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COMMENSURATE SCALE RELATIONS AND THE ABELIAN CORRESPONDENCE PRINCIPLE *

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Commensurate scale relations are perturbative QCD predictions which relate observable to observable at fixed relative scales, independent of the choice of intermediate renormalization scheme or other theoretical conventions. A prominent example is the "generalized Crewther relation" which connects the Bjorken and Gross-Llewellyn Smith deep inelastic scattering sum rules to measurements of the e^+e^- annihilation cross section. Commensurate scale relations also provide an extension of the standard minimal subtraction scheme which is analytic in the quark masses, has non-ambiguous scale-setting properties, and inherits the physical properties of the effective charge $\alpha_V(Q^2)$ defined from the heavy quark potential. I also discuss a property of perturbation theory, the "Abelian correspondence principle", which provides an analytic constraint on non-Abelian gauge theory for $N_C \to 0$.

1 Introduction

Quantum Chromodynamics provides an elegant and fundamental description of hadronic and nuclear interactions in terms of quark and gluon degrees of freedom. A common goal of particle and nuclear physics has been to test QCD in all of its manifestations to as high precision as possible. A central focus of QCD studies in high energy physics has been the determination of the strength of the quark-gluon interaction, as characterized by the $\alpha_{\overline{\rm MS}}(\mu)$ coupling, defined by convention in a particular dimensional regularization scheme. However, the precision of determining α_s is limited due to questions of principle in relating physical measurements to the $\overline{\rm MS}$ coupling. These problems include apparent renormalization scale ambiguities, implementation of finite quark mass effects, and the question of the convergence of perturbative expansions which have divergent "renormalon" $n!$ growth [1,2,3,4]. Resummations of such divergent series have been proposed, which in turn highlight the uncertainties in the behavior of the $\overline{\rm MS}$ coupling at low momentum scales. The ambiguities introduced by the scale ambiguities and scheme conventions of the $\overline{MS}$ scheme are amplified in processes involving more than one physical scale such as jet observables and semi-inclusive reactions. In this talk I will discuss three new theoretical tools which bypass the above difficulties and have the

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potential to greatly increase the precision of QCD tests: (1) "commensurate scale relations", scale-fixed QCD predictions which relate observable to observable; (2) the "Abelian correspondence principle", which provides new analytic constraints on QCD predictions; and (3) the adoption of the effective charge $\alpha_V(Q^2)$ defined from the heavy quark potential as a replacement for expansions in the standard $\overline{\rm MS}$ coupling. Commensurate scale relations also provide an extension of the standard minimal subtraction scheme which is analytic in the quark masses, has non-ambiguous scale-setting properties, and inherits the physical properties of α_V.

2 Commensurate Scale Relations

Commensurate scale relations relate one physical observables to another physical observable, and thus must be independent of theoretical conventions such as the choice of intermediate renormalization scheme. For example, the "generalized Crewther relation", discussed below, provides a rigorous all-orders relation between the Bjorken and Gross Llewellyn-Smith sum rules for deep inelastic scattering at a given momentum transfer Q to the annihilation cross section $\sigma_{e^+e^-\to \rm hadrons}(s)$, at a specific "commensurate" energy scale [5,6]. The relations between the physical scales Q and $\sqrt{s}$ reflects the fact that the radiative corrections to the sum rules and annihilation cross section have different heavy quark thresholds. The generalized Crewther relation can be derived by calculating the radiative corrections to both the sum rules and $R_{e^+e^-}$ in the modified minimal subtraction scheme $\overline{\rm MS}$ and then algebraically eliminating $\alpha_{\overline{MS}}(\mu)$. BLM scale setting is then used to eliminate the β-dependence of the coefficients. However, the relation between observables at any given order of perturbation theory is independent of the choice of the choice of renormalization scheme and the initial choice of scale μ: obviously, relations between physical observables cannot depend on conventions which theorists choose. QCD can then be tested in a new fundamental and precise way by checking that the observables track both in their relative normalization and in their commensurate scale dependence.

A helpful tool and notation for relating physical quantities is the effective charge. Any perturbatively calculable physical quantity can be used to define an effective charge [7,8,9] by incorporating the entire radiative correction into its definition. All effective charges $\alpha_A(Q)$ satisfy the Gell-Mann-Low renormalization group equation with the same β_0 and β_1; different schemes or effective charges only differ through the third and higher coefficients of the β function. Thus, any effective charge can be used as a reference running coupling constant in QCD to define the renormalization procedure. More generally, each

effective charge or renormalization scheme, including $\overline{\rm MS}$, is a special case of the universal coupling function $\alpha(Q,\beta_n)$ [10,11]. Peterman and Stückelberg have shown [10] that all effective charges are related to each other through a set of evolution equations in the scheme parameters β_n.

For example, consider the Adler function [12] for the e^+e^- annihilation cross section

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2),\ \Pi(Q^2) = -\frac{Q^2}{12\pi^2}\int_{4m_\pi^2}^{\infty}\frac{R_{e^+e^-}(s)ds}{s(s+Q^2)}. \qquad (1)$$

The entire radiative correction to this function is defined as the effective charge $\alpha_D(Q^2)$:

$$\begin{aligned} D\left(Q^2/\mu^2, \alpha_{\rm s}(\mu^2/\Lambda^2_{\overline{\rm MS}})\right) &= D\left(1, \alpha_{\rm s}(Q^2/\Lambda^2_{\overline{\rm MS}})\right) \qquad (2)\\ &\equiv 3\sum_f Q_f^2\left[1+\frac{\alpha_D(Q^2/\Lambda_D^2)}{\pi}\right] + (\sum_f Q_f)^2 C_{\rm L}(Q^2)\\ &\equiv 3\sum_f Q_f^2 C_D(Q^2) + (\sum_f Q_f)^2 C_{\rm L}(Q^2), \end{aligned}$$

where Λ_D is the scheme-independent effective scale parameter. The coefficient $C_{\rm L}(Q^2)$ appears at the third order in perturbation theory and is related to the "light-by-light scattering type" diagrams. (Hereafter $\alpha_{\rm s}$ will denote the $\overline{\rm MS}$ scheme strong coupling constant.)

Similarly, we can define the entire radiative correction to the Bjorken sum rule as the effective charge $\alpha_{g_1}(Q^2)$ where Q is the corresponding momentum transfer:

$$\int_0^1 dx\left[g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)\right] \equiv \frac{1}{6}\left|\frac{g_A}{g_V}\right| C_{\rm Bj}(Q^2) = \frac{1}{6}\left|\frac{g_A}{g_V}\right|\left[1-\frac{\alpha_{g_1}(Q^2/\Lambda^2_{g_1})}{\pi}\right]. \qquad (3)$$

It is straightforward to algebraically relate $\alpha_{g_1}(Q^2)$ to $\alpha_D(Q^2)$ using the known expressions to three loops in the $\overline{\rm MS}$ scheme. If one chooses the renormalization scale to re-sum all quark and gluon vacuum polarization corrections into $\alpha_D(Q^2)$, then the final result turns out to be remarkably simple: ($\widehat{\alpha} = 3/4\, C_F\ \alpha/\pi = \alpha/\pi$) [6]

$$\widehat{\alpha}_{g_1}(Q) = \widehat{\alpha}_D(\overline{Q}^*) - \widehat{\alpha}_D^2(\overline{Q}^*) + \widehat{\alpha}_D^3(\overline{Q}^*) + \cdots, \qquad (4)$$

Here

$$\ln\left(\frac{\overline{Q}^{*2}}{Q^2}\right) = \frac{7}{2} - 4\zeta(3) + \left(\frac{\alpha_D(\overline{Q}^*)}{4\pi}\right)\left[\left(\frac{11}{12}+\frac{56}{3}\zeta(3) - 16\zeta^2(3)\right)\beta_0\right.$$

$$+\frac{26}{9}C_{\rm A} - \frac{8}{3}C_{\rm A}\zeta(3) - \frac{145}{18}C_{\rm F} - \frac{184}{3}C_{\rm F}\zeta(3) + 80C_{\rm F}\zeta(5)\Bigg]. \quad (5)$$

where in QCD $C_{\rm A} = 3$, $C_{\rm F} = 4/3$. This relation shows how the coefficient functions for these two different processes are related to each other at their respective commensurate scales. The evaluation of one of them at the appropriate physical scale gives us information about the second one at the different physical scale. Notice also that all the $\zeta(3)$ and $\zeta(5)$ dependencies have been absorbed into the renormalization scale $\overline{Q}^*$. We emphasize that the $\overline{\rm MS}$ renormalization scheme is used only for calculational convenience; it serves simply as an intermediary between observables. The renormalization group property [13,10,14] ensures that the forms of the CSR relations in perturbative QCD are independent of the choice of an intermediate renormalization scheme.

The Crewther relation was originally derived assuming that the theory is conformally invariant; *i.e.*, for zero β function. In the physical case, where the QCD coupling runs, the non-conformal effects are resummed into the energy and momentum transfer scales of the effective couplings α_R and α_{g1}. The coefficients in the relation between the two effective charges

$$1 - \frac{\alpha_{g_1}(Q)}{\pi} = \left[1 + \frac{\alpha_D(\overline{Q}^*)}{\pi}\right]^{-1} . \quad (6)$$

This relation generalizes Crewther's relation to non-conformal QCD. Notice that the coefficients which appear in the perturbative expansion form a simple geometric series, and thus do not have a divergent renormalon behavior $n!\alpha_s^n$. This is again a special advantage of relating observable to observable. The coefficients are independent of color and are the same in Abelian, non-Abelian, and conformal gauge theory. The non-Abelian structure of the theory is reflected in the expression for the scale $\overline{Q}^*$.

The generalized Crewther relation can also be written in the form

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(s) C_{\rm Bj}(Q^2) = 1 + \varepsilon_1(Q^2), \quad (7)$$

and

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(s) C_{\rm GLS}(Q^2) = 1 + \varepsilon_2(Q^2), \quad (8)$$

where ε_1 and ε_2 are small quantities from NNLO corrections; *e.g.* light-by-light scattering contributions. The experimental measurements of the R-ratio above the thresholds for the production of $c\overline{c}$-bound states, together with the

theoretical fit performed by Mattingly and Stevenson [15], provide the empirical constraint

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(\sqrt{s} = 5.0 \text{ GeV}) \simeq \frac{3}{10}(3.6 \pm 0.1) = 1.08 \pm 0.03. \tag{9}$$

and thus

$$\frac{\alpha_R^{\rm exp}(\sqrt{s} = 5.0 \text{ GeV})}{\pi} \simeq 0.08 \pm 0.03. \tag{10}$$

The prediction for the effective coupling in the deep inelastic sum rules at the commensurate momentum transfer Q is

$$\frac{\alpha_{g_1}^{\rm exp}(Q = 12.33 \pm 1.20 \text{ GeV})}{\pi} \simeq \frac{\alpha_{\rm GLS}^{\rm exp}(Q = 12.33 \pm 1.20 \text{ GeV})}{\pi} \simeq 0.074 \pm 0.026\,. \tag{11}$$

Measurements of the Gross-Llewellyn Smith sum rule have been carried out only at relatively small values of Q^2 [16,17]; however, one can use the results of the theoretical extrapolation [18] of the experimental data presented in [19]:

$$\frac{\alpha_{\rm GLS}^{\rm extrapol}(Q = 12.25 \text{ GeV})}{\pi} \simeq 0.093 \pm 0.042. \tag{12}$$

This interval overlaps with the prediction [20] from the generalized Crewther relation. It is clear that higher precision measurements will be necessary to fully test these fundamental relations.

Commensurate scale relations allow one to relate any perturbatively calculable observable, such as the annihilation ratio $R_{e^+e^-}$, the heavy quark potential and the radiative corrections to structure function sum rules, to each other without any renormalization scale or scheme ambiguity [5]. Commensurate scale relations can also be applied in grand unified theories to make scale-fixed, scheme invariant predictions which relate physical observables in different sectors of the theory. In each case, commensurate scale relations connecting the effective charges for observables A and B have the form $\alpha_A(Q_A) = \alpha_B(Q_B)\left(1 + r_{A/B}\frac{\alpha_B}{\pi} + \cdots\right)$, where the coefficient $r_{A/B}$ is independent of the number of flavors n_F contributing to coupling constant renormalization. The scales of the effective charges that appear in commensurate scale relations are thus fixed by the requirement that the couplings sum all of the effects of the non-zero β function; the coefficients in the perturbative expansions in the commensurate scale relations are thus identical to those of a corresponding conformally-invariant theory with $\beta = 0$. The method thus has the important advantage of isolating and "pre-summing" the large and strongly divergent terms in the PQCD series which grow as $n!(\beta_0\alpha_s)^n$, *i.e.*,

the infrared renormalons associated with coupling-constant renormalization [1,2,3,4]. The renormalization scales Q^* in the BLM method are physical in the sense that they reflect the mean virtuality of the gluon propagators [21,22,20,23]. The ratio of scales $\lambda_{A/B} = Q_A/Q_B$ is unique at leading order and guarantees that the observables A and B pass through new quark thresholds at the same physical scale. One also can show that the commensurate scales satisfy the transitivity rule $\lambda_{A/B} = \lambda_{A/C}\lambda_{C/B}$, which ensures that predictions are independent of the choice of an intermediate renormalization scheme or observable C.

3 Implementation of α_V Scheme

The physics of commensurate scale relations illuminates the importance of using an effective charge defined from a physical observable to characterize QCD. The central advantage of such a procedure is that predictions which relate one physical observable to another observable have no ambiguities from theoretical conventions such as the choice of renormalization scale or scheme.

The heavy-quark potential $V(Q^2)$ is defined as the two-particle-irreducible scattering amplitude of test charges; *i.e.* the scattering of two infinitely-heavy quark and antiquark at momentum transfer $t = -Q^2$. The relation $V(Q^2) = -4\pi C_F \alpha_V(Q^2)/Q^2$ with C_F given by $C_F = (N_C^2 - 1)/2N_C = 4/3$ then defines the effective charge $\alpha_V(Q)$. This coupling can provide a physically-based alternative to the usual $\overline{\rm MS}$ scheme. As in the corresponding case of Abelian QED, the scale Q of the coupling $\alpha_V(Q)$ is identified with the exchanged momentum. There is thus never any ambiguity in the interpretation of the scale. All vacuum polarization corrections due to fermion pairs are incorporated in α_V terms of the usual vacuum polarization kernels which are functions of the physical mass thresholds. An similar alternative is the effective charge defined from heavy quark radiation [24].

The-relation of $\alpha_V(Q^2)$ to the conventional $\overline{MS}$ coupling is now known to NNLO [25]. Recently, Gill, Melles, Rathsman and I [26] have derived the required connection in the form of a single-scale commensurate scale relation [27].

$$\alpha_{\overline{\rm MS}}(Q) = \alpha_V(Q^*) + \frac{2}{3}N_C\frac{\alpha_V^2(Q^*)}{\pi} + \left\{ -\left(\frac{5}{144} + \frac{24\pi^2 - \pi^4}{64} - \frac{11}{4}\zeta_3\right) N_C^2 + \left(\frac{385}{192} - \frac{11}{4}\zeta_3\right) C_F N_C \right\} \frac{\alpha_V^3(Q^*)}{\pi^2}$$

$$= \alpha_V(Q^*) + 2\frac{\alpha_V^2(Q^*)}{\pi} + 4.625\frac{\alpha_V^3(Q^*)}{\pi^2}, \tag{13}$$

above or below the quark mass threshold. The coefficients in the perturbation expansion have their conformal values, *i.e.*, the same coefficients would occur even if the theory had been conformally invariant with $\beta = 0$ and thus do not contain the diverging $(\beta_0\alpha_s)^n n!$ growth characteristic of an infrared renormalon [27]. The next-to leading order (NLO) coefficient $\frac{2}{3}N_C$ is a feature of the non-Abelian couplings of QCD and is not present in QED. Here

$$Q^* = Q\exp\left[\frac{5}{6} + [(\frac{35}{32} - \frac{3}{2}\zeta_3)C_F - (\frac{19}{48} - \frac{7}{4}\zeta_3)N_C]\frac{\alpha_V}{\pi} + \cdots\right] \tag{14}$$

For $N_C = 3$ we have $\ln Q^*/Q = 5/6 + 4.178\alpha_V/\pi$. The factor $e^{5/6} \simeq 0.4346$ is the ratio of commensurate scales between the two schemes to leading order. It arises because of the convention used in defining the modified minimal subtraction scheme. The scale in the $\overline{\rm MS}$ scheme is thus a factor ~ 0.4 smaller than the physical scale. The coefficient $2N_C/3$ in the NLO coefficient is a feature of the non-Abelian couplings of QCD; the same coefficient occurs even if the theory were conformally invariant with $\beta_0 = 0$.

Using the above QCD results, we can transform any NNLO prediction given in $\overline{MS}$ scheme as a scale-fixed expansion in $\alpha_V(Q)$ [28]. We can derive the connection between the $\overline{MS}$ and α_V schemes for Abelian perturbation theory using the limit $N_C \to 0$ with $C_F\alpha_s$ and N_F/C_F held fixed [29] (see Section 4). In this case

$$\widehat{\alpha}_{\overline{\rm MS}}(Q) = \widehat{\alpha}_V(Q^*) \tag{15}$$

with

$$Q^* = Q\exp\left[\frac{5}{6} + [(\frac{35}{32} - \frac{3}{2}\zeta_3)\frac{\widehat{\alpha}_V}{\pi} + \cdots\right] \ . \tag{16}$$

The use of α_V as the expansion parameter with BLM scale-fixing has been found to be valuable in lattice gauge theory, greatly increasing the convergence of perturbative expansions relative to those using the bare lattice coupling [22]. Recent lattice calculations of the Υ- spectrum [30] have been used to determine the normalization of the static heavy quark potential and its effective charge $\alpha_V^{(3)}(8.2{\rm GeV}) = 0.196(3)$ where the effective number of light flavors is $n_f = 3$. A recent determination [30] of the corresponding modified minimal subtraction coupling evolved to the Z mass is given by $\alpha_{\overline{\rm MS}}^{(5)}(M_Z) = 0.1174(24)$.

Thus a high precision value for $\alpha_V(Q^2)$ at a specific scale is available. Predictions for other QCD observables can be directly referenced to this value,

without the scale or scheme ambiguities, greatly increasing the precision of QCD tests. We can anticipate that eventually nonperturbative methods such as lattice gauge theory or discretized light-cone quantization will provide a complete form for the heavy quark potential in QCD. It is reasonable to assume that $\alpha_V(Q)$ will not diverge at small space-like momenta. One possibility is that α_V stays relatively constant $\alpha_V(Q) \simeq 0.4$ at low momenta, consistent with fixed-point behavior. There is, in fact, empirical evidence for freezing of the α_V coupling from the observed systematic dimensional scaling behavior of exclusive reactions [31]. If this is in fact the case, then the range of QCD predictions can be extended to quite low momentum scales, a regime normally avoided because of the apparent singular structure of perturbative extrapolations.

There are other advantages of the V-scheme:

1. Perturbative expansions in $\alpha_V(Q^*)$ cannot have any β-function dependence in their coefficients since all vacuum polarization contributions to the running are already summed into the definition of the potential and the effective coupling. There is thus never any scale ambiguities. The value of the scale Q^* reflects the mean virtuality of the exchanged gluons in the Feynman amplitude. Since coefficients involving β_0 cannot occur in an expansions in α_V, diverging infrared renormalons of the form $\alpha_V^n \beta_0^n n!$ cannot occur. The general convergence properties of the scale Q^* as an expansion in α_V is not known [2].

2. The effective coupling $\alpha_V(Q^2)$ incorporates vacuum polarization contributions with finite fermion masses. When continued to timelike momenta, the coupling has the correct analytic dependence dictated by particle production in the t channel. Thus since α_V incorporates quark mass effects exactly, it avoids the problem of explicitly computing and resuming quark mass corrections.

3. Eq. (13) is technically only valid far above and below a heavy quark threshold. However, the same equation can be used to define an analytically-extended $\overline{MS}$ scheme at any scale Q. The new modified scheme inherits all of the good properties of the α_V scheme, including its correct analytic properties as a function of the quark masses and unambiguous scale fixing [26].

4. The use of α_V at any stage allows a simple connection to the Abelian theory via the $N_C \to 0$ limit. I discuss this further in the next section.

5. Computations in different sectors of the Standard Model have been traditionally carried out using different renormalization schemes. The tra-

ditional QED scheme is equivalent to α_V. However, in a grand unified theory, the forces between all of the particles in the fundamental representation should become universal above the grand unification scale. Thus it is natural to use α_V as the effective charge for all sectors of a grand unified theory since unification should occur in $\alpha_V(Q^2)$ rather than in a convention-dependent coupling such as $\alpha_{\overline{MS}}$.

6. The α_V coupling is the natural expansion parameter for processes involving non-relativistic momenta, such as heavy quark production at threshold where the Coulomb interactions, which are enhanced at low relative velocity v as $\pi\alpha_V/v$, need to be re-summed. [32,33,34] The threshold corrections to heavy quark production in e^+e^- annihilation depend directly on α_V at specific scales Q^*. Two distinct ranges of scales arise as arguments of α_V near threshold: the relative momentum of the quarks governing the soft gluon exchange responsible for the Coulomb potential, and a high momentum scale approximately equal to twice the quark mass for the corrections induced by hard gluon exchange [33]. One thus can use threshold production to obtain a direct determination of α_V even at low scales. The corresponding QED results for τ pair production allow for a measurement of the magnetic moment of the τ and could be tested at a future τ-charm factory [32,33].

7. The effective NRQCD Hamiltonian is effectively written in α_V scheme.

One can also apply commensurate scale relations in α_V to the domain of exclusive processes at large momentum transfer such as the form factors and the photon-to-pion transition form factor at large momentum transfer [21,31] and exclusive weak decays of heavy hadrons in QCD [35]. Each gluon propagator with four-momentum k^μ in the hard-scattering quark-gluon scattering amplitude is associated with the coupling $\alpha_V(k^2)$ since the gluon exchange propagators closely resembles the interactions encoded in the effective potential $V(Q^2)$. [In Abelian theory this is exact.] Commensurate scale relations can then be established which connect the hard-scattering subprocess amplitudes which control exclusive processes to other QCD observables.

4 QCD in the Limit of Small Number of Colors.

A remarkable property of perturbative QCD, first demonstrated by 't Hooft [1] is that the theory is dominated by diagrams with planar topology in the limit $N_C \to \infty$. In this limit, the dynamics of the theory is effectively constrained by the color degrees of freedom. Recently Patrick Huet and I [29] have explored the general properties of perturbative QCD expressions taken as analytic functions

of $z = N_C^2$. We found several unexpected features of the $SU(N_C)$ theory which provide useful constraints on non-Abelian gauge theory, including an interesting Abelian limit for $N_C^2 = 0$

It is useful to introduce rescaled couplings and flavor number $\widehat{\alpha}_s = C_F \alpha_s$, $\widehat{n}_f = T n_F / C_F$, where $C_F = \frac{N_C^2 - 1}{2N_C}$ is the fundamental Casimir constant and $T = 1/2$. At large N_C^2, $\widehat{\alpha}$ incorporates the rescaling of the coupling advocated by 't Hooft.

The expansion of QED predictions for color-averaged quantities in the rescaled coupling have the form

$$C_{n,\ell} = \frac{\widehat{\alpha}_s^n \widehat{n}_f^\ell}{(N_C^2 - 1)^{\ell - n}} \Sigma_{i=1}^{2^n} (-1)^{n - e_i} (N_C^2)^{\widetilde{\omega}_i}, \tag{17}$$

where $\widetilde{\omega}_i$ is an index computed from the topology of the component color graph which is obtained by replacing the gluons by e_i "double lines" using the Cvitanovic-Mandula rules. The maximum value for the index occurs when all gluons are replaced with double $q\overline{q}$ lines as in a $U(N)$ theory. For planar graphs, $C_{n,\ell}$ grows maximally at large N_C as $\widehat{\alpha}_s^n \widehat{n}_f^\ell N_C^2$ which is the 't Hooft limit. On the other hand, for $N_C^2 \to 0$, the component diagrams which dominate the color factor have $\widetilde{\omega} = 0$ and occur only from color graphs which have a "tree structure." Thus for $N_C^2 \to 0$, the coefficient of $\widehat{\alpha}_s^n \widehat{n}_f^\ell$ is a finite constant and is identical to the coefficients of an Abelian theory. The two limits essentially bound the polynomial behavior of perturbation theory for large and small color. The only analytic singularity occurs at $N_C^2 \to 1$ where $SU(N_C)$ becomes undefined.

The $N_C^2 \to 0$ limit reduces the non-Abelian theory to an Abelian theory dominated by the coupling of the $N_C - 1$ diagonal gluons of the adjoint representation. The small-N_C limit of $SU(N_C)$ reflects the coupling of $N_C - 1$. Abelian gluons and thus has the group structure $\lim_{N_C \to 0} [U(1)^{N_C - 1} \sim U(1)^{-1}$, *i.e.*: -1 Abelian gluons. The $N_C \to 0$ theory resembles QED; however, in high order graphs involving fermion loops, there is an "offset" factor relative to the QED value calculated with $\widehat{\alpha}_s \to \alpha_{QED}$ and $\widehat{n}_f \to n_{\rm leptons}$ in QED. The offset factor is easily evaluated by counting the number of tree diagrams contributing to the color weight. For example, a color diagram originating from a ring of $\ell = 3$ fermion loops interconnected pair-wise by p, q, and r gluons has the offset factor $(p-1)(q-1)(r-1) - pqr$.

As an example, consider the QCD prediction for the ratio of the annihilation cross section to the point-like limit $R_{e^+e^-}$ in the $\overline{MS}$ scheme. In terms of

$\widehat{\alpha}_s$ and $\widehat{n}_f$,

$$R_{e^+e^-}(Q^2)/(N_C^2-1) = \sum_I^{\widehat{n_f}} Q_I^2 \{1 + \frac{\widehat{\alpha}_s(Q)}{\pi}\widehat{F}_2 + \frac{\widehat{\alpha}_s^2(Q)}{\pi^2}\widehat{F}_3 + \cdots\} + \cdots \quad (18)$$

with $\widehat{F}_n = F_n/C_F^{n-1}$. Specifically,

$$\widehat{F}_2 = \frac{3}{4} \quad \text{and} \quad \widehat{F}_3 = -\frac{3}{32} + \left(\frac{123}{32} - \frac{11}{4}\zeta(3)\right)\left(\frac{N_C^2}{N_C^2-1}\right) + \left(\frac{-11}{8} + \zeta(3)\right)\widehat{n_f}\,. \quad (19)$$

For $N_C^2 \to 0$, these forms coincide with the QED coefficients F_2^{QED} and F_3^{QED} with $\alpha_{QED} = \widehat{\alpha}_s$ and $n_{\rm leptons} = \widehat{n}_f$. The coefficients of α_s^3 in the expansion above has been computed and the corresponding $\widehat{F}_4$ also coincides with its QED counterpart [36]. In the next order where the Casimir d_{abc}^2 appears, the QED result is 1/2 of the $N_C^2 \to 0$ limit of the QCD production due to the offset factor.

The simple structure of the color coefficients in the rescaled quantities $\widehat{\alpha}_s$ and $\widehat{n}_f$ provides a constraint on Padé and other methods which resum perturbation theory since no coefficient can grow faster then N_C^2.

5 The Abelian Correspondence Principle

The non-trivial analytical limit of perturbative QCD expressions at small number of colors provides a new type of "correspondence principle": QCD predictions must coincide analytically with predictions of the corresponding Abelian theory at $N_C \to 0$. In addition to providing a boundary condition and useful check on non-Abelian analyses, there are a number of important physical implications:

1. Perturbative QCD results, such as factorization theorems for hard inclusive and exclusive reactions, evolution equations, and results derived from the operator product expansion are immediately applicable to QED. Similarly, physical principles controlling the high energy interactions of hadrons in QCD such as, diffraction, hard pomeron and odderon exchange, color transparency and intrinsic heavy particle Fock states all have physical analogs for the interactions of neutral atoms in QED. Conversely, phenomena in QED atoms such as van der Waals interactions, co-mover coalescence, and the Lamb shift, predict analogous phenomena in QCD.

2. The treatment of renormalization schemes and scales in perturbative QCD must match those of QED at $N_C \to 0$. In QED it is traditional to define the fundamental effective charge of the theory $\alpha_{QED}(Q^2)$ as the coupling which appears in the potential between two massive test charges: $V(Q^2) = -\frac{4\pi Z_1 Z_2 \alpha_{QED}(Q^2)}{Q^2}$ where $Q^2 = -q^2$ is the space-like momentum transfer squared. and normalize it to the measured value at $Q^2 = 0 : [\alpha_{WED}(0)]^{-1} = 137.0359895(61)$ [37]. In the QED scheme, all vacuum polarization effects which normalize the photon propagator are summed into $\Pi(Q^2)$. There is thus no scale ambiguity and fermion pair masses are treated exactly. As we have seen in Section 3, these constraints are fulfilled when α_V is used as the effective charge in QCD: perturbative QCD expressions in the α_V scheme have the correct Abelian correspondence limit with QED expressions in the α_{QED} scheme.

The above analyses of the color weights of $SU(N_C)$ gauge theory and the Abelian limit at $N_C \to 0$ apply to any order in perturbation theory. The coefficients in $\widehat{\alpha}$ and $\widehat{n}$ which are finite in N_C are given by the Abelian theory. Alternatively, we can use the general N_C analysis to expand QCD expressions at small N_C, starting with the QED prediction as the initial approximation. The most interesting questions center on whether the simple analytic properties of perturbation theory also hold for nonperturbative QCD predictions, such as those calculated from instanton effects. More generally, does confinement or QCD phase transitions lead to non-analytic behavior in N_C^2 not present in all-order perturbative analyses?

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