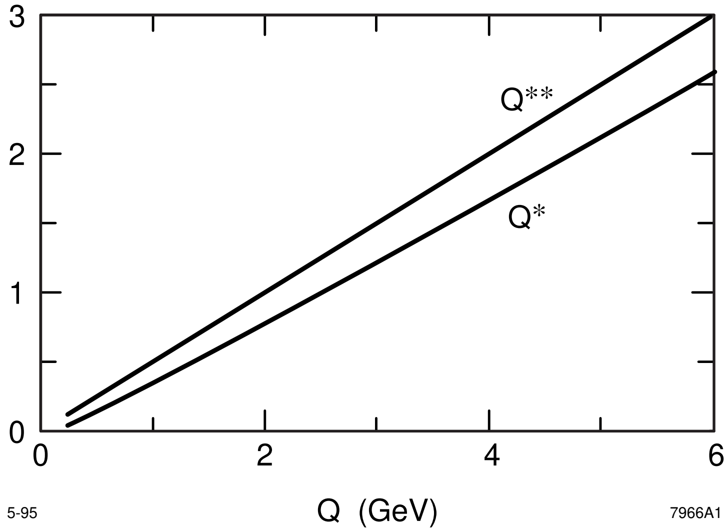

Figure 1: The commensurate scales Q^* and Q^{**} for the case of Bjorken sum rule expressed in terms of $\alpha_R(Q)$.

In Fig. 1 we plot the scales Q^* and Q^{**} as function of Q for in the range $0 \le Q \le 6$. We can see that the scales Q^* and Q^{**} are of the same order as Q but roughly a factor 1/2 to 1/3 smaller.

In Fig. 2 we plot the prediction for the value of the Bjorken sum rule using as input the values of $\alpha_R(Q)$ as given by Mattingly and Stevenson [34]. We use $Q^{***} = Q^{**}$ here. Notice that the prediction has a very smooth and flat behavior, even at $Q^2 \sim 2$ GeV^2 since the effective charge $\alpha_R(Q)$ as obtained by Mattingly and Stevenson incorporates the "freezing" of the strong coupling constant.

Broadhurst and Kataev have recently observed a number of interesting relations between $\alpha_R(Q)$ and $\alpha_{g_1}(Q)$ (the "Seven Wonders") [27]. In particular, they have shown the factorization of the beta function in the correction to Crewther's relation thus establishing a non-trivial connection between the total e^+e^- annihilation cross section and the polarized Bjorken sum rule. The simple form of Eq. (7) also points

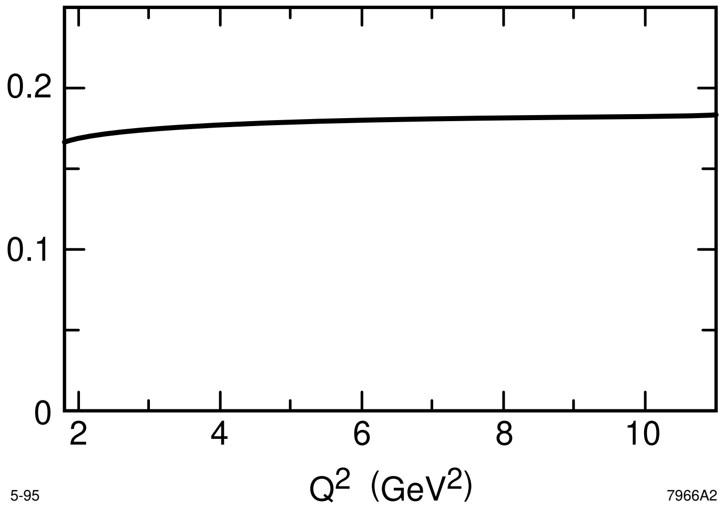

Figure 2: Prediction of the Bjorken sum rule from $R_{e^+e^-}$ according to the commensurate scale relation and using Mattingly and Stevenson's result for $\alpha_R(Q)$.

to the existence of a "secret symmetry" between $\alpha_R(Q)$ and $\alpha_{g_1}(Q)$ which is revealed after the application of the NLO BLM scale setting procedure. In fact, as pointed out by Kataev and Broadhurst [27], in the conformally invariant limit, *i.e.*, for vanishing beta functions, Crewther's relation becomes

$$(1+\widehat{\alpha}_R^{\rm eff})(1-\widehat{\alpha}_{g_1}^{\rm eff}) = 1. \tag{11}$$

Thus Eq. (7) can be regarded as the extension of the Crewther relation to non-conformally invariant gauge theory.

The commensurate scale relation between α_{g_1} and α_R given by Eq. (7) implies that the radiative corrections to the annihilation cross section and the Bjorken (or Gross-Llewellyn Smith) sum rule cancel at their commensurate scales. The relations between the physical cross sections can be written in the forms:

$$\frac{R_{e^+e^-}(s)}{3\sum e_q^2}\;\frac{\int_0^1 dx g_1^p(x,Q^2) - g_1^n(x,Q^2)}{\frac{1}{3}g_A/g_V} = 1 - \Delta\beta_0 \widehat{a}^3 \tag{12}$$

and

$$\frac{R_{e^+e^-}(s)}{3\sum e_q^2}\ \frac{\int_0^1 dx F_3^{\nu p}(x,Q^2)+F_3^{\overline{\nu}p}(x,Q^2)}{6} = 1-\Delta\beta_0\widehat{a}^3, \tag{13}$$

provided that the annihilation energy in $R_{e^+e^-}(s)$ and the momentum transfer Q appearing in the deep inelastic structure functions are commensurate at NLO: $\sqrt{s} = Q^* = Q\exp[\frac{7}{4}-2\zeta_3+(\frac{11}{96}+\frac{7}{3}\zeta_3-2\zeta_3^2-\frac{\pi^2}{24})\beta_0\widehat{a}(Q)]$. The light-by-light correction to the CSR for the Bjorken sum rule vanishes for three flavors. The term $\Delta\beta_0\widehat{a}^3$ with $\Delta = \ell n\,(Q^{**}/Q^*)$ is the third-order correction arising from the difference between Q^{**} and Q^*; in practice this correction is negligible: for a typical value $\widehat{a} = \alpha_R(Q)/\pi = 0.14$, $\Delta\beta_0\widehat{a}^3 = 0.007$. Thus at the magic energy $\sqrt{s} = Q^*$, the radiative corrections to the Bjorken and GLLS sum rules almost precisely cancel the radiative corrections to the annihilation cross section. This allows a practical test and extension of the Crewther relation to non-conformal QCD.

As an initial test of Eq. (13), we can compare the CCFR measurement [35] of the Gross-Llewellyn Smith sum rule $1-\widehat{\alpha}_{F_3} = \frac{1}{6}\int_0^1 dx[F_3^{\nu p}(x,Q^2)+F_3^{\overline{\nu}p}(x,Q^2)] = \frac{1}{3}(2.5\pm 0.13)$ at $Q^2 = 3\ \mathrm{GeV}^2$ and the parameterization of the annihilation data [34] $1+\widehat{\alpha}_R = R_{e^+e^-}(s)/3\sum e_q^2 = 1.20$. at the commensurate scale $\sqrt{s} = Q^* = 0.38\,Q = 0.66$ GeV. The product is $(1+\widehat{\alpha}_R)(1-\widehat{\alpha}_{F_3}) = 1.00\pm 0.04$, which is a highly nontrivial check of the theory at very low physical scales. More recently, the E143 [36] experiment at SLAC has reported a new value for the Bjorken sum rule at $Q^2 = 3\ \mathrm{GeV}^2$: $\Gamma_1^p-\Gamma_1^n = 0.163\pm 0.010(\mathrm{stat})\pm 0.016(\mathrm{syst})$. The Crewther product in this case is also consistent with QCD: $(1+\widehat{\alpha}_R)(1-\widehat{\alpha}_{g_1}) = 0.93\pm 0.11$.

In a recent paper with Gabadadze and Kataev [19] we show that it is also possible and convenient to choose one unique mean scale $\overline{Q}^*$ in $\alpha_R(Q)$ so that the perturbative expansion will also reproduce the coefficients of the geometric progression. The possibility of using a single scale in the generalization of the BLM prescription beyond the next-to-leading order (NLO) was first considered by Grunberg and Kataev [37].

The new single-scale Crewther relation has the form:

$$\widehat{\alpha}_{g_1}(Q) = \widehat{\alpha}_R(\overline{Q}^*) - \widehat{\alpha}_R^2(\overline{Q}^*) + \widehat{\alpha}_R^3(\overline{Q}^*) + \cdots, \tag{14}$$

The generalized Crewther relation provides an important test of QCD. Since the Crewther formula written in the form of the CSR relates one observable to another observable, the predictions are independent of theoretical conventions, such as the choice of renormalization scheme. It is clearly very interesting to test these fundamental self-consistency relations between the polarized Bjorken sum rule or the Gross-Llewellyn Smith sum rule and the e^+e^--annihilation R-ratio. Present data are consistent with the generalized Crewther relations, but measurements at higher precision and energies will be needed to decisively test these fundamental connections in QCD.

It is worthwhile to point out that all of the results presented here are derived within the framework of perturbation theory in leading twist and do not involve the nonperturbative contributions to the Adler's function $D(Q^2)$ [38] and the R-ratio, as well as to the polarized Bjorken and the Gross-Llewellyn Smith sum rules [39, 40]. These nonperturbative contributions are expected to be significant at small energies and momentum transfer. In order to make these contributions comparatively negligible, one should choose relatively large values for s and Q^2. In order to put the analysis of the experimental data for lower energies on more solid ground, it will be necessary to understand whether there exist any Crewther-type relations between non-perturbative order $O(1/Q^4)$-corrections to the Adler's D-function [38] and the order $O(1/Q^2)$ higher twist contributions to the deep-inelastic sum rules [39, 40].

The direct measurements of the polarized Bjorken sum rule (or of the Gross-Llewellyn Smith sum rule) can be useful for the study of the intriguing question whether the experimental data can "sense" the violation of the initial conformal invariance caused by the renormalization procedure. In the language of the Crewther relation this question can be reformulated in the following manner: what will happen if we put the scales of $R_{e^+e^-}$ and the corresponding sum rules to be equal to each other? Will the experimental data produce the conformal invariant limit, if we put $s = |Q^2|$? Recall, that in this case, the theoretical expression for the generalized Crewther relation will differ from the conformal invariant result starting from the proportional to the well- known factor $\beta(\alpha_s)/\alpha_s$ the α_s^2-order corrections [41], which presumably reflects the violation of the conformal symmetry by the procedure of renormalization [25, 42, 43]. Notice, however, that the size of the perturbative contribution proportional to the QCD β - function is rather small [41].

Commensurate scale relations such as the generalized Crewther relation discussed here open up additional possibilities for testing QCD. One can compare two observables by checking that their effective charges agree both in normalization and in their scale dependence. The ratio of leading-order commensurate scales $\lambda_{A/B}$ is fixed uniquely: it ensures that both observables A and B pass through heavy quark thresholds at precisely the same physical point. The same procedure can be applied to multi-scale problems; in general, the commensurate scales Q^*, Q^{**}, etc. will depend on all of the available scales.

An important computational advantage in the commensurate scale relations is that one only needs to compute the flavor dependence of the higher order terms in order to specify the lower order scales in the commensurate scale relations. We have shown [14] that in many cases the application of the NLO BLM formulas to relate known physical observables in QCD leads to results with surprising elegance and simplicity. The commensurate scale relations for some of the observables ($\alpha_R, \alpha_\tau, \alpha_{g_1}$ and α_{F_3}) are universal in the sense that the coefficients of $\widehat{\alpha}_s$ are independent of color; in fact, they are the same as those for Abelian gauge theory. Thus much information on the structure of the non- Abelian commensurate scale relations can be obtained from much simpler Abelian analogs. In fact, in the examples we have discussed here, the non-Abelian nature of gauge theory is reflected in the β-function coefficients and the choice of second-order scale Q^{**}. The commensurate scale relations between observables possibly can be tested at quite low momentum transfers, even where PQCD relationships are expected to break down. It is possible that some of the higher twist contributions common to the two observables are also correctly represented by the commensurate scale relations. In contrast, expansions of any observable in $\alpha_{\overline{\rm MS}}(Q)$ must break down at low momentum transfer since $\alpha_{\overline{\rm MS}}(Q)$ becomes singular at $Q = \Lambda_{\overline{\rm MS}}$. (For example, in the 't Hooft scheme where the higher order $\beta_n = 0$ for $n = 2, 3, \ldots$, $\alpha_{\overline{\rm MS}}(Q)$ has a simple pole at $Q = \Lambda_{\overline{\rm MS}}$.) The commensurate scale relations allow tests of QCD in terms of finite effective charges without explicit reference to singular schemes such as $\overline{\rm MS}$.

The coefficients in a CSR are identical to the coefficients in a conformal theory where explicit renormalon behavior does not appear. It is thus reasonable to expect that the series expansions appearing in the CSR are convergent when one relates finite observables to each other. Thus commensurate scale relations between observables allow tests of perturbative QCD with higher and higher precision as the perturbative expansion grows.

3. The Connection between the Heavy Quark Potential and Heavy Quark Production near Threshold

As we have noted in the introduction, the coupling $\alpha_V(Q)$ is in many ways the natural physical coupling of QCD. As in QED, the scale Q of this running coupling is the physical momentum transfer. Furthermore, α_V can be directly determined in lattice gauge theory from the heavy quarkonium spectrum.

In this section, which is based on the analysis of Brodsky, Hoang, Kühn, and Teubner [44], we show how the heavy quark coupling can be measured from the angular anisotropy of heavy quarks at threshold through the strong final-state rescattering corrections. We first calculate the anisotropy for QED, and then generalize it to the non-Abelian case. We argue that the angular distributions of the heavy quarks will be reflected in the angular distribution of open charm and beauty production, even in the domain of exclusive channels. The scale-fixed determination of the $\overline{\rm MS}$ coupling is then determined using the commensurate scale relation between α_V and $\alpha_{\overline{\rm MS}}$.

The amplitude for the creation of a massive fermion pair from a virtual photon is characterized by the Dirac (F_1) and Pauli (F_2) form factors:

$$\overline{u}\,\Lambda_\mu\,v = ieQ_f\,\overline{u}\,[\,\gamma_\mu\,F_1(q^2) + \frac{i}{2\,m}\,\sigma_{\mu\nu}\,q^\nu\,F_2(q^2)\,]\,v \tag{15}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The photon momentum flowing into the vertex is denoted by q, the fermion mass by m. The resulting angular distribution is conveniently expressed in terms of the electric and magnetic form factors G_e and G_m [45]:

$$\frac{{\rm d}\,\sigma({\rm e^+e^-} \to {\rm f\overline{f}})}{{\rm d}\,\Omega} = \frac{\alpha^2\,Q_f^2\,\beta}{4\,s}\left[\frac{4\,m^2}{s}\,|G_e|^2\,\sin^2\theta + |G_m|^2\,(1+\cos^2\theta)\right] \tag{16}$$

with

$$G_e = F_1 + \frac{s}{4\,m^2}F_2\,, \qquad G_m = F_1 + F_2\,. \tag{17}$$

The anisotropy is thus given by

$$\begin{aligned} A &= \frac{|G_m|^2 - (1-\beta^2)|G_e|^2}{|G_m|^2 + (1-\beta^2)|G_e|^2} \\ &= \frac{\widetilde{A}}{1-\widetilde{A}}\,, \end{aligned} \tag{18}$$

where

$$\widetilde{A} = \frac{\beta^2}{2}\,\frac{|F_1|^2\,(1-\beta^2) - |F_2|^2}{|F_1+F_2|^2\,(1-\beta^2)}\,. \tag{19}$$

In Born approximation $A_{\rm Born} = \beta^2/(2-\beta^2)$. Notice that for $F_2 = 0$, the anisotropy is identical to the Born prediction, independent of F_1. Thus the form

$$\frac{\widetilde{A}}{\widetilde{A}_{\rm Born}} - 1 = -\,2\,F_2(s)\left[1 + \mathcal{O}(\frac{\alpha}{\pi})\right], \qquad \widetilde{A}_{\rm Born} = \frac{\beta^2}{2} \tag{20}$$

isolates $F_2(s)$. This provides a way to experimentally determine the timelike Pauli form factor of the τ lepton. The QED prediction is

$$F_2(4\,m^2) = -\,\frac{\alpha}{2\,\pi} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^2\right) \tag{21}$$

which is, up to the sign, equal to the familiar Schwinger result $F_2(0) = \alpha/2\pi$. Away from threshold the one-loop QED prediction is

$$F_2(\beta) = \frac{\alpha}{\pi}\left[\frac{1-\beta^2}{4\,\beta}\,\ell n\frac{1-\beta}{1+\beta}\right]. \tag{22}$$

This type of higher twist correction will be neglected in the following.

In QED, the order α correction to the Dirac form factor δF_1 in the timelike region exhibits an infrared singularity which can be regulated using a nonvanishing photon mass λ:

$$\delta F_1 = \delta F_1^{\rm fin} + \delta F_1^{\rm IR}\,\ell n\frac{s}{\lambda^2} \tag{23}$$

with

$$\begin{aligned} \delta F_1^{\rm fin} &= \frac{\alpha\,\pi}{4\,\beta} - \frac{3}{2}\,\frac{\alpha}{\pi} + \frac{\alpha\,\pi}{4}\,\beta + \mathcal{O}(\beta^2)\,, & (24)\\ \delta F_1^{\rm IR} &= -\,\frac{2}{3}\,\frac{\alpha}{\pi}\,\beta^2 + \mathcal{O}(\beta^4)\,. & (25) \end{aligned}$$

The leading term of $F_1^{\rm fin}$ is proportional $\pi\alpha/\beta$ and exhibits the familiar Coulomb singularity. Also the constant term and the term linear in β are infrared finite. The infrared singular part of F_1 is strongly suppressed at threshold $\propto \beta^2$, giving rise to a β^3 contribution to the rate. The correction to the Pauli form factor δF_2 is infrared finite and approaches a constant value at threshold:

$$\delta F_2 = -\,\frac{1}{2}\,\frac{\alpha}{\pi} + \mathcal{O}(\beta^2)\,. \tag{26}$$

Real radiation, in contrast, vanishes as β^3 in the threshold region, where two powers of β result from the square of the dipole matrix element, and one power of β comes from phase space. It exhibits the same logarithmic dependence on the infrared cutoff as the F_1 form factor and the same leading β dependence as the infrared singular part of the virtual correction. As a consequence of the strong suppression $\propto \beta^3$ it can be neglected in the threshold region, together with the corresponding infrared divergent part of the form factor. The angular distribution and, similarly, the correction to the total cross section in the threshold region are therefore determined by the the infrared finite parts of the form factors. To order α one thus finds for the coefficient describing the angular dependent piece

$$A = A_{\mathrm{Born}} \left(1 + \frac{2}{2-\beta^2} \frac{\alpha}{\pi} \right) . \tag{27}$$

As we shall show there are interesting modifications of the anisotropy due to the running of the QCD coupling, and the dependence of the renormalization scale on $\sqrt{s}$ and $|\vec{q}|$ will be crucial.

The $\mathcal{O}(\alpha^2)$-QED corrections to the form factors, induced by light fermion loops, have been calculated analytically [46]. In the threshold region one obtains

$$\begin{aligned} F_1 &= 1 + \frac{\alpha\,\pi}{4\,\beta} \left[1 + \left(\frac{\alpha}{\pi}\right) \sum_{i=1}^{n_f} \frac{1}{3} \left(\ell n \frac{s\,\beta^2}{m_i^2} - \frac{8}{3} \right) \right] \\ &\quad - \frac{3}{2}\frac{\alpha}{\pi} \left[1 + \left(\frac{\alpha}{\pi}\right) \sum_{i=1}^{n_f} \frac{1}{3} \left(\ell n \frac{s}{4\,m_i^2} - \frac{1}{2} \right) \right] , \end{aligned} \tag{28}$$

$$\begin{aligned} F_2 &= \frac{\alpha\,\pi}{4\,\beta} \left[\left(\frac{\alpha}{\pi}\right) \frac{n_f}{3} \right] \\ &\quad - \frac{1}{2}\frac{\alpha}{\pi} \left[1 + \left(\frac{\alpha}{\pi}\right) \sum_{i=1}^{n_f} \frac{1}{3} \left(\ell n \frac{s}{4\,m_i^2} - \frac{13}{6} \right) \right] . \end{aligned} \tag{29}$$

The calculation has been performed in the limit where the mass of the light virtual fermion m_f is far smaller than m, a situation appropriate for the subsequent translation to QCD. The factor n_f is introduced to allow for several light fermions and, in our case, to single out the fermion-induced terms. These formulae provide the first step on the way to a full two-loop calculation in order α^2. As we shall see, the results require two conceptually different scales in the argument of the running coupling, a scale of order s from the hard virtual correction from transverse photons and a soft scale of order $s\,\beta^2$ from the Coulomb rescattering. Supplemented by the BLM prescription they even determine the dominant two-loop gluon-induced terms in QCD.

The linear combination appearing in the denominator of $\widetilde{A}$ in Eq. (19) is thus given by

$$\begin{aligned} F_1 + F_2 &= 1 + \frac{\alpha\,\pi}{4\,\beta}\left[1 + \left(\frac{\alpha}{\pi}\right)\sum_{i=1}^{n_f}\frac{1}{3}\left(\ell n\frac{s\,\beta^2}{m_i^2} - \frac{5}{3}\right)\right] \\ &\quad - 2\,\frac{\alpha}{\pi}\left[1 + \left(\frac{\alpha}{\pi}\right)\sum_{i=1}^{n_f}\frac{1}{3}\left(\ell n\frac{s}{4\,m_i^2} - \frac{11}{12}\right)\right]\,. \end{aligned} \tag{30}$$

The n_f terms arise from the vacuum polarization insertions and thus can be resummed into the QED running coupling:

$$\alpha(Q^2) = \alpha\,\left[1 + \left(\frac{\alpha}{\pi}\right)\sum_{i=1}^{n_f}\frac{1}{3}\left(\ell n\frac{Q^2}{m_i^2} - \frac{5}{3}\right)\right]\,. \tag{31}$$

The constant $5/3$ is the usual term in the Serber-Uehling vacuum polarization $\Pi(Q^2)$ at large Q^2. This corresponds to the usual QED scheme where $V(Q^2) = -\,4\pi\,\alpha(Q^2)/Q^2$ is the QED potential for the scattering of heavy test charges. One thus obtains

$$\begin{aligned} F_1 + F_2 &= 1 + \frac{\alpha(s\,\beta^2)\,\pi}{4\,\beta} - 2\,\frac{\alpha(s\,e^{3/4}/4)}{\pi} \\ &\cong \left(1 - 2\,\frac{\alpha(s\,e^{3/4}/4)}{\pi}\right)\left(1 + \frac{\alpha(s\,\beta^2)\,\pi}{4\,\beta}\right)\,. \end{aligned} \tag{32}$$

Two distinctly different correction factors arise. The first originates from hard transverse photon exchange, with the scale set by the short distance process; the second from the instantaneous Coulomb potential. It is remarkable and non-trivial that the non-logarithmic terms in the $\pi\alpha/\beta$ corrections are absorbed if the relative momentum is adopted as the scale for the coupling. Up to two loops the running coupling governing the Coulomb singularity is thus identical to the running coupling in the potential. This will provide an important guide for the application of these results to QCD.

The proper resummation of the $1/\beta$ terms based on Sommerfeld's rescattering formula then leads to

$$|F_1 + F_2|^2 = \left(1 - 4\,\frac{\alpha(m^2\,e^{3/4})}{\pi}\right)\,\frac{x}{1-e^{-x}} \tag{33}$$

with

$$x = \frac{\alpha(4\,m^2\,\beta^2)\,\pi}{\beta}\,. \tag{34}$$

In a similar way one finds for the relevant combination in the numerator of (19)

$$|F_1|^2 - |F_2|^2 \cong \left(1 - 3\,\frac{\alpha(m^2\,e^{7/6})}{\pi}\right)\,\frac{x'}{1-e^{-x'}} \tag{35}$$

with

$$x' = \frac{\alpha(4\,m^2\,\beta^2/e)\,\pi}{\beta}\,. \tag{36}$$

The $|F_2|^2$ term in the numerator can actually be ignored in the present approximation. The scales of the effective coupling differ in the numerator and denominator of (19). In particular, in the factor arising from Coulomb exchange the scale is significantly smaller in the numerator than in the denominator. This behavior is consistent with qualitative considerations based on the relative distances relevant for S- versus P-waves in the Coulomb part. In the factor arising from hard photon exchange the scales are quite comparable, with a slightly larger value in the numerator.

One thus arrives at the prediction in QED for the anisotropy which involves four scales:

$$A = \frac{\widetilde{A}}{1-\widetilde{A}}\,, \qquad \widetilde{A} = \frac{\beta^2}{2}\,\frac{\left(1 - 3\,\frac{\alpha(m^2 e^{7/6})}{\pi}\right)}{\left(1 - 4\,\frac{\alpha(m^2 e^{3/4})}{\pi}\right)}\,\frac{1-e^{-x}}{1-e^{-x'}}\,\frac{\alpha(4\,m^2\,\beta^2/e)}{\alpha(4\,m^2\,\beta^2)}\,. \tag{37}$$

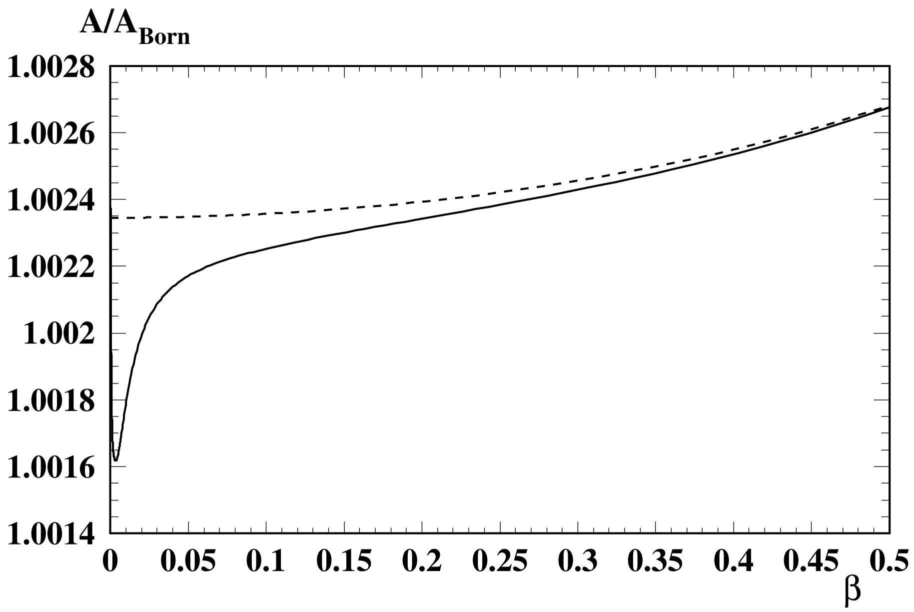

Figure 3: Ratio between the anisotropy A and the Born prediction $A_{\rm Born}$ as function of β for the process $e^+e^- \to \tau^+\tau^-$. Dashed curve: constant α; solid curve: including the running of α.

To display the effects more clearly, the ratio of the anisotropy to the Born prediction $A/A_{\rm Born}$ is shown in Fig. 3 for the case of τ pair production. The dashed curve gives the prediction for constant $\alpha_{\rm QED}$; the solid curve shows the effect of the lepton vacuum polarization $\Pi(Q^2)$ in the QED running coupling. The vacuum polarization affects the anisotropy for small β because two different scales appear in the S- and P-wave Coulomb rescattering corrections. Away from threshold A essentially measures the anomalous magnetic moment.

The QED coupling $\alpha(Q^2)$ translates into the the QCD coupling $\alpha_{\rm V}(Q^2)$, defined as the effective charge in the potential

$$V(Q^2) = -\,\frac{4}{3}\,\frac{4\,\pi\,\alpha_{\rm V}(Q^2)}{Q^2} \qquad (38)$$

for the scattering of two heavy quarks in a color-singlet state. In the BLM procedure all terms arising from the non-zero beta- function are resummed into $\alpha_{\rm V}(Q^2)$. For example, all n_f-dependent coefficients vanish in the $\pi\alpha/\beta$ terms if the scale of the relative momentum is adopted. This is, in fact, a result expected on general grounds: threshold physics is governed by the nonrelativistic instantaneous potential. Below threshold, the potential leads to bound states, above threshold it affects the cross section through final state interactions. It is, therefore, natural to take for the QCD case the coupling governing the QCD potential at the momentum scale involved in the rescattering.

In a similar way, BLM scale-fixing is adopted for the correction from hard gluon exchange. In the radiative correction, there still remain $\mathcal{O}(\alpha_s^2)$ terms, identical to the radiative corrections for the theory with a fixed coupling constant. With the same scheme convention for the coupling as above, one arrives at

$$\widetilde{A} = \frac{\beta^2}{2} \frac{\left(1 - 4\,\frac{\alpha_{\rm V}(m^2 e^{7/6})}{\pi}\right)}{\left(1 - \frac{16}{3}\,\frac{\alpha_{\rm V}(m^2 e^{3/4})}{\pi}\right)} \, \frac{1 - e^{-x_s}}{1 - e^{-x_s'}} \, \frac{\alpha_{\rm V}(4\,m^2\,\beta^2/e)}{\alpha_{\rm V}(4\,m^2\,\beta^2)} \tag{39}$$

where

$$x_s = \frac{4\,\pi}{3}\,\frac{\alpha_{\rm V}(4\,m^2\beta^2)}{\beta}, \qquad x_s' = \frac{4\,\pi}{3}\,\frac{\alpha_{\rm V}(4\,m^2\beta^2/e)}{\beta}. \tag{40}$$

The anisotropy A is plotted in Fig. 4 versus the velocity β in the range $0.2 < \beta < 0.5$ for charmed, bottom, and top quarks. For comparison, the tree level prediction is also shown. For charmed quarks, only β values above 0.4 are admitted in order to allow for the simultaneous production of $D\overline{D}$ and $D^*\overline{D}$. The charm prediction is particularly sensitive to the QCD parameters, since very low scales are accessible. Measurements of the anisotropy for $e^+e^- \to c\bar{c}$ thus have the potential of determining $\alpha_{\rm V}$ in the regime where perturbation theory begins to fail.

The curves are based on an input value $\alpha_{\overline{\rm MS}}^{(n_f=5)}(M_Z^2) = 0.115$. We use the two-loop beta-function to evolve $\alpha_{\overline{\rm MS}}$ to lower momenta and then used Eq. (3) to calculate $\alpha_{\rm V}(Q^2)$. To investigate the sensitivity of the predictions for bottom quarks, the input value for $\alpha_{\overline{\rm MS}}(M_Z^2)$ has been varied by ± 0.008 from the central value of 0.115. As

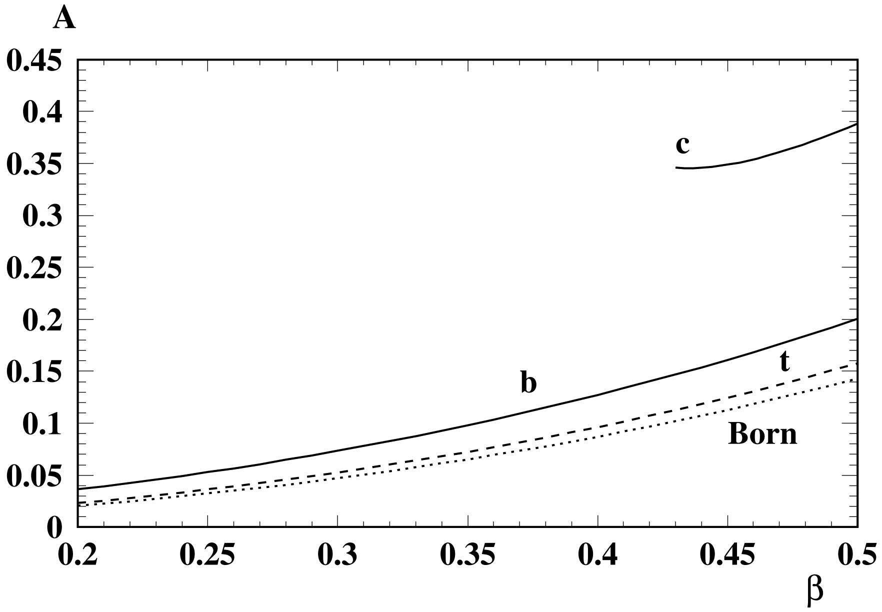

Figure 4: Anisotropy for charmed, bottom and top quark production as a function of β. Also shown is the Born prediction. We have assumed the effective quark masses $m_c = 1.7$ GeV, $m_b = 5$ GeV and $m_t = 175$ GeV.

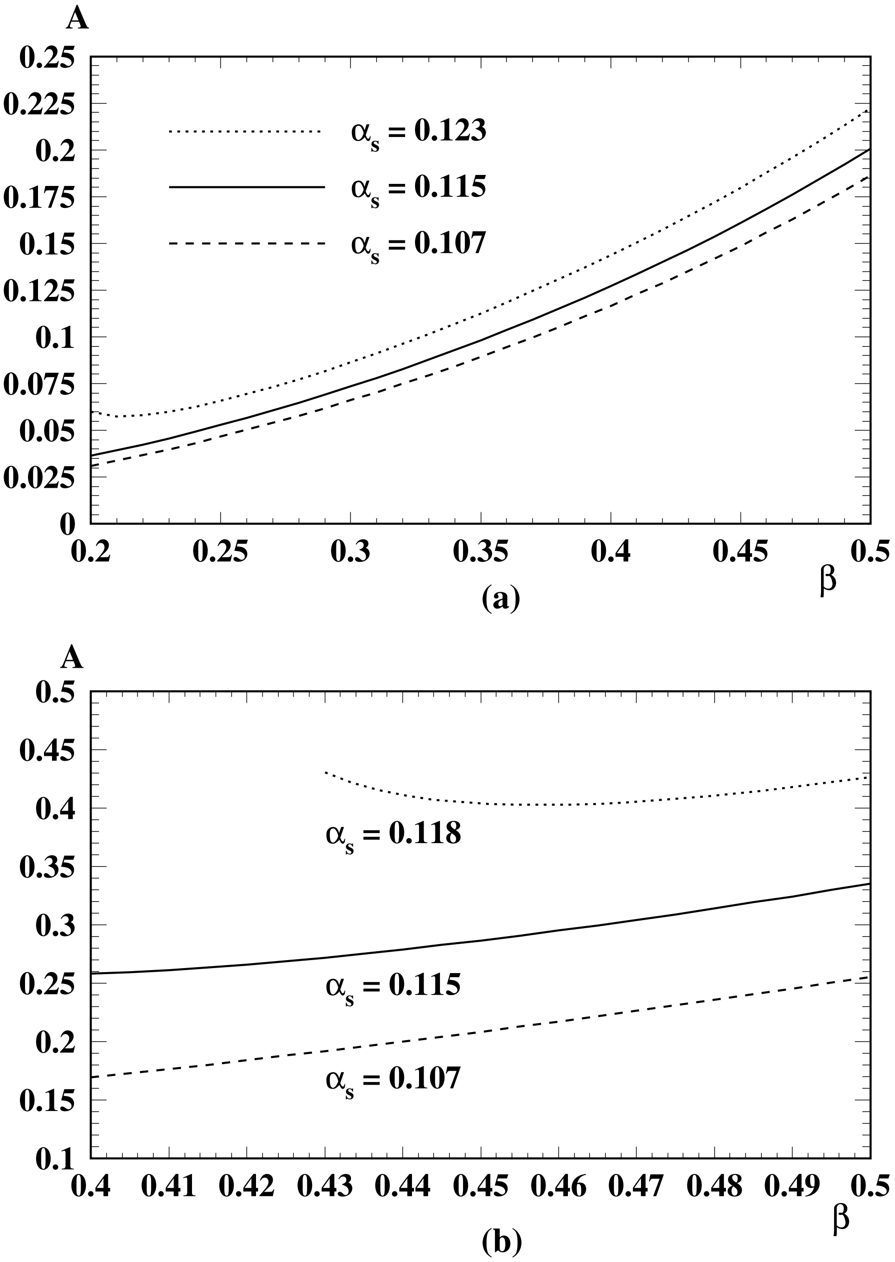

Figure 5: Sensitivity of the anisotropy A for (a) $e^+e^- \to b\bar{b}$ and (b) $e^+e^- \to c\bar{c}$ to changes in $\alpha_{\overline{\rm MS}}(M_Z^2)$.

demonstrated in Fig. 5a the variation of the anisotropy parameter amounts to about 10%, and could therefore be accessible experimentally. The charm predictions (see Fig. 5b) are even more sensitive.

The anisotropy $A(\beta^2)$ in the center-of-mass angular distribution $\mathrm{d}\sigma(e^+e^- \to Q\overline{Q})/\mathrm{d}\Omega \propto 1 + A\cos^2\theta$ of heavy quarks produced near threshold is sensitive to the QCD coupling $\alpha_{\mathrm{V}}(Q^2)$ at specific scales determined by the quark relative momentum $p_{\mathrm{cm}} = \sqrt{s}\,\beta$. The coupling $\alpha_{\mathrm{V}}(Q^2)$ is the physical effective charge defined through heavy quark scattering. An important consequence of heavy quark kinematics is that the production angle of a heavy hadron follows the direction of the parent heavy quark. This applies not only at Born approximation, but also after QCD corrections have been applied. The predictions thus provide a connection between two types of observables, the heavy quark potential and the angular distribution of heavy hadrons, independent of theoretical scale and scheme conventions.

An important feature of this analysis is the use of BLM scale-fixing, in which all higher-order corrections associated with the beta-function are resummed into the scale of the coupling. The resulting scale for $\alpha_{\mathrm{V}}(Q^2)$ corresponds to the mean gluon virtuality. In the case of the soft rescattering corrections to the S-wave, the BLM scale is $s\,\beta^2 = p_{\mathrm{cm}}^2$. One thus has sensitivity to the running coupling over a range of momentum transfers within the same experiment. The anisotropy measurement thus can provide a check on other determinations of $\alpha_{\mathrm{V}}(Q^2)$, *e.g.* from heavy quark lattice gauge theory, or from the conversion of $\alpha_{\overline{\mathrm{MS}}}$ determinations to α_{V}.

Our analysis also shows that the running coupling appears within the cross section with several different scales. This is particularly apparent at low β where the physical origin of the $\mathcal{O}(\alpha_s)$ corrections can be traced to gluons with different polarization and virtuality.

In principle, the anisotropy of τ pairs produced in $e^+e^- \to \tau^+\tau^-$ could be used to measure the Pauli form factor $F_2(s)$ near threshold $s \gtrsim 4m_\tau^2$. A highly precise measurement of the anisotropy thus could provide a measurement of a fundamental parameter of the τ lepton: its timelike anomalous magnetic moment.

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