SLAC–PUB–8022
December 1998

Scheme-Independent Predictions in QCD: Commensurate Scale Relations and Physical Renormalization Schemes *

S. J. Brodsky and J. Rathsman

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309, USA
e-mail: sjbth@slac.stanford.edu – rathsman@slac.stanford.edu

Contribution to the Proceedings of the
IVth INTERNATIONAL SYMPOSIUM ON RADIATIVE CORRECTIONS
Universitat Autonoma de Barcelona
Barcelona (Catalonia, Spain)
September 8–12, 1998

*Work partially supported by the Department of Energy, contract DE–AC03–76SF00515.

Abstract

Commensurate scale relations are perturbative QCD predictions which relate observable to observable at fixed relative scale, such as the "generalized Crewther relation", which connects the Bjorken and Gross-Llewellyn Smith deep inelastic scattering sum rules to measurements of the e^+e^- annihilation cross section. All non-conformal effects are absorbed by fixing the ratio of the respective momentum transfer and energy scales. In the case of fixed-point theories, commensurate scale relations relate both the ratio of couplings and the ratio of scales as the fixed point is approached. The relations between the observables are independent of the choice of intermediate renormalization scheme or other theoretical conventions. Commensurate scale relations also provide an extension of the standard minimal subtraction scheme, which is analytic in the quark masses, has non-ambiguous scale-setting properties, and inherits the physical properties of the effective charge $\alpha_V(Q^2)$ defined from the heavy quark potential. The application of the analytic scheme to the calculation of quark-mass-dependent QCD corrections to the Z width is also reviewed.

1 Introduction

One of the central problems in constructing precision tests of a quantum field theory such as quantum chromodynamics is the elimination of theoretical ambiguities such as the dependence on the renormalization scale μ in perturbative expansions in the coupling $\alpha_s(\mu)$. However, any prediction which relates one physical quantity to another cannot depend on theoretical conventions such as the choice of renormalization scheme or renormalization scale. This is the principle underlying "commensurate scale relations" (CSR) [1], which are general QCD predictions relating physical observables to each other. For example, the "generalized Crewther relation", which is discussed in more detail below, provides a scheme-independent relation between the QCD corrections to the Bjorken (or Gross Llewellyn-Smith) sum rule for deep inelastic lepton-nucleon scattering, at a given momentum transfer Q, to the radiative corrections to the annihilation cross section $\sigma_{e^+e^-\to\mathrm{hadrons}}(s)$, at a corresponding "commensurate" energy scale $\sqrt{s}$. [1, 2] The specific relation between the physical scales Q and $\sqrt{s}$ reflects the fact that the radiative corrections to each process have distinct quark mass thresholds.

The generalized Crewther relation can be derived by calculating the QCD radiative corrections to the deep inelastic sum rules and $R_{e^+e^-}$ in a convenient renormalization scheme such as the modified minimal subtraction scheme $\overline{\mathrm{MS}}$. One then algebraically eliminates $\alpha_{\overline{MS}}(\mu)$. Finally, BLM scale-setting [3] is used to eliminate the β-function dependence of the coefficients. The form of the resulting relation between the observables thus matches the result which would have been obtained had QCD been a conformal theory with zero β function. The final result relating the observables is independent of the choice of intermediate $\overline{\mathrm{MS}}$ renormalization scheme.

In quantum electrodynamics, the running coupling $\alpha_{QED}(Q^2)$, defined from the Coulomb scattering of two heavy test charges at the momentum transfer $t = -Q^2$, is taken as the standard observable. Similarly, one can take the momentum-dependent coupling $\alpha_V(Q^2)$, defined from the potential scattering for heavy color charges, as a standard QCD observable. Commensurate scale relations between α_V and the QCD radiative corrections to other observables have no scale or scheme ambiguity, even in multiple-scale problems such as multijet production. As is the case in QED, the momentum scale which appears as the argument of α_V reflect the mean virtuality of the exchanged gluons. Furthermore, we can write a commensurate scale relation between α_V and an analytic extension of the $\alpha_{\overline{MS}}$ coupling, thus transferring all of the unambiguous scale-fixing and analytic properties of the physical α_V scheme to the $\overline{MS}$ coupling.

Commensurate scale relations thus provide fundamental and precise scheme-independent tests of QCD, predicting how observables track not only in relative normalization, but also in their commensurate scale dependence.

2 The Generalized Crewther Relation

Any perturbatively calculable physical quantity can be used to define an effective charge [4, 5, 6] by incorporating the entire radiative correction into its definition. All such effective charges $\alpha_A(Q)$ satisfy the Gell-Mann-Low renormalization group equation. In the case of massless quarks, the first two terms in the perturbative expansion for the β function of each effective charge, β_0 and β_1, are universal; different schemes or effective charges only differ through the third and higher coefficients. Any effective charge can be used as a reference running coupling constant in QCD to define the renormalization procedure. More generally, each effective charge or renormalization scheme, including $\overline{\mathrm{MS}}$, is a special case of the universal coupling function $\alpha(Q, \beta_n)$.

For example, consider the Adler function [7] for the e^+e^- annihilation cross section

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2),\ \Pi(Q^2) = -\frac{Q^2}{12\pi^2}\int_{4m_\pi^2}^{\infty} \frac{R_{e^+e^-}(s)ds}{s(s+Q^2)}. \tag{1}$$

The entire radiative correction to this function is defined as the effective charge $\alpha_D(Q^2)$:

$$D\left(Q^2/\mu^2, \alpha_{\mathrm{s}}(\mu^2)\right) \;=\; D\left(1, \alpha_{\mathrm{s}}(Q^2)\right) \tag{2}$$

$$
\begin{aligned}
&\equiv 3\sum_f Q_f^2\left[1+\frac{3}{4}C_F\frac{\alpha_D(Q^2)}{\pi}\right]+(\sum_f Q_f)^2 C_{\rm L}(Q^2)\\
&\equiv 3\sum_f Q_f^2 C_D(Q^2)+(\sum_f Q_f)^2 C_{\rm L}(Q^2),
\end{aligned}
$$

where $C_F=\frac{N_C^2-1}{2N_C}$. The coefficient $C_{\rm L}(Q^2)$ appears at the third order in perturbation theory and is related to the "light-by-light scattering type" diagrams. (Hereafter $\alpha_{\rm s}$ will denote the $\overline{\rm MS}$ scheme strong coupling constant.) Similarly, we can define the entire radiative correction to the Bjorken sum rule as the effective charge $\alpha_{g_1}(Q^2)$ where Q is the corresponding momentum transfer:

$$
\int_0^1 dx\left[g_1^{ep}(x,Q^2)-g_1^{en}(x,Q^2)\right]\equiv\frac{1}{6}\left|\frac{g_A}{g_V}\right|C_{\rm Bj}(Q^2)=\frac{1}{6}\left|\frac{g_A}{g_V}\right|\left[1-\frac{3}{4}C_F\frac{\alpha_{g_1}(Q^2)}{\pi}\right]. \quad (3)
$$

It is straightforward to algebraically relate $\alpha_{g_1}(Q^2)$ to $\alpha_D(Q^2)$ using the known expressions to three loops in the $\overline{\rm MS}$ scheme. If one chooses the renormalization scale to resum all of the quark and gluon vacuum polarization corrections into $\alpha_D(Q^2)$, then the final result turns out to be remarkably simple [2] ($\widehat{\alpha}=3/4\,C_F\;\alpha/\pi$) :

$$
\widehat{\alpha}_{g_1}(Q)=\widehat{\alpha}_D(Q^*)-\widehat{\alpha}_D^2(Q^*)+\widehat{\alpha}_D^3(Q^*)+\cdots, \quad (4)
$$

where

$$
\begin{aligned}
\ln\left(\frac{Q^{*2}}{Q^2}\right) &= \frac{7}{2}-4\zeta(3)+\left(\frac{\alpha_D(Q^*)}{4\pi}\right)\left[\left(\frac{11}{12}+\frac{56}{3}\zeta(3)-16\zeta^2(3)\right)\beta_0\right.\\
&\left.+\frac{26}{9}C_{\rm A}-\frac{8}{3}C_{\rm A}\zeta(3)-\frac{145}{18}C_{\rm F}-\frac{184}{3}C_{\rm F}\zeta(3)+80C_{\rm F}\zeta(5)\right]. \quad (5)
\end{aligned}
$$

where in QCD, $C_{\rm A}=N_C=3$ and $C_{\rm F}=4/3$. This relation shows how the coefficient functions for these two different processes are related to each other at their respective commensurate scales. We emphasize that the $\overline{\rm MS}$ renormalization scheme is used only for calculational convenience; it serves simply as an intermediary between observables. The renormalization group ensures that the forms of the CSR relations in perturbative QCD are independent of the choice of an intermediate renormalization scheme.

The Crewther relation was originally derived assuming that the theory is conformally invariant; *i.e.*, for zero β function. In the physical case, where the QCD coupling runs, all non-conformal effects are resummed into the energy and momentum transfer scales of the effective couplings α_R and α_{g1}. The general relation between these two effective charges for nonconformal theory thus takes the form of a geometric series

$$1 - \widehat{\alpha}_{g_1} = \left[1 + \widehat{\alpha}_D(Q^*)\right]^{-1} \ . \tag{6}$$

We have dropped the small light-by-light scattering contributions. This is again a special advantage of relating observable to observable. The coefficients are independent of color and are the same in Abelian, non-Abelian, and conformal gauge theory. The non-Abelian structure of the theory is reflected in the expression for the scale Q^*.

Is experiment consistent with the generalized Crewther relation? Fits [8] to the experimental measurements of the R-ratio above the thresholds for the production of $c\bar{c}$ bound states provide the empirical constraint: $\alpha_R(\sqrt{s} = 5.0 \text{ GeV})/\pi \simeq 0.08 \pm 0.03$. The prediction for the effective coupling for the deep inelastic sum rules at the commensurate momentum transfer Q is then $\alpha_{g_1}(Q = 12.33 \pm 1.20 \text{ GeV})/\pi \simeq \alpha_{\rm GLS}(Q = 12.33 \pm 1.20 \text{ GeV})/\pi \simeq 0.074 \pm 0.026$. Measurements of the Gross-Llewellyn Smith sum rule have so far only been carried out at relatively small values of Q^2 [9, 10]; however, one can use the results of the theoretical extrapolation [11] of the experimental data presented in [12]: $\alpha_{\rm GLS}^{\rm extrapol}(Q = 12.25 \text{ GeV})/\pi \simeq 0.093 \pm 0.042$. This range overlaps with the prediction from the generalized Crewther relation. It is clearly important to have higher precision measurements to fully test this fundamental QCD prediction.

3 General Form of Commensurate Scale Relations

In general, commensurate scale relations connecting the effective charges for observables A and B have the form

$$\alpha_A(Q_A) = \alpha_B(Q_B)\left(1 + r^{(1)}_{A/B}\frac{\alpha_B(Q_B)}{\pi} + r^{(2)}_{A/B}\frac{\alpha_B(Q_B)}{\pi}^2 + \cdots\right), \tag{7}$$

where the coefficients $r^n_{A/B}$ are identical to the coefficients obtained in a conformally invariant theory with $\beta_B(\alpha_B) \equiv (d/d\ln Q^2)\alpha_B(Q^2) = 0$. The ratio of the scales

Q_A/Q_B is thus fixed by the requirement that the couplings sum all of the effects of the non-zero β function. In practice the NLO and NNLO coefficients and relative scales can be identified from the flavor dependence of the perturbative series; *i.e.* by shifting scales such that the N_F-dependence associated with $\beta_0 = 11/3C_A - 4/3T_F N_F$ and $\beta_1 = -34/3C_A^2 + \frac{20}{3}C_A T_F N_F + 4C_F T_F N_F$ does not appear in the coefficients. Here $C_A = N_C$, $C_F = (N_C^2 - 1)/2N_C$ and $T_F = 1/2$. The shift in scales which gives conformal coefficients in effect pre-sums the large and strongly divergent terms in the PQCD series which grow as $n!(\beta_0 \alpha_s)^n$, *i.e.*, the infrared renormalons associated with coupling-constant renormalization. [13, 14, 15, 16]

The renormalization scales Q^* in the BLM method are physical in the sense that they reflect the mean virtuality of the gluon propagators. This scale-fixing procedure is consistent with scale fixing in QED, in agreement with in the Abelian limit, $N_C \to 0$. [17] [3, 18, 19, 20] The ratio of scales $\lambda_{A/B} = Q_A/Q_B$ guarantees that the observables A and B pass through new quark thresholds at the same physical scale. One can also show that the commensurate scales satisfy the transitivity rule $\lambda_{A/B} = \lambda_{A/C}\lambda_{C/B}$, which ensures that predictions are independent of the choice of an intermediate renormalization scheme or intermediate observable C.

4 Commensurate Scale Relations and Fixed Points

In general, we can write the relation between any two effective charges at arbitrary scales μ_A and μ_B as a correction to the corresponding relation obtained in a conformally invariant theory:

$$\alpha_A(\mu_A) = C_{AB}[\alpha_B(\mu_B)] + \beta_B[\alpha_B(\mu_B)] F_{AB}[\alpha_B(\mu_B)] \tag{8}$$

where

$$C_{AB}[\alpha_B] = \alpha_B + \sum_{n=1} C_{AB}^{(n)} \alpha_B^n \tag{9}$$

is the functional relation when $\beta_B[\alpha_B] = 0$. In fact, if α_B approaches a fixed point $\overline{\alpha}_B$ where $\beta_B[\overline{\alpha}_B] = 0$, then α_A tends to a fixed point given by

$$\alpha_A \to \overline{\alpha}_A = C_{AB}[\overline{\alpha}_B]. \tag{10}$$

The commensurate scale relation for observables A and B has a similar form, but in this case the relative scales are fixed such that the non-conformal term F_{AB} is zero. Thus the commensurate scale relation $\alpha_A(Q_A) = C_{AB}[\alpha_B(Q_B)]$ at general commensurate scales is also the relation connecting the values of the fixed points for any two effective charges or schemes. Furthermore, as $\beta \to 0$, the ratio of commensurate scales Q_A^2/Q_B^2 becomes the ratio of fixed point scales $\overline{Q}_A^2/\overline{Q}_B^2$ as one approaches the fixed point regime.

5 Implementation of α_V Scheme

Is there a preferred effective charge which we should use to characterize the coupling strength in QCD? In QED, the running coupling $\alpha_{QED}(Q^2)$, defined from the potential between two infinitely heavy test charges, has traditionally played that role. In the case of QCD, the heavy-quark potential $V(Q^2)$ is defined as the two-particle-irreducible scattering amplitude of test color charges; *i.e.* the scattering of an infinitely heavy quark and antiquark at momentum transfer $t = -Q^2$. The relation $V(Q^2) = -4\pi C_F \alpha_V(Q^2)/Q^2$ then defines the effective charge $\alpha_V(Q)$. This coupling can provide a physically based alternative to the usual $\overline{MS}$ scheme. As in the corresponding case of Abelian QED, the scale Q of the coupling $\alpha_V(Q)$ is identified with the exchanged momentum. Thus there is never any ambiguity in the interpretation of the scale. All vacuum polarization corrections due to fermion pairs are incorporated in α_V through the usual vacuum polarization kernels which depend on the physical mass thresholds. Of course, other observables could be used to define the standard QCD coupling, such as the effective charge defined from heavy quark radiation. [21]

The relation of $\alpha_V(Q^2)$ to the conventional $\overline{MS}$ coupling is now known to NNLO, [22] but in the following only the NLO relation will be used. The commensurate scale relation is given by [23]

$$\begin{aligned} \alpha_{\overline{\rm MS}}(Q) &= \alpha_V(Q^*) + \frac{2}{3} N_C \frac{\alpha_V^2(Q^*)}{\pi} \\ &= \alpha_V(Q^*) + 2\frac{\alpha_V^2(Q^*)}{\pi} , \end{aligned} \tag{11}$$

which is valid for $Q^2 \gg m^2$. The coefficients in the perturbation expansion have their conformal values, *i.e.*, the same coefficients would occur even if the theory had been conformally invariant with $\beta = 0$. The commensurate scale is given by

$$Q^* \quad = \quad Q \exp \left[\frac{5}{6} \right] \ . \tag{12}$$

The scale in the $\overline{MS}$ scheme is thus a factor ~ 0.4 smaller than the physical scale. The coefficient $2N_C/3$ in the NLO coefficient is a feature of the non-Abelian couplings of QCD; the same coefficient occurs even if the theory were conformally invariant with $\beta_0 = 0$.

Using the above QCD results, we can transform any NLO prediction given in $\overline{MS}$ scheme to a scale-fixed expansion in $\alpha_V(Q)$. We can also derive the connection between the $\overline{MS}$ and α_V schemes for Abelian perturbation theory using the limit $N_C \to 0$ with $C_F \alpha_s$ and N_F/C_F held fixed. [17]

The use of α_V and related physically defined effective charges such as α_p (to NLO the effective charge defined from the (1,1) plaquette, α_p is the same as α_V) as expansion parameters has been found to be valuable in lattice gauge theory, greatly increasing the convergence of perturbative expansions relative to those using the bare lattice coupling. [18] Recent lattice calculations of the Υ- spectrum [24] have been used with BLM scale-fixing to determine a NLO normalization of the static heavy quark potential: $\alpha_V^{(3)}(8.2\mathrm{GeV}) = 0.196(3)$ where the effective number of light flavors is $n_f = 3$. The corresponding modified minimal subtraction coupling evolved to the Z mass and five flavors is $\alpha_{\overline{MS}}^{(5)}(M_Z) = 0.1174(24)$. Thus a high precision value for $\alpha_V(Q^2)$ at a specific scale is available from lattice gauge theory. Predictions for other QCD observables can be directly referenced to this value without the scale or scheme ambiguities, thus greatly increasing the precision of QCD tests.

One can also use α_V to characterize the coupling which appears in the hard scattering contributions of exclusive process amplitudes at large momentum transfer, such as elastic hadronic form factors, the photon-to-pion transition form factor at large momentum transfer [3, 25] and exclusive weak decays of heavy hadrons.[26] Each gluon propagator with four-momentum k^μ in the hard-scattering quark-gluon scattering amplitude T_H can be associated with the coupling $\alpha_V(k^2)$ since the gluon exchange propagators closely resembles the interactions encoded in the effective potential $V(Q^2)$. [In Abelian theory this is exact.] Commensurate scale relations can then be established which connect the hard-scattering subprocess amplitudes which control exclusive processes to other QCD observables.

We can anticipate that eventually nonperturbative methods such as lattice gauge theory or discretized light-cone quantization will provide a complete form for the heavy quark potential in QCD. It is reasonable to assume that $\alpha_V(Q)$ will not diverge at small space-like momenta. One possibility is that α_V stays relatively constant $\alpha_V(Q) \simeq 0.4$ at low momenta, consistent with fixed-point behavior. There is, in fact, empirical evidence for freezing of the α_V coupling from the observed systematic dimensional scaling behavior of exclusive reactions. [25] If this is in fact the case, then the range of QCD predictions can be extended to quite low momentum scales, a regime normally avoided because of the apparent singular structure of perturbative extrapolations.

There are a number of other advantages of the V-scheme:

1. Perturbative expansions in α_V with the scale set by the momentum transfer cannot have any β-function dependence in their coefficients since all running coupling effects are already summed into the definition of the potential. Since coefficients involving β_0 cannot occur in an expansions in α_V, the divergent infrared renormalon series of the form $\alpha_V^n \beta_0^n n!$ cannot occur. The general convergence properties of the scale Q^* as an expansion in α_V is not known. [14]

2. The effective coupling $\alpha_V(Q^2)$ incorporates vacuum polarization contributions with finite fermion masses. When continued to time-like momenta, the coupling has the correct analytic dependence dictated by the production thresholds in the t channel. Since α_V incorporates quark mass effects exactly, it avoids the problem of explicitly computing and resumming quark mass corrections.

3. The α_V coupling is the natural expansion parameter for processes involving nonrelativistic momenta, such as heavy quark production at threshold where the

Coulomb interactions, which are enhanced at low relative velocity v as $\pi\alpha_V/v$, need to be re-summed. [27, 28, 29] The effective Hamiltonian for nonrelativistic QCD is thus most naturally written in α_V scheme. The threshold corrections to heavy quark production in e^+e^- annihilation depend on α_V at specific scales Q^*. Two distinct ranges of scales arise as arguments of α_V near threshold: the relative momentum of the quarks governing the soft gluon exchange responsible for the Coulomb potential, and a high momentum scale, induced by hard gluon exchange, approximately equal to twice the quark mass for the corrections. [28] One thus can use threshold production to obtain a direct determination of α_V even at low scales. The corresponding QED results for τ pair production allow for a measurement of the magnetic moment of the τ and could be tested at a future τ-charm factory. [27, 28]

We also note that computations in different sectors of the Standard Model have been traditionally carried out using different renormalization schemes. However, in a grand unified theory, the forces between all of the particles in the fundamental representation should become universal above the grand unification scale. Thus it is natural to use α_V as the effective charge for all sectors of a grand unified theory, rather than in a convention-dependent coupling such as $\alpha_{\overline{MS}}$.

6 The Analytic Extension of the $\overline{MS}$ Scheme

The standard $\overline{MS}$ scheme is not an analytic function of the renormalization scale at heavy quark thresholds; in the running of the coupling the quarks are taken as massless, and at each quark threshold the value of N_F which appears in the β function is incremented. Thus Eq. (11) is technically only valid far above a heavy quark threshold. However, we can use this commensurate scale relation to define an extended $\overline{MS}$ scheme which is continuous and analytic at any scale. The new modified scheme inherits all of the good properties of the α_V scheme, including its correct analytic properties as a function of the quark masses and its unambiguous scale fixing. [23]

Thus we define

$$\widetilde{\alpha}_{\overline{\rm MS}}(Q) = \alpha_V(Q^*) + \frac{2N_C}{3}\frac{\alpha_V^2(Q^{**})}{\pi} + \cdots , \tag{13}$$

for all scales Q. This equation not only provides an analytic extension of the $\overline{MS}$ and similar schemes, but it also ties down the renormalization scale to the physical masses of the quarks as they enter into the vacuum polarization contributions to α_V.

The modified scheme $\widetilde{\alpha}_{\overline{\rm MS}}$ provides an analytic interpolation of conventional $\overline{MS}$ expressions by utilizing the mass dependence of the physical α_V scheme. In effect, quark thresholds are treated analytically to all orders in m^2/Q^2; *i.e.*, the evolution of the analytically extended coupling in the intermediate regions reflects the actual mass dependence of a physical effective charge and the analytic properties of particle production. Just as in Abelian QED, the mass dependence of the effective potential and the analytically extended scheme $\widetilde{\alpha}_{\overline{\rm MS}}$ reflects the analyticity of the physical thresholds for particle production in the crossed channel. Furthermore, the definiteness of the dependence in the quark masses automatically constrains the renormalization scale. There is thus no scale ambiguity in perturbative expansions in α_V or $\widetilde{\alpha}_{\overline{\rm MS}}$.

In leading order the effective number of flavors in the modified scheme $\widetilde{\alpha}_{\overline{\rm MS}}$ is given to a very good approximation by the simple form [23]

$$\widetilde{N}^{(0)}_{F,\overline{\rm MS}}\left(\frac{m^2}{Q^2}\right) \cong \left(1 + \frac{5m^2}{Q^2 \exp(\frac{5}{3})}\right)^{-1} \cong \left(1 + \frac{m^2}{Q^2}\right)^{-1} . \tag{14}$$

Thus the contribution from one flavor is $\simeq 0.5$ when the scale Q equals the quark mass m_i. The standard procedure of matching $\alpha_{\overline{\rm MS}}(\mu)$ at the quark masses serves as a zeroth-order approximation to the continuous N_F.

Adding all flavors together gives the total $\widetilde{N}^{(0)}_{F,\overline{\rm MS}}(Q)$ which is shown in Fig. 1. For reference, the continuous N_F is also compared with the conventional procedure of taking N_F to be a step-function at the quark-mass thresholds. The figure shows clearly that there are hardly any plateaus at all for the continuous $\widetilde{N}^{(0)}_{F,\overline{\rm MS}}(Q)$ in between the quark masses. Thus there is really no scale below 1 TeV where $\widetilde{N}^{(0)}_{F,\overline{\rm MS}}(Q)$ can be approximated by a constant; for all Q below 1 TeV there is always one quark with mass m_i such that $m_i^2 \ll Q^2$ or $Q^2 \gg m_i^2$ is not true. We also note that if one would use any other scale than the BLM-scale for $\widetilde{N}^{(0)}_{F,\overline{\rm MS}}(Q)$, the result would be to

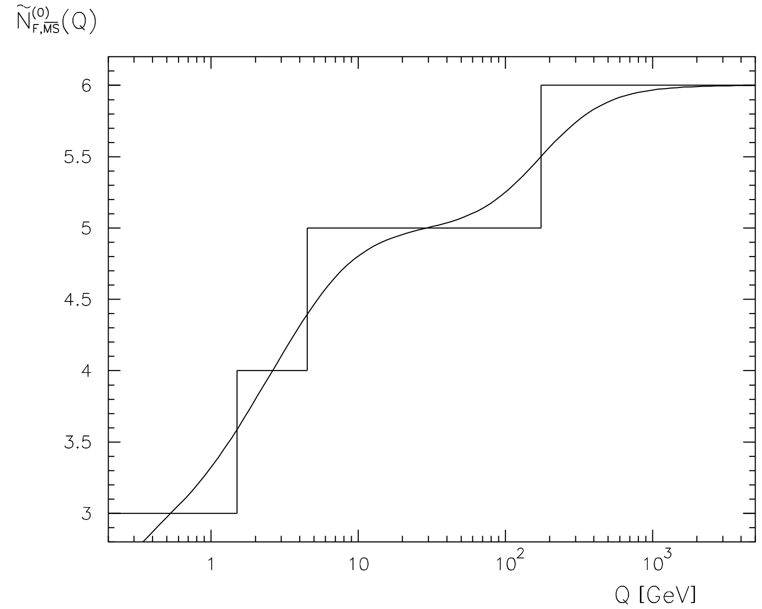

Figure 1: The continuous $\widetilde{N}^{(0)}_{F,\overline{\rm MS}}$ in the analytic extension of the $\overline{\rm MS}$ scheme as a function of the physical scale Q. (For reference the continuous N_F is also compared with the conventional procedure of taking N_F to be a step-function at the quark-mass thresholds.)

increase the difference between the analytic N_F and the standard procedure of using the step-function at the quark-mass thresholds.

Figure 2 shows the relative difference between the two different solutions of the 1-loop renormalization group equation, *i.e.* $(\tilde{\alpha}_{\overline{\rm MS}}(Q) - \alpha_{\overline{\rm MS}}(Q))/\tilde{\alpha}_{\overline{\rm MS}}(Q)$. The solutions have been obtained numerically starting from the world average [30] $\alpha_{\overline{\rm MS}}(M_Z) = 0.118$. The figure shows that taking the quark masses into account in the running leads to effects of the order of one percent, most especially pronounced near thresholds.

To illustrate how to compute an observable using the analytic extension of the $\overline{\rm MS}$ scheme and compare with the standard treatment in the $\overline{\rm MS}$ scheme we consider the QCD corrections to the quark part of the non-singlet hadronic width of the Z-boson,

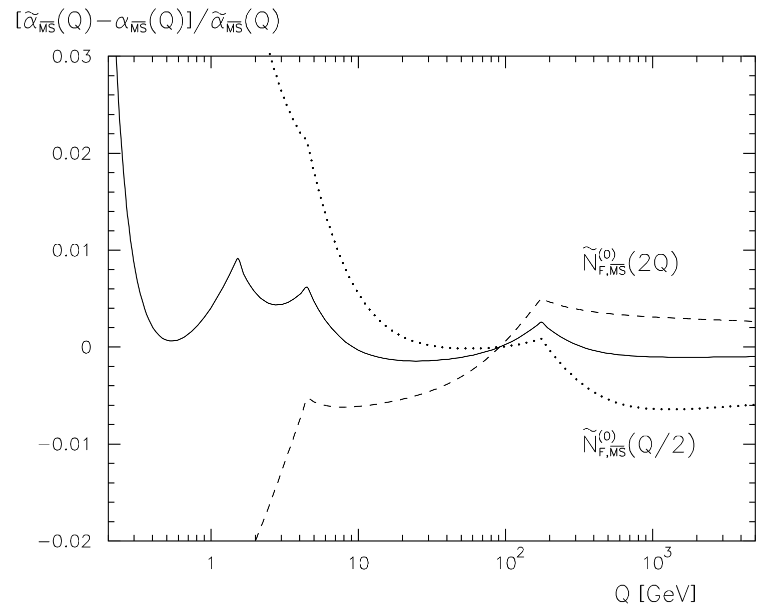

Figure 2: The solid curve shows the relative difference between the solutions to the 1-loop renormalization group equation using continuous N_F, $\widetilde{\alpha}_{\overline{\rm MS}}(Q)$, and conventional discrete theta-function thresholds, $\alpha_{\overline{\rm MS}}(Q)$. The dashed (dotted) curves shows the same quantity but using the scale $2Q$ $(Q/2)$ in $\widetilde{N}^{(0)}_{F,\overline{\rm MS}}$. The solutions have been obtained numerically starting from the world average [30] $\alpha_{\overline{\rm MS}}(M_Z) = 0.118$.

$\Gamma^{NS}_{had,q}$. Writing the QCD corrections in terms of an effective charge we have

$$\Gamma^{NS}_{had,q} = \frac{G_F M_Z^3}{2\pi\sqrt{2}} \sum_q \{(g_V^q)^2 + (g_A^q)^2\} \left[1 + \frac{3}{4} C_F \frac{\alpha^{NS}_{\Gamma,q}(s)}{\pi}\right] \tag{15}$$

where the effective charge $\alpha^{NS}_{\Gamma,q}(s)$ contains all QCD corrections,

$$\begin{aligned} \frac{\alpha^{NS}_{\Gamma,q}(s)}{\pi} &= \frac{\alpha^{(N_L)}_{\overline{\rm MS}}(\mu)}{\pi}\Bigg\{1 + \frac{\alpha^{(N_L)}_{\overline{\rm MS}}(\mu)}{\pi} \\ &\times \left[\sum_{q=1}^{N_L}\left(-\frac{11}{12} + \frac{2}{3}\zeta_3 + F\left(\frac{m_q^2}{s}\right) - \frac{1}{3}\ln\left(\frac{\mu}{\sqrt{s}}\right)\right)\right. \\ &\left. + \sum_{Q=N_L+1}^{6} G\left(\frac{m_Q^2}{s}\right)\right] + \ldots\Bigg\} \end{aligned} \tag{16}$$

To calculate $\alpha_{\Gamma,q}^{NS}(s)$ in the analytic extension of the $\overline{\rm MS}$ scheme one first applies the BLM scale-setting procedure in order to absorb all the massless effects of non-zero N_F into the running of the coupling. This gives

$$
\begin{aligned}
\frac{\alpha_{\Gamma,q}^{NS}(s)}{\pi} &= \frac{\alpha_{\overline{\rm MS}}^{(N_L)}(Q^*)}{\pi} \\
&\quad \times \left\{ 1 + \frac{\alpha_{\overline{\rm MS}}^{(N_L)}(Q^*)}{\pi} \left[\sum_{q=1}^{N_L} F\left(\frac{m_q^2}{s}\right) + \sum_{Q=N_L+1}^{6} G\left(\frac{m_Q^2}{s}\right) \right] + \ldots \right\}
\end{aligned}
\tag{17}
$$

where

$$
Q^* = \exp\left[3\left(-\frac{11}{12} + \frac{2}{3}\zeta_3\right)\right]\sqrt{s} = 0.7076\sqrt{s}. \tag{18}
$$

Operationally, one then simply drops all the mass dependent terms in the above expression and replaces the fixed N_F coupling $\alpha_{\overline{\rm MS}}^{(N_L)}$ with the analytic $\widetilde{\alpha}_{\overline{\rm MS}}$. (For an observable calculated with massless quarks this step reduces to replacing the coupling.) In this way both the massless N_F contribution, as well as the mass-dependent contributions from double bubble diagrams, are absorbed into the coupling. We are thus left with a very simple expression,

$$
\frac{\alpha_{\Gamma,q}^{NS}(s)}{\pi} = \frac{\widetilde{\alpha}_{\overline{\rm MS}}(Q^*)}{\pi}, \tag{19}
$$

reflecting the fact that the QCD effects of quarks in the perturbative coefficients, both massless and massive, should be absorbed into the running of the coupling.

In order to compare the analytic extension of the $\overline{\rm MS}$ scheme with the standard $\overline{\rm MS}$ result for $\alpha_{\Gamma,q}^{NS}(s)$, we will apply the BLM scale-setting procedure also for the standard $\overline{\rm MS}$ scheme. This is to ensure that any differences are due to the different ways of treating quark masses and not due to the scale choice. In other words we want to compare Eqs. (17) and (19). As the normalization point we use $\alpha_{\overline{\rm MS}}^{(5)}(M_Z) = 0.118$ which we evolve down to $Q^* = 0.7076 M_Z$ using leading order massless evolution with $N_F = 5$. This value is then used to calculate $\alpha_{\Gamma,q}^{NS}(M_Z) = 0.1243$ in the $\overline{\rm MS}$ scheme using Eq. (17). Finally, Eq. (19) gives the normalization point for $\widetilde{\alpha}_{\overline{\rm MS}}(Q^*)$.

Figure 3 shows the relative difference between the two expressions for $\alpha_{\Gamma,q}^{NS}(s)$ given by Eqs. (17) and (19) respectively. As can be seen from the figure the relative difference is remarkably small, less than 0.2% for scales above 1 GeV. Thus the

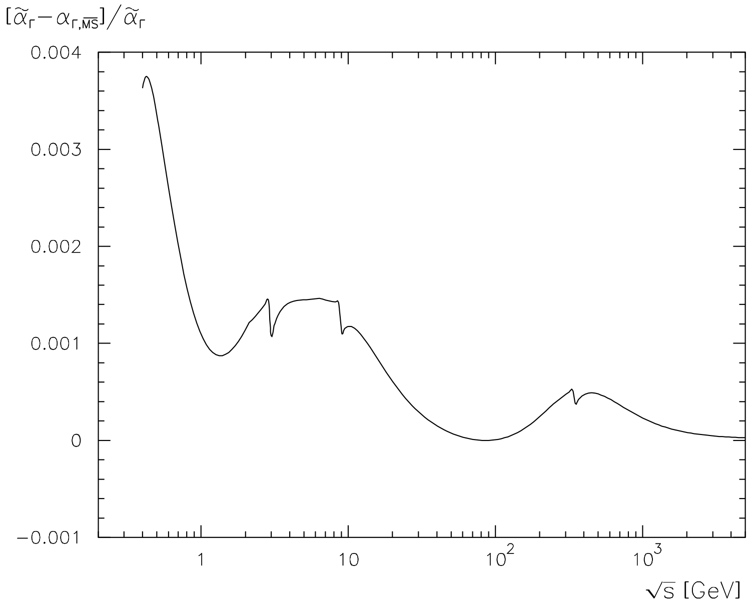

Figure 3: The relative difference between the calculation of $\alpha^{NS}_{\Gamma,q}(s)$ in the analytic extension of the $\overline{\rm MS}$ scheme and the standard treatment of masses in the $\overline{\rm MS}$ scheme. The discontinuities are due to the mismatch between the s/m^2 and m^2/s expansions of the functions F and G.

analytic extension of the $\overline{\rm MS}$ scheme takes the mass corrections into account in a very simple way without having to include an infinite series of higher dimension operators or doing complicated multi-loop diagrams with explicit masses.

The form of $N_F(Q)$ at NNLO has recently been computed to two loop order in QCD for the α_V scheme. The application to the analytic extension of $\overline{\rm MS}$ scheme will be discussed in a forthcoming paper. [31]

7 Conclusion

Commensurate scale relations have a number of attractive properties:

1. The ratio of physical scales Q_A/Q_B which appears in commensurate scale rela-

tions reflects the relative position of physical thresholds, *i.e.* quark anti-quark pair production.

2. The functional dependence and perturbative expansion of the CSR are identical to those of a conformal scale-invariant theory where $\beta_A(\alpha_A) = 0$ and $\beta_B(\alpha_B) = 0$.

3. In the case of theories approaching fixed-point behavior $\beta_A(\overline{\alpha}_A) = 0$ and $\beta_B(\overline{\alpha}_B) = 0$, the commensurate scale relation relates both the ratio of fixed point couplings $\overline{\alpha}_A/\overline{\alpha}_B$, and the ratio of scales as the fixed point is approached.

4. Commensurate scale relations satisfy the Abelian correspondence principle [17]; *i.e.* the non-Abelian gauge theory prediction reduces to Abelian theory for $N_C \to 0$ at fixed $C_F\alpha_s$ and fixed N_F/C_F.

5. The perturbative expansion of a commensurate scale relation has the same form as a conformal theory, and thus has no $n!$ renormalon growth arising from the β-function. It is an interesting conjecture whether the perturbative expansion relating observables to observable are in fact free of all $n!$ growth. The generalized Crewther relation, where the commensurate relation's perturbative expansion forms a geometric series to all orders, has convergent behavior.

Virtually any perturbative QCD prediction can be written in the form of a commensurate scale relation, thus eliminating any uncertainty due to renormalization scheme or scale dependence. Recently it has been shown [32] how the commensurate scale relation between the radiative corrections to τ-lepton decay and $R_{e^+e^-}(s)$ can be generalized and empirically tested for arbitrary τ mass and nearly arbitrarily functional dependence of the τ weak decay matrix element.

An essential feature of the α_V(Q) scheme is the absence of any renormalization scale ambiguity, since Q^2 is, by definition, the square of the physical momentum transfer. The α_V scheme naturally takes into account quark mass thresholds, which is of particular phenomenological importance to QCD applications in the mass region close to threshold. As we have seen, commensurate scale relations provide an analytic extension of the conventional $\overline{\mathrm{MS}}$ scheme in which many of the advantages of the α_V scheme are inherited by the $\widetilde{\alpha}_{\overline{\mathrm{MS}}}$ scheme, but only minimal changes have to be made. Given the commensurate scale relation connecting $\widetilde{\alpha}_{\overline{\mathrm{MS}}}$ to α_V expansions in $\widetilde{\alpha}_{\overline{\mathrm{MS}}}$ are effectively expansions in α_V to the given order in perturbation theory at a corresponding commensurate scale. Taking finite quark mass effects into account analytically in the running, rather than using a fixed flavor number N_F between thresholds, leads to effects of the order of 1% for the one-loop running coupling, with the largest differences occurring near thresholds. These differences are important for observables which are calculated neglecting quark masses, and could turn out to be significant when comparing low and high energy measurements of the strong coupling.

Unlike the conventional $\alpha_{\overline{\mathrm{MS}}}$ scheme, the modified $\widetilde{\alpha}_{\overline{\mathrm{MS}}}$ scheme is analytic at quark mass thresholds, and it thus provides a natural expansion parameter for perturbative representations of observables. In addition, the extension of the $\overline{\mathrm{MS}}$ scheme, including quark mass effects analytically, reproduces the standard treatment of quark masses in the $\overline{\mathrm{MS}}$ scheme to within a fraction of a percent. The standard treatment amounts to either calculating multi-loop diagrams with explicit quark masses or adding higher dimension operators to the effective Lagrangian. These corrections can be viewed as compensating for the fact that the number of flavors in the running is kept constant between mass thresholds. By utilizing the BLM scale setting procedure, based on the massless N_F contribution, the analytic extension of the $\overline{\mathrm{MS}}$ scheme correctly absorbs both massless and mass dependent quark contributions from QCD diagrams, such as the double bubble diagram, into the running of the coupling. This gives the opportunity to convert any calculation made in the $\overline{\mathrm{MS}}$ scheme with massless quarks into an expression which includes quark mass corrections from QCD diagrams by using the BLM scale and replacing $\alpha_{\overline{\mathrm{MS}}}$ with $\widetilde{\alpha}_{\overline{\mathrm{MS}}}$.

Finally, we note the potential importance of utilizing the α_V effective charge or the equivalent analytic $\widetilde{\alpha}_{\overline{\mathrm{MS}}}$ scheme in supersymmetric and grand unified theories, particularly since the unification of couplings and masses would be expected to occur in terms of physical quantities rather than parameters defined by theoretical convention.

Acknowledgments

Much of this work is based on collaborations with Michael Melles, Mandeep Gill, Hung Jung Lu, Andrei Kataev, and Gregory Gabadadze. We also thank the organizers of RADCOR98, particularly Professor Joan Sola of the Universitat Autonoma de Barcelona, for their outstanding arrangements and hospitality.

References

[1] S. J. Brodsky and H. J. Lu, *Phys. Rev.* **D 51** (1995) 3652.

[2] S. J. Brodsky , G. T. Gabadadze, A. L. Kataev, and H. J. Lu. *Phys. Lett.* **B 372** (1996) 133.

[3] S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, *Phys. Rev.* **D 28** (1983) 228.

[4] G. Grunberg, *Phys. Lett.* **B 95** (1980) 70; *Phys. Lett.* **B 110** (1982) 501; *Phys. Rev.* **D D29** (1984) 2315.

[5] A. Dhar and V. Gupta, *Phys. Rev.* **D D29** (1984) 2822.

[6] V. Gupta, D. V. Shirkov and O. V. Tarasov, *Int. J. Mod. Phys.* **A6** (1991) 3381.

[7] S. Adler, *Phys. Rev.* **182** (1969) 1517.

[8] A. C. Mattingly and P. M. Stevenson, *Phys. Rev.* **D 49** (1994) 437.

[9] CCFR Collaboration, W.C. Leung, *et al.*, *Phys. Lett.* **B317** (1993) 655;

[10] CCFR and NuTeV Collaboration, presented by D. Harris at XXX Recontre de Moriond, 1995, presented by J. H. Kim at the European Conference on High Energy Physics, Brussels, July 1995.

[11] A. L. Kataev, A.V. Sidorov, *Phys. Lett.* **B331** (1994) 179.

[12] CCFR Collaboration, P.Z. Quinta, *et al.*, *Phys. Rev. Lett.* **71** (1993) 1307.

[13] G. 't Hooft, in the Proceedings of the International School, Erice, Italy, 1977, edited by A. Zichichi, Subnuclear Series Vol. 15 (Plenum, New York, 1979).

[14] A. H. Mueller, *Phys. Lett.* **B 308** (1993) 355.

[15] H. J. Lu, *Phys. Rev.* **D 45** (1992) 1217.

[16] M. Beneke and V. M. Braun, *Phys. Lett.* **B 348** (1995) 513.

[17] S. J. Brodsky and P. Huet, *Phys. Lett.* **B 417** (1998) 145.

[18] G. Peter Lepage and P. B. Mackenzie, *Phys. Rev.* **D 48** (1993) 2250.

[19] M. Neubert, *Phys. Rev.* **D 51** (1995) 5924.

[20] P. Ball, M. Beneke and V. M. Braun, *Nucl. Phys.* **B 452** (1995) 563.

[21] A. Czarnecki, K. Melnikov, and N. Uraltsev, Phys. Rev. Lett. **80** (1998) 3189. Yu. L. Dokshitser and B. R. Webber, *Phys. Lett.* **B404** (1997) 321.

[22] M. Peter, *Nucl. Phys.* **B501** (1997) 471.

[23] S. J. Brodsky, M. S. Gill, M. Melles, and J. Rathsman, *Phys. Rev.* **D 58** (1998) 116006, hep-ph/9801330

[24] C.T.H. Davies, *et al.*, *Phys. Lett.* **B 345** (() 1995)42; *Phys. Rev.* **D 56** (1997) 2755.

[25] S. J. Brodsky, C.-R. Ji, A. Peng and D. G. Robertson, *Phys. Rev.* **D 57** (1998) 245.

[26] A. Szczepaniak, E. M. Henley, S. J. Brodsky, *Phys. Lett.* **B243** (1990) 287.

[27] B. H. Smith, M. B. Voloshin, *Phys. Lett.* **B324** (1994) 117. Erratum-ibid. **B333** (1994) 564.

[28] S.J. Brodsky, A. H. Hoang, J. H. Kuhn, and T. Teubner, *Phys. Lett.* **B359** (1995) 355.

[29] V. S. Fadin, V. A. Khoze, A. D. Martin, and W. J. Stirling, *Phys. Lett.* **B363** (1995) 112.

[30] P. N. Burrows, Acta Phys. Polon. **B28**, 701 (1997).

[31] S. Brodsky, M. Melles, and J. Rathsman, SLAC-PUB-8019, in preparation.

[32] S. J. Brodsky, J. R. Pelaez, and N. Toumbas, hep-ph/9810424