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Beam break-up with tune-chirp for an arbitrary wakefield

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Abstract

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I. INTRODUCTION

Transverse beam-break-up (BBU) has become a very popular subject over the last thirty or so years, particularly for high current relativistic electron beams.[1] Cumulative collective instability arises for beams in plasmas,[2] bunched beams in rf linacs,[3] pulsed beams in induction linacs,[4] in large-scale free-electron lasers,[5] and in other venues.[6,7] Many methods have been proposed to control BBU,[8] and all involve either taming the wake within each cell[9,10] or over many cells,[11,12] or taming the beam itself, by tampering with the focusing mechanism---introducing non-linearities,[13] or a variation in focusing strength along the beam. The latter effect, a “chirp” in betatron tune, can be accomplished via "conditioning"[14] or by way of rf quadrupole or energy variation along the beam, a technique proposed by Balakin, Novokhatsky and Smirnov (BNS).[15] The BNS effect in rf linacs has since been analyzed by a number of workers,[16,17] and has proven invaluable for practical operation in a large collider.[18]

Here the effect of tune chirp is analyzed for an unbunched, coasting beam, for an arbitrary wake. The calculation is motivated by the need for a simple appraisal of tune chirp for comparison with other mechanisms, as they might apply apply to a large induction linac and its dominant wakefields, a resonant mode,[19] and the resistive wall wake.[20] While work presented here is of special interest for intense beams, as in the “two beam accelerator”,[21] this work is also of general interest in that it provides, for the first time, an analytic closed-form expression quantifying growth in the presence of an *arbitrary* wakefield. The problem is formulated in

terms of a Green's function expressed in closed form for an arbitrary wake in Sec. II, and illustrated for the example of the resistive wall in Sec. III and for a resonator mode in Sec. IV.

II. FORMULATION

Beam break-up is described by an equation for the transverse displacement of the beam centroid, ξ, in its simplest form[3]

$$\left(\frac{\partial}{\partial z}\gamma\frac{\partial}{\partial z}+\gamma k_\beta^2\right)\xi(z,\zeta)=\int_0^\zeta d\zeta' W(\zeta-\zeta')\nu(\zeta')\xi(z,\zeta'), \qquad (1)$$

where $\zeta=t-z/c$ is the displacement along the ultra-relativistic beam and varies from 0 at the beam head to τ at the beam tail, with τ the pulse length, t the time, z the axial displacement and c the speed of light. Beam electrons remain at a fixed ζ, as they propagate in z down the beam pipe. The wake $W(\zeta-\zeta')$, is the Green's function which determines the Lorentz force on an electron at ζ due to the electric and magnetic fields generated by beam segments at ζ'. The Budker parameter $\nu=I/I_0$ where the beam current is I and is assumed constant for $\zeta>0$ ("unbunched beam"), and zero for $\zeta<0$. The Alfven constant I_0~17 kA. The Lorentz factor γ is assumed constant in z ("coasting beam"). The betatron wavenumber is assumed to take the form $k_\beta(\zeta)=k_0+\Delta k(\zeta/\tau)$, where $k_0=2\pi/\lambda_0$, λ_0 is the betatron wavelength at the beam head.

With k_β varying along the beam, solution of Eq. (1) appears intractable in general. With a linear chirp however, the problem can

be solved in closed form. We express ξ in terms of a complex envelope χ,

$$\xi(z,\zeta) = Im\{\chi(z,\zeta)\, exp(ik_\beta(\zeta)z)\}. \tag{2}$$

and consider the limit where the envelope χ varies little on the λ_0 length scale, *i.e.*, the "strong focusing" limit, $L_g >> \lambda_0$, with L_g the instability growth length. In this case an eikonal approximation is appropriate; substituting Eq. (2) for ξ in Eq. (1), there results

$$\frac{\partial\chi}{\partial z}(z,\zeta) \approx \frac{\nu}{2i\gamma_0 k_0}\int_0^\zeta d\zeta' W(\zeta-\zeta')\,\chi(z,\zeta') exp\left(-i\,\Delta k\, z\,\frac{\zeta-\zeta'}{\tau}\right), \tag{3}$$

and Laplace transforming Eq. (3) in ζ, we find

$$\frac{\partial\tilde{\chi}}{\partial z}(z,p) = \frac{\nu}{2i\gamma_0 k_0}\tilde{W}\left(p + i\frac{\Delta k\, z}{\tau}\right)\tilde{\chi}(z,p), \tag{4}$$

where p is the Laplace transform variable and the overtilde denotes the Laplace transform. When $\Delta k=0$ frequency components on the beam are continuously driven by their counterpart in the impedance. When $\Delta k\neq 0$, and after travelling a distance z, the beam will have sampled the impedance over a bandwidth $\Delta kz/\tau$.

Integrating Eq. (4) in z and inverting the Laplace transform produces the solution

$$\chi(z,\zeta) = \int_0^{\zeta} d\zeta' G(z,\zeta-\zeta';\Delta k)\chi(0,\zeta'), \tag{5}$$

where the Green's function is given by

$$G(z,\zeta;\Delta k) = \frac{1}{2\pi i}\int_{-i}^{+i} dp \exp\theta(z,\zeta,p;\Delta k), \tag{6}$$

the contour is to the right of all poles of the integrand in the complex p-plane, and the exponential phase is

$$\theta = p\zeta + \frac{\nu}{2 i \gamma_0 k_0}\int_0^{z} dz' \tilde{W}\left(p + i\frac{\Delta k z'}{\tau}\right). \tag{7}$$

The asymptotic dependence of χ on z and ζ is determined by the method of steepest descent from the point(s) of stationary phase in the p-plane, where

$$\theta^{(1)}(p) = \zeta - \frac{\nu\tau}{2\gamma_0 k_0 \Delta k}\left\{\tilde{W}\left(p + i\frac{\Delta k z}{\tau}\right) - \tilde{W}(p)\right\} = 0. \tag{8}$$

($^{(n)} = \partial^n/\partial p^n$). The asymptotic form for χ may then be obtained from Eq. (5) and depends on the character of the dominant stationary phase point(s). The contribution from a single such point is

$$\chi(z,\zeta) \approx \tilde{\chi}(0,p)\left(2\pi\theta^{(2)}\right)^{-1/2}\exp\theta, \tag{9}$$

when this is not an inflection point, and where θ and $\theta^{(2)}$ are evaluated at the stationary point. In the case of an inflection point one has

$$\chi(z,\zeta) \approx \tilde{\chi}(0,p)\frac{\Gamma\left(\frac{1}{3}\right)}{2^{2/3}3^{1/6}\pi}\left(\theta^{(3)}\right)^{-1/3}\exp\theta, \qquad (10)$$

where $\Gamma(1/3)\sim 2.6789$. In general the amplitude may include a sum over stationary points.

It will be helpful to notice that if $p(z,\zeta;\Delta k)$ is a stationary point satisfying Eq. (8), then

$$p(z,\zeta;-\Delta k) = p(z,\zeta;\Delta k) + i\frac{\Delta k z}{\tau}, \qquad (11)$$

is a stationary point of Eq. (8) with the opposite sign of Δk. Substituting this in Eq. (7) (with the sign of Δk reversed) one can show that the real part of the asymptotic growth exponent is *independent* of the sign of Δk. Moreover one can show that $|\theta^{(n)}|$ is also independent of the sign of Δk. Consequently asymptotic growth, as given by expressions like Eq. (9) depends on the sign of Δk only through the algebraic dependence on the incident beam spectrum. To illustrate these considerations and to derive some practical scalings, we analyze two practical examples.

III. RESISTIVE WALL

We consider the impedance due to a resistive pipe,[22]

$$W(\zeta-\zeta') \approx \frac{4}{\sqrt{\pi}} \frac{1}{\tau_D^{1/2} b^2} \frac{1}{\sqrt{\zeta-\zeta'}}, \tag{12}$$

where $\tau_D = 4\pi\sigma b^2/c^2$, is a diffusion time-scale, the pipe conductivity is σ, and b is the pipe radius.[23] It is convenient to introduce a dimensionless wakefield parameter

$$w = \frac{4}{\pi^{1/2}} \left(\frac{\nu}{\gamma}\right) \frac{1}{k_0^2 b^2} \left(\frac{\tau}{\tau_D}\right)^{1/2}. \tag{13}$$

The envelope expressed as a function of $k_0 z$ and ζ/τ is parameterized by $\Delta k/k_0$ and w. For example, for $\Delta k/k_0 \to 0$ the envelope varies with exponent

$$Re\,\theta \approx \frac{3\pi^{1/3}}{2^{7/3}} w^{2/3} \left(\frac{\zeta}{\tau}\right)^{1/3} (k_0 z)^{2/3}, \tag{14}$$

and the beam tail experiences growth as $exp(z/L_g)^{2/3}$ where $L_g \sim 0.2\lambda_0/w$. The strong focusing approximation then limits us to the range $w<0.2$ or so. Parameters of practical interest would be $\lambda_0 \sim 1$ m, $I \sim 1.5$ kA, $\sigma \sim 1\text{x}10^{17}$ sec^{-1}, $b \sim 2$ cm, $\gamma \sim 20$, $\tau \sim 20$ ns, and $L \sim 50$ m, and these correspond to $w \sim 1x10\text{-}3$.[20] We'll consider larger w to test the limits of the strong focusing approximation.

To simplify algebraic expressions we introduce two additional parameters,

$$\mu = \left| \frac{k_0}{\Delta k} \right|^{3/2} \left(\frac{\tau}{\zeta} \right) \frac{2\pi^{1/2} w}{(k_0 z)^{1/2}}, \tag{15}$$

$$\delta = \frac{\Delta k z}{2\tau}. \tag{16}$$

We exchange the variable p for q such that $p = |\delta| q - i\delta$ and let

$$r = \frac{2}{\mu} \sqrt{q^2 + 1}. \tag{17}$$

In terms of these quantities, the stationary point is determined by

$$\sqrt{2}\, r = \sqrt{q - i} - \sqrt{q + i}, \tag{18}$$

which implies that r is a root of the quartic,

$$r^4 + \mu r^3 + 1 = 0. \tag{19}$$

In terms of r and q, the phase and second derivative are

$$\theta = \zeta (q|\delta| - i\delta + \mu r |\delta|), \tag{20}$$

$$\theta^{(2)} = 4\zeta \frac{q + \frac{1}{4}\mu r}{\mu^2 r^2 |\delta|}. \tag{21}$$

Thus the root r determines the asymptotic growth, and there is only one root of Eq. (19) satisfying Eq. (18). For $\mu<<1$ (large z) it is approximately

$$r \approx exp\left(-i\frac{\pi}{4}\right) - \frac{\mu}{4} \quad . \tag{22}$$

For detailed comparisons this approximate form is not adequate and we note the exact solution for $\mu<1.75$,

$$r = \frac{1}{4}\left(u^{3/2} - \mu\right) - \frac{i}{2}\left(u - \frac{1}{2}\mu^2 + \frac{\mu^3}{2u^{3/2}}\right)^{1/2} \quad , \tag{23}$$

where u is the root of the associated cubic ($u^3-4u-\mu^2=0$) satisfying $u>2$, *i.e.*, $u=4sin(\varphi-\pi/6)/3^{1/2}$ with $cos(3\varphi)=3^{3/2}\mu^2/16$ and $\pi/2<\varphi<5\pi/6$. (In practice, faced as one may be, with many quartics, it is simplest to rely on a quartic solving subroutine.)

As expected, the sign of the tune chirp does not appear in the real part of the exponent. Nevertheless growth does depend markedly on this sign. Taking a unit displacement of the beam at $z=0$ for illustration, we have $\tilde{\chi}(0,p) = i/p$, with $p=q|\delta|-i\delta$, and considering the small μ limit, we find

$$|\chi| \approx \frac{1}{\sqrt{\pi\varepsilon}}\, exp\left(\sqrt{\frac{z}{L}} - \varepsilon\right) \times \begin{cases} 1 & ; \Delta k < 0 \\ \dfrac{\mu^2}{16} & ; \Delta k > 0 \end{cases} \tag{24}$$

where

$$\varepsilon = \frac{\pi}{4} w^2 \left(\frac{\tau}{\zeta}\right)\left(\frac{k_0}{\Delta k}\right)^2 , \tag{25}$$

$$L = \frac{1}{\pi^2 w^2}\left|\frac{\Delta k}{k_0}\right|\lambda_0 . \tag{26}$$

To check these results, we solve Eq. (1) numerically. In Fig.1 are depicted analytic and numerical solutions for χ at the beam tail as a function of z, for wake strength parameter w=0.1 and fractional tune chirps $\Delta k/k_0$=±0.05, 0. Clearly the amplitude is little reduced for negative tune chirp. In Fig. 2, comparison is made of analytic and numerical results, for several wake strengths w, as a function of $\Delta k/k_0$. (Comparison is made at the beam tail, since for these parameters, the maximum in amplitude along the beam is quite close to that at the tail.) The gradual divergence for large w, of the analytic result is due to the break down of the strong focusing approximation, and confirms the constraint w<0.2.

A similar analysis shows that, in general, for impedances varying asymptotically as p^{-r}, with r<1, there occurs a transition, from an exponent varying as z^n, with $n=1/(1+r)$, to one varying as $z^{(1-r)}$, representing more gradual growth. There is *no saturation* in the case of such idealized "broadband" impedances.

IV. RESONATOR WAKE

At the opposite extreme from the broadband resistive wall impedance is the resonator wake of the generic form

$$W(\zeta) = W_0 \frac{\omega_0^2}{\Omega} exp(-\frac{\omega_0 \zeta}{2Q}) sin(\Omega\zeta) , \quad (27)$$

for which the Laplace transform is

$$\tilde{W}(p) = W_0 \frac{\omega_0^2}{\omega_0^2 + p\frac{\omega_0}{Q} + p^2} , \quad (28)$$

with $\Omega = \omega_0(1-1/4Q^2)^{1/2}$, and Q the quality factor. Such a wake appears in the case of a single, dominant, TM_{11} mode of a microwave cavity for which the factor W_0 may be expressed in terms of a shunt impedance per unit length $r_\perp$,[24] $W_0 = r_\perp \omega_0/Q$. In the absence of tune chirp, the solution as a function of $k_0 z$ and $\omega_0 \zeta$ is characterized by two parameters, the quality factor Q and the dimensionless wake amplitude,

$$w = \frac{\nu W_0 \omega_0}{\gamma_0 k_0^2 \Omega} . \quad (29)$$

Parameters of practical interest would be λ_0~1 m, I~1.5 kA, γ~20, τ ~20 ns, Q~6, $\omega_0/2\pi$~1GHz, $r_\perp/Q$~8Ω/m and L~50 m, and these correspond to $w \sim 4x10^{-3}$. As noted in previous works there are several regimes of growth[1] and we will specialize to the limit of a long pulse $\Omega\tau >> 1$. In this case, with no tune chirp, the asymptotic exponent is

$$Re\,\theta = \left(k_0 z\,\omega_0 \tau w\right)^{1/2} - \frac{\omega_0 \tau}{2Q}, \tag{30}$$

corresponding to a growth length $L_g \sim \lambda_0/2\pi w \omega_0 \tau$. The condition for adiabatic growth is $w < 1/2\pi\omega_0\tau$, or $w < 1\text{x}10^{-3}$ for the "typical" parameters. We'll consider $w \sim 1\text{x}10^{-2}$ to test the limits of the strong focusing approximation.

To account for chirp it is convenient to introduce the dimensionless parameters

$$\delta = \frac{\Delta k z}{2\,\Omega\tau}, \tag{31}$$

$$\varepsilon = \frac{vW_0\omega_0^2}{4\,\gamma_0\Omega}\frac{1}{\Delta k} = \frac{1}{4\,\Delta k L_g}. \tag{32}$$

The term ε is small when phase-mixing is rapid on the scale of a growth length. We make the change of variable,

$$p = i\Omega(r - \delta) - \frac{\omega_0}{2Q}, \tag{33}$$

in terms of which the exponent in Eq. (7) takes the form

$$\theta = -i\delta\Omega\zeta - \frac{\omega_0}{2Q}\zeta + i\Omega r + i\varepsilon\,ln\left(\frac{1 - r - \delta}{1 - r + \delta}\frac{1 + r - \delta}{1 + r + \delta}\right), \tag{34}$$

with the logarithm defined by analytic continuation from the region of r imaginary and negative. After differentiation and some algebra, Eq. (8) results in a two-parameter quartic polynomial in r,

$$r^4 - 2r^2(1+\delta^2) - 8r\delta\left(\frac{\varepsilon}{\Omega\zeta}\right) + (1-\delta^2)^2 = 0 \quad , \tag{35}$$

with coefficients independent of the sign of the tune chirp. Only one root of this quartic results in growth; illustrative solutions for Imr are depicted in Fig. 3.

It is instructive to consider explicit analytic scalings. For small ζ such that $\varepsilon/\Omega\zeta >> 1$, Re$r$~-$|\delta|$, Im$r$~-$(2|\varepsilon|/\Omega\zeta)^{1/2}$ and the influence of the mode resonance at Ω is diminished. Effectively the wake is linear and the impedance varies as $1/p^2$. In this case saturation occurs after a range $|\delta|$~2-3, usually well past the range of interest~$50\lambda_0$. Accurate prediction of the amplitude in the intermediate regime, prior to saturation, requires the exact solution of the quartic.

More typically one would be interested in the limit $\varepsilon/\Omega\zeta$ small; this corresponds, at the beam tail, to the condition $|\Delta k|/k_0 > w$, a chirp of perhaps 1% or more. In this limit Imr vanishes beyond $\delta \sim 2\varepsilon/\Omega\zeta$, corresponding to an inflection point in the phase lying just short of the point of maximum amplitude. This provides a rough estimate of the length for saturation for the beam tail $z_{sat} \sim 1/\Delta k^2 L_g$. As one can see from Eq. (33) (taking δ small, and r~-1) the root p does not exhibit a strong asymmetry in this limit, since regardless of the sign of the chirp -Imp lies near Ω. To obtain an estimate of this amplitude at saturation we apply Eq. (10) and after some algebra find

$$|\chi|_{sat} \approx 0.6\frac{|\varepsilon^{2/3}|}{\Omega\zeta}\exp\left(\pi|\varepsilon| - \frac{\omega_0\zeta}{2Q}\right). \tag{36}$$

To check these results we may again solve Eq. (1) numerically. Illustrative results are shown in Figs. 4-6. Parameters are fixed at $\tau=40\pi/\omega_0$, and Q=6. Figure 4 compares the asymptotic result of Eq. (9) with simulation for a +5% chirp and $w=1\text{x}10^{-2}$, indicating that close agreement can be obtained after just a few betatron periods.

Figure 5 compares the asymptotic form with the rms envelopes from the simulation, versus position for ±5% and 0 chirp, with $w=1\text{x}10^{-2}$. One can see that there is only a small asymmetry in chirp; one can also see the inflection point just preceding saturation (the small peaks, where Eq. (9) breaks down). In Fig. 6 the maximum in amplitude at the beam tail in the course of propagation through $50\lambda_0$ is depicted versus chirp, for several different wake amplitudes w. Overlayed are the corresponding analytic results: Eq. (10) when $z_{sat}<50\lambda_0$, otherwise Eq. (9) evaluated at $50\lambda_0$. The dashed curves are the maximum in amplitude over the entire beam, and show that for larger chirps the peak amplitude occurs within the body of the pulse. Evidently there is no dramatic dependence on the sign of the chirp.

V. CONCLUSIONS

Beam break-up growth has been computed up to quadrature for an arbitrary wake in the presence of a linear tune chirp. This result reduces the problem of computing asymptotic growth to that of identifying and characterizing stationary points. The result was applied to two representative practical examples, the broadband resistive wall impedance, and the narrow-band resonator impedance.

In the case of a broadband impedance we saw that tune chirp does not in general produce saturation as one would expect from a damping mechanism (viz. Landau damping). For a broadband impedance, varying as $1/p^r$ with $r<1$ no saturation results, although growth can be drastically diminished. In the case of a resonator mode, saturation always results, although the form of the amplitude differs depending on the size of the chirp $|\Delta k|/k_0$ versus the wake amplitude w.

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FIG. 1. Comparison of the numerical solution of Eq. (1) and the analytic result of Eq. (9) for $|\chi(z,\tau)|$ at as a function of z, for wake strength parameter w=0.1 and fractional tune chirps $\Delta k/k_0$=±0.05, 0.

FIG. 2. Comparison of the numerical solution for the maximum in $|\chi(z,\tau)|$ over the course of $50\lambda_0$, with the analytic result for z=$50\lambda_0$, for several wake strengths w, as a function of fractional tune chirp along the beam $\Delta k/k_0$.

FIG. 3. Solutions of Eq. (35) for the root r determining asymptotic growth in the case of a resonator wake, as a function of δ. Solutions correspond to **(a)** $\varepsilon/\Omega\zeta$<1/2 and **(b)** $\varepsilon/\Omega\zeta$>1/2. For small $\varepsilon/\Omega\zeta$ saturation (Im$r\rightarrow$0) occurs for $\delta\sim 2\varepsilon/\Omega\zeta$. For large $\varepsilon/\Omega\zeta$, Im$r\rightarrow -(2\varepsilon/\Omega\zeta)^{1/2}$, for δ~2-3.

FIG. 4. Depicted is the result of Eq. (9) (dashed curve) overlayed with simulation for a +5% chirp, for a pulse 20 resonator periods in length, Q=6 and wake strength w=0.01. Evidently the asymptotic form converges after just a few betatron periods.

FIG. 5. Shown are the asymptotic forms for the envelopes from the simulation, versus position for ±5% and 0 chirp, with w=0.01 and Q=6. Overlayed are the results of Eq. (9) (dashed curves). The small peaks on the analytic curves (just preceding saturation) are in the vicinity of the inflection point where Eq. (9) breaks down.

FIG. 6. Shown here is a survey of the maximum in amplitude over the course of propagation through $50\lambda_0$ versus chirp, for several different wake amplitudes w, with Q=6. Overlayed is the analytic prediction, either the saturated or unsaturated result, as explained in the text. The slopes on the curves change slightly at the transition from saturation within $50\lambda_0$ to unsaturated.

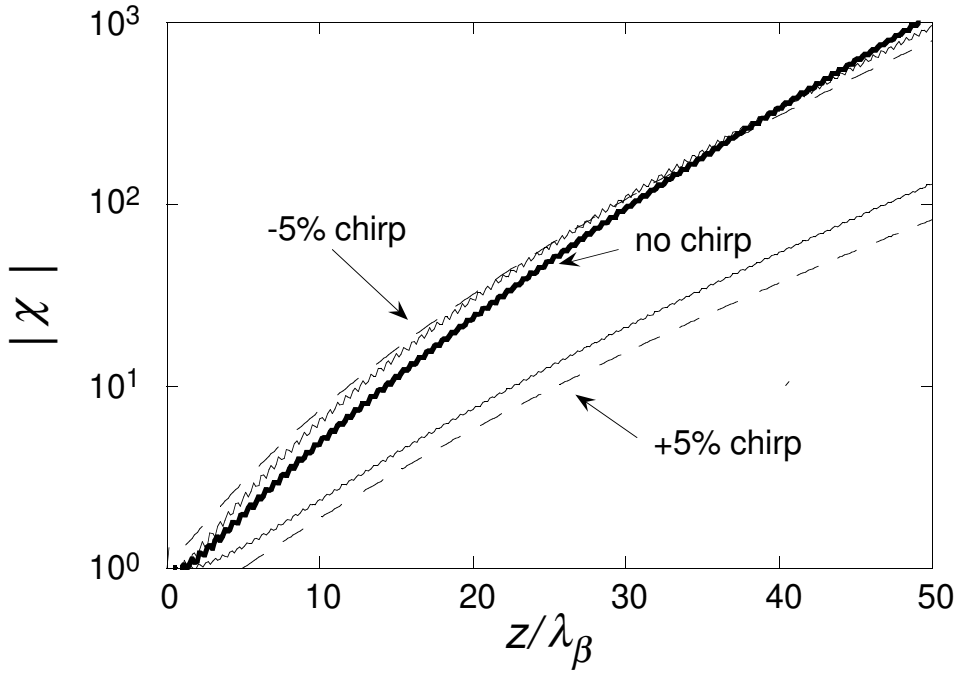
10^3
10^2
10^1
10^0
$|\chi|$
-5% chirp
no chirp
+5% chirp
0
10
20
30
40
50
z/λ_β

FIG.1

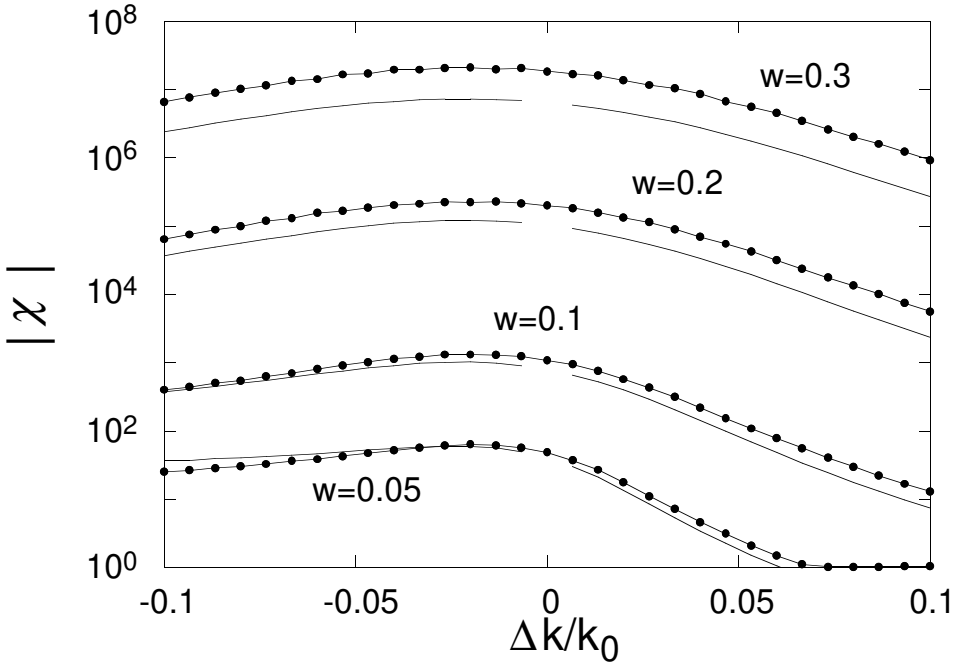
10^8
10^6
10^4
10^2
10^0
$|\chi|$
w=0.3
w=0.2
w=0.1
w=0.05
-0.1
-0.05
0
0.05
0.1
$\Delta k/k_0$

FIG.2

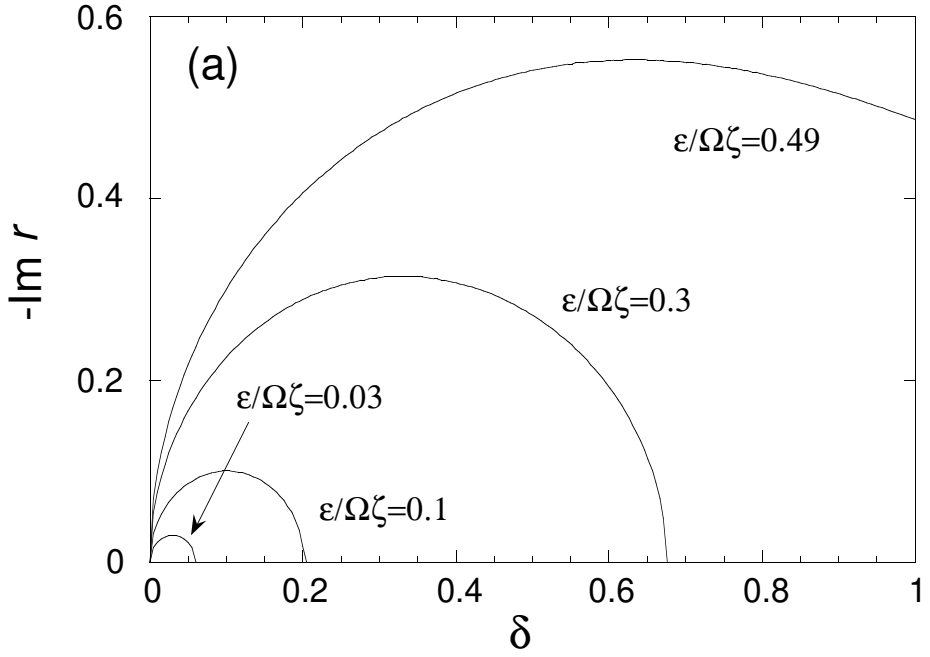
(a)
ε/Ωζ=0.49
ε/Ωζ=0.3
ε/Ωζ=0.03
ε/Ωζ=0.1
-Im r
δ
0
0.2
0.4
0.6
0.8
1
0.6
0.4
0.2
0

FIG.3(a)

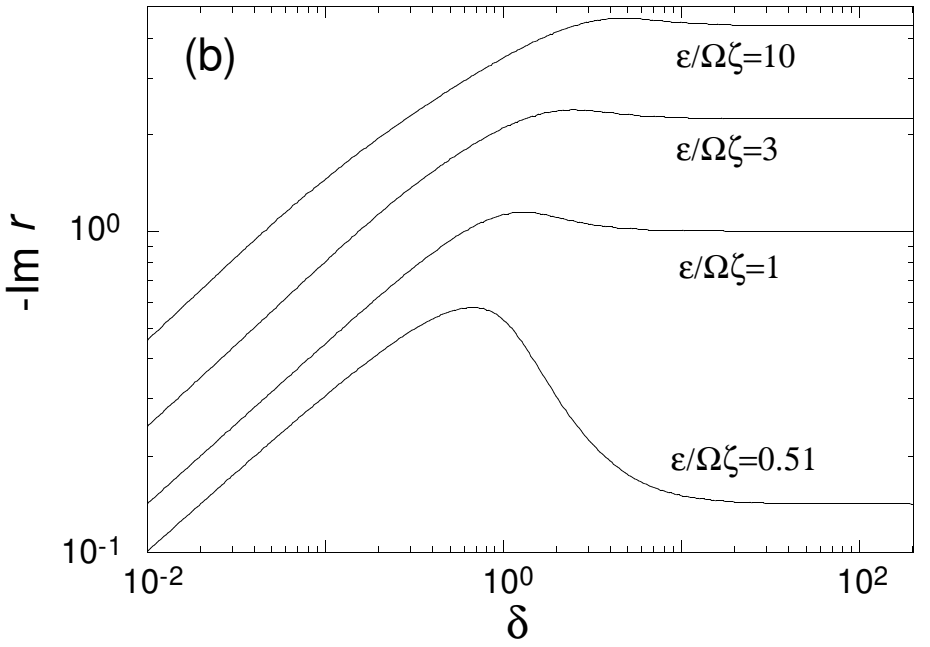
(b)
ε/Ωζ=10
ε/Ωζ=3
ε/Ωζ=1
ε/Ωζ=0.51
-Im r
δ
10^0
10^-1
10^-2
10^0
10^2

FIG.3 (b)

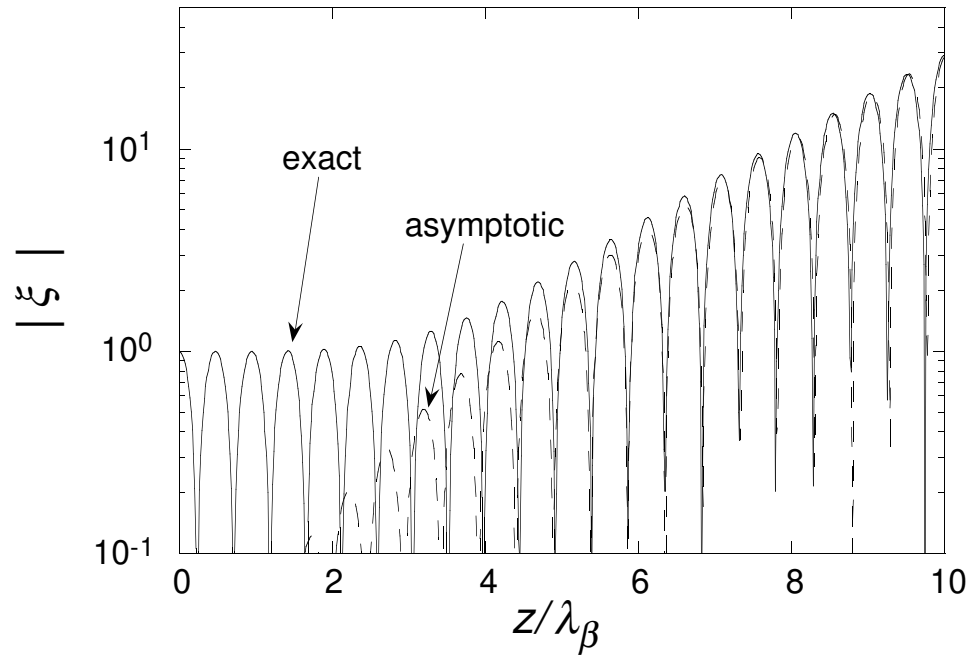
exact
asymptotic
$|\xi|$
10^1
10^0
10^{-1}
0
2
4
6
8
10
z/λ_β

FIG.4

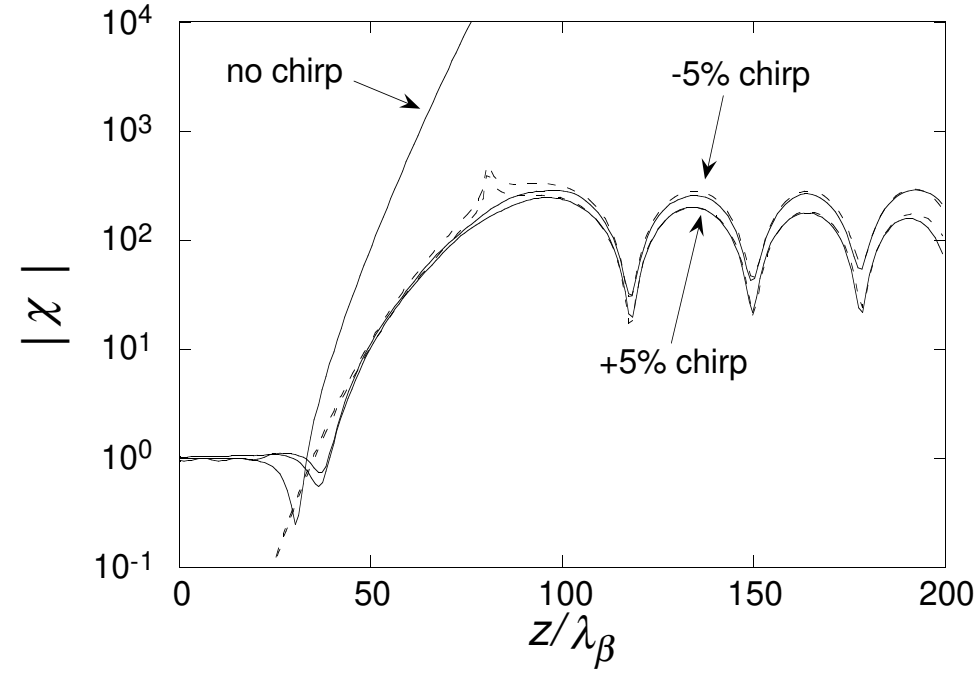
no chirp
-5% chirp
+5% chirp
$|\chi|$
10^4
10^3
10^2
10^1
10^0
10^{-1}
0
50
100
150
200
z/λ_β

FIG.5

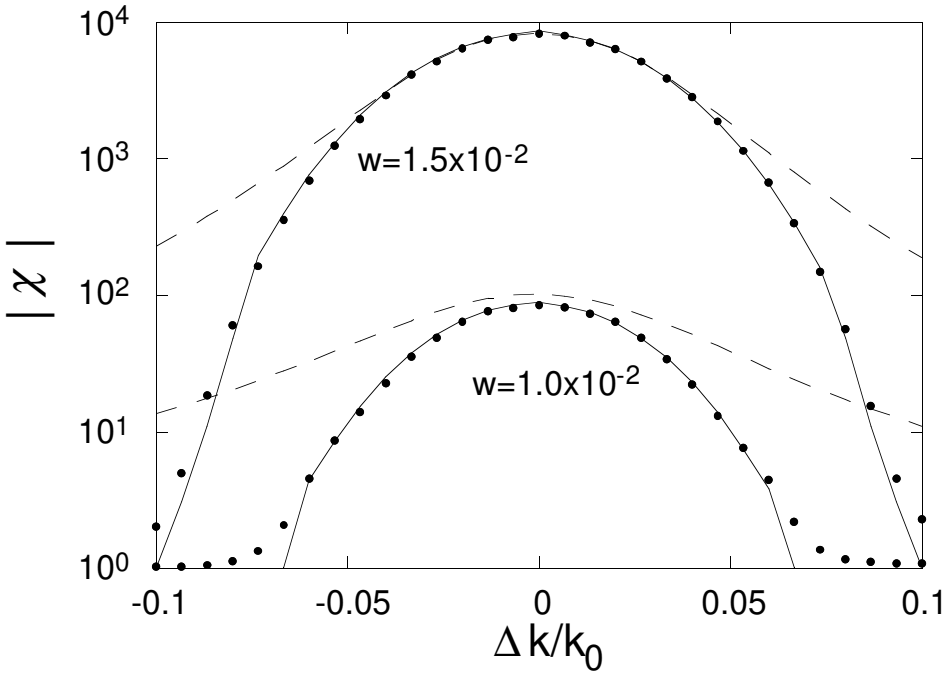
10^4
10^3
10^2
10^1
10^0
$|\chi|$
w=1.5x10^{-2}
w=1.0x10^{-2}
-0.1
-0.05
0
0.05
0.1
$\Delta k/k_0$

FIG.6