SLAC–PUB–8144
May 1999

Chaos in Accelerators*

Alex Chao
Stanford Linear Accelerator Center, Stanford Unversity, CA 94309

Abstract

Chaos is a general phenomenon in nonlinear dynamical systems. Accelerators – storage rings in particular – in which particles are stored for 10^{10} revolutions constitute a particularly intricate nonlinear dynamical system. (In comparison, the earth has revolved around the sun for only 10^9 turns.) Storage rings therefore provide an ideal testing ground for chaos physics. In fact, it is the chaos phenomenon that imposes one of the key design criteria for these accelerators. One might arguably say that the demise of the Superconducting Super Collider project originated from a misjudgement in its chaos analysis at one point along its design path, leading to its first substantial cost escalation. This talk gives an elementary introduction to the study of chaos in accelerators.

Invited talk at Overseas Chinese Physics Association Conference on Recent Advances and Cross-Century Outlooks in Physics, American Physical Society Centennial Meeting
Atlanta, Georgia
March 18-20, 1999

*Work supported by Department of Energy contract DE–AC03–76SF00515.

CHAOS IN ACCELERATORS

ALEX CHAO
Stanford Linear Accelerator Center, P. O. Box 4349, Stanford, CA 94309
E-mail: achao@slac.stanford.edu

Chaos is a general phenomenon in nonlinear dynamical systems. Accelerators – storage rings in particular – in which particles are stored for 10^{10} revolutions constitute a particularly intricate nonlinear dynamical system. (In comparison, the earth has revolved around the sun for only 10^9 turns.) Storage rings therefore provide an ideal testing ground for chaos physics. In fact, it is the chaos phenomenon that imposes one of the key design criteria for these accelerators. One might arguably say that the demise of the Superconducting Super Collider project originated from a misjudgement in its chaos analysis at one point along its design path, leading to its first substantial cost escalation. This talk gives an elementary introduction to the study of chaos in accelerators.

1 Introduction

The problem being studied is illustrated in Fig.1. Consider a particle that starts on the "design orbit" as in Fig.1(a). It returns exactly to its starting point turn after turn. This particle is stable, i.e. it always stays within the accelerator confinement. However, in Fig.1(b), a particle starting with a small deviation (x, x') from the design orbit will not return to where it starts from even in its next turn. As it circulates around the accelerator, its deviation from the design orbit may stay confined, or may grow with time. The question to be addressed is: *does the accelerator provide an environment for particles with small deviations to stay in the accelerator confinement for* $\tau = 10^{10}$ *turns*?

Note that accelerator physicists are not interested in $\tau = \infty$, which would be a very different question perhaps of academic interest. In this sense, the question being raised here is a rather pragmatic one.

Instability results from nonlinearities. Particle motion is stable if the accelerator is perfectly linear (assuming the accelerator is properly designed). This leads to the practice that all accelerators are designed to be "as linear as possible" with the hope that nonlinearities can be treated as small perturbations. However, would the small nonlinearities cause an ever-so-slow growth of the oscillation amplitudes, and thus lead to an eventual loss of particles in 10^{10} turns?

There are many sources of nonlinearities in an accelerator. For example, the magnets may not have perfect field profiles. (Another example is the so-called beam-beam interaction – see later.) Of course, one tries to build

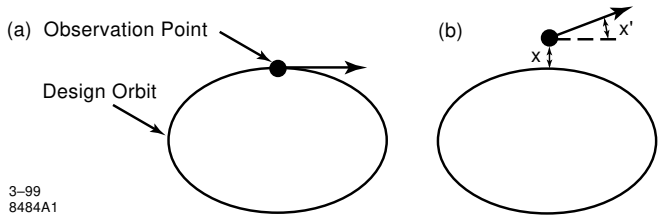

Figure 1. A particle moving (a) along the design orbit, (b) with a small deviation from the design orbit.

magnets as perfectly as possible, but that would be expensive. To minimize cost, one must build the magnets at their lowest tolerable degree of linearity, while providing 10^{10}-turn lifetime of the beam. A miscalculation or misjudgement, like what happended to the SSC, can be a serious matter. Typically, thanks to the advances made by the magnet builders, the magnet field nonlinearities are very small, at the level of 10^{-4}. But unfortunately, even that does not assure a 10^{10}-turn lifetime – the existence of the *KAM invariant surfaces* is not of much help here.

2 Effects of Nonlinearities

Consider the particle in Fig.1(b). Let it start with (x_0, x_0') and register its motion turn after turn as (x_1, x_1'), (x_2, x_2'), ... [See Fig.2(a) – note that (x, x') forms the *phase space* of the dynamical system. Observing the particle motion at discrete periodic times is called the *Poincaré section*.] If the accelerator is perfectly linear, (x_i, x_i') will trace out a circle as shown in Fig.2(b), with

$$\begin{aligned} \text{Amplitude } \sqrt{x_i^2 + {x_i'}^2} &= \text{constant } A \\ (\text{phase angle advance per turn})/2\pi &= \text{tune } \nu_0 \end{aligned} \tag{1}$$

In a linear system, different particles have different amplitudes, but all particles have the same ν_0, i.e. $\nu = \nu_0 \neq \nu(A)$. We have assumed ν_0 is an irrational number. (The tune is sometimes called the *winding number* in other fields.) If ν_0 is a rational number, then a particle trajectory traces out a chain of discrete unconnected dots in the phase space.

We now consider the effects introduced by a small nonlinearity in the accelerator. The first thing that happens is that the tune will no longer be the same for all amplitudes, i.e. now there is a tune spread, and $\nu = \nu(A)$. As soon as this happens, the tune no longer stays irrational, and takes on rational

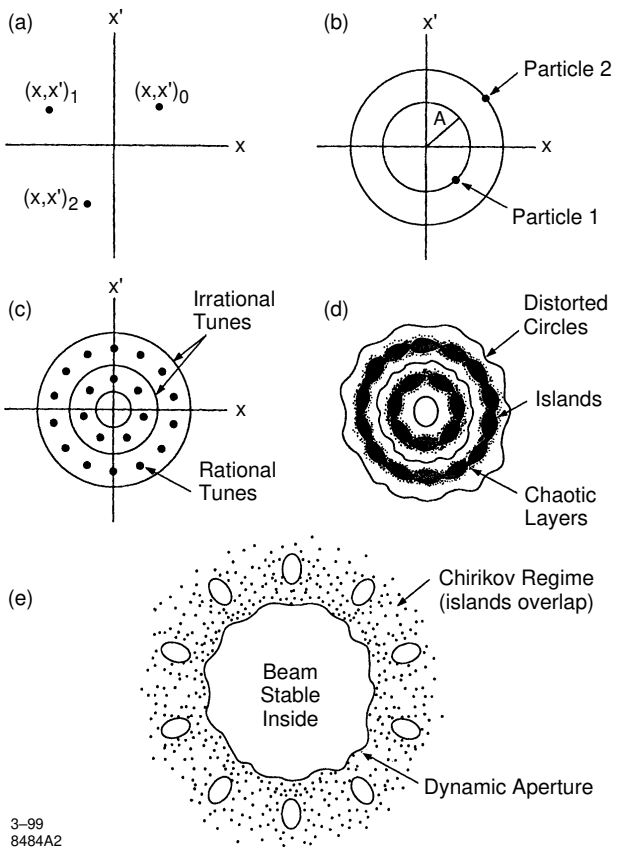

Figure 2. Observing particle motion in the phase space (x, x'). (a) Registering the observation turn after turn. (b) If the accelerator system is perfectly linear, the trajectories trace out a circle in phase space. Two circles are indicated, each traced out by one particle. (c) When a tune spread is introduced, the phase space spontaneously transforms into an infinitely layered topology. (d) Around each layer corresponding to a rational tune, a chain of islands with thin chaotic coating emerges. (e) The width of the islands increases with amplitude A. Beyond a certian maximum A, the dynamic aperture, the islands overlap and particles are lost from the accelerator.

and irrational values depending on the amplitude of the particle under consideration. The phase space thus spontaneously degenerate into an infinitely layered sandwich of circles and dot-chains, as shown in Fig.2(c). Each circle corresponds to one irrational tune, while each chain of dots corresponds to one rational tune.

In spite of its intricacy, the system remains integrable (and soluble) at this point. However, those "dots" in Fig.2(c) in fact have finite sizes, i.e. they are actually "islands". There is one chain of islands for each *resonance condition*

$$\nu = \text{rational number } n/m \tag{2}$$

The finite sizes of the islands necessarily affect the neighboring circles. Some circles are more resistent and the only thing happening to them is that their shapes are distorted. For some other less resistant circles, however, their turn-by-turn trajectories break into chaotic layers. For those particles, their motion is now nonintegrable. Fig.2(d) shows some distorted circles, some islands, and some chaotic layers.

Whether a circle is break-resistant or not depends on its tune value. It turns out that not all irrational numbers are created equal; some are more irrational than others, and the more resistant circles correspond to the more irrational tunes, which turn out to take the form

$$\nu(\text{most irrational}) = \frac{n + \ell\sqrt{5}}{m} \tag{3}$$

(It is somewhat surprising that $\sqrt{5}$ has this fundamental significance.) Circles whose tunes are of the form (3) tend to break last. Unbroken circles, although distorted, have the significance of *invariant surfaces*. Finding an invariant surface is always a significant step because all particles inside the invariant surface will stay inside for all times (in a 1-D system) and are therefore stable.

It is a property of accelerator magnet field errors that the nonlinearity increases as one deviates more from the design orbit. Increasing A therefore implies the corresponding islands get larger, which in turn means more neighboring circles get broken. When A is increased further, it can reach a value A_{DA} beyond which islands "overlap" into a continuum and no invariant surfaces exist beyond it. Particles with $A > A_{DA}$ are therefore unstable. This limiting amplitude A_{DA} – the last invariant surface of the system – is called the *dynamic aperture* of the accelerator.

3 Chirikov Criterion and Beam-Beam Interaction

The criterion of large-scale chaos due to overlapping resonances in a 1-D system is called the *Chirikov criterion*.[1] To apply the Chirikov criterion, one typically first performs a simplified calculation of the widths of all resonances, valid to first order of the nonlinearity strength and assuming each resonance is isolated from all others. (This calculation is doable because under these

assumptions the system is integrable.) One then adds up the widths of all resonances to obtain a total width $\delta\nu_{tot}$. A large-scale chaos occurs when $\delta\nu_{tot} \geq 1$.

The above calculation has been applied, for example, to the important problem of *beam-beam interaction*. In a collider storage ring, two beams of charged particles collide at an interaction point. At this interaction point, each beam is strongly perturbed by the electromagnetic field of the on-coming beam. This perturbation is highly nonlinear, and its strength is characterized by a parameter called the *beam-beam tune shift* ξ. The beam-beam induced nonlinear resonance widths, and therefore $\delta\nu_{tot}$, are proportional to ξ. An application of the Chirikov criterion to a 1-D beam-beam problem yields the conclusion that the beams are unstable if $\xi > 0.095$.[2]

Unfortunately, the maximum ξ's achieved in the many existing colliders range only from 0.03 to 0.06 for electron beams and even lower for proton beams. One possible explanation is that an accelerator operates in a 3-D world, while the Chirikov criterion applies to a 1-D world. In a real 3-D accelerator system, nonlinearities may cause other more subtle instabilities such as the *Arnold diffusion*.[3]

Another important complication is due to noise or ripple in the magnet settings or ground motion (see later). When these effects are included, the dynamic aperture determination again becomes more subtle.

4 Chirikov Criterion and Incompressible Fluid

Chirikov criterion	Incompressible fluid
particle motion in phase space	fluid motion in real space
Hamilton equation	Navier-Stokes equation
smooth contours	laminar flow
chaos	turbulence
overlapping resonances	Reynolds' condition

Table 1. Analogy between the Chirikov criterion in 1-D nonlinear dynamics and the Reynolds' condition for an incompressible fluid.

As suggested by Teng, it is interesting to draw an analogy between chaos in 1-D nonlinear dynamics and turbulence in fluid dynamics.[4] The idea is based on the observation (Liouville theorem in a Hamiltonian system) that the beam distribution in phase space behaves as a viscous incompressible fluid in the real space. By writing down the Hamilton equation on the one hand and the fluid equation on the other, it is possible to establish the analogy as given in Table 1. Chirikov criterion is then equivalent to the Reynolds

condition in fluid dynamics that the viscosity must be large enough in order to prevent turbulence from occurring. This offers an alternative derivation of the Chirikov criterion – although the result takes a form which is very close but not exactly identical.

5 Accelerator Modeling and Simulation

Numerical simulation is an essential tool to study chaos physics. To simulate the nonlinear dynamics in an accelerator, one must first have a dependable computer model of the accelerator. Typically the accelerator is modeled as a string of beam-line elements. (A beam-line element could be a dipole magnet, a quadrupole magnet, a drift space, an rf cavity, etc.) Each beam-line element is then represented as a *map* which relates the incoming and outgoing beam dynamical variables as a particle traverses through the element:

$$\vec{X}_{out} = \vec{F}(\vec{X}_{in}) \tag{4}$$

Explicit expressions of the function $\vec{F}$ can be obtained from the Hamiltonian (and can be a *Lie map*[5] or a *Taylor map* depending on whether it is expressed in a Lie algebraic language or as a Taylor series). Given the initial launching conditions $\vec{X}_0$, the subsequent trajectory of a particle in the accelerator can be followed numerically by sequentially applying the known element maps (4).

Unfortunately, this *element-by-element tracking* is severely limited by the computer capacity. Take the SSC for example. The SSC contains a string of 5000 superconducting magnets, each having its own set of field errors. An element-by-element model of the SSC therefore contains 5000 nonlinear kicks per revolution, and it took 200 CRAY hours (1993) to track the SSC for 10^6 turns. Tracking for 10^{10} turns is obviously not practical.

To proceed, one then presents the tracking results in the form as Fig.3.[6] Such a presentation is called a *survival plot*. The number of turns a particle survives in the accelerator is plotted against the launching amplitude of the particle. As the launching amplitude is lowered, the survival time increases until a point when computer time becomes a limit. What one wished is to be able to extrapolate the curve to find out the launching amplitude when the survival time reaches 10^{10} turns, and that amplitude is then identified as the dynamic aperture.

It is obvious that the survival plot technique is not too assuring, and one might ask if there are ways to speed up the simulation. One such method has been the *concatenation* techniques. The 5000 nonlinear maps in the SSC, for example, can be concatenated into one single nonlinear map, which represents the *one-turn map* of the SSC. To save speed, this one-turn map must be

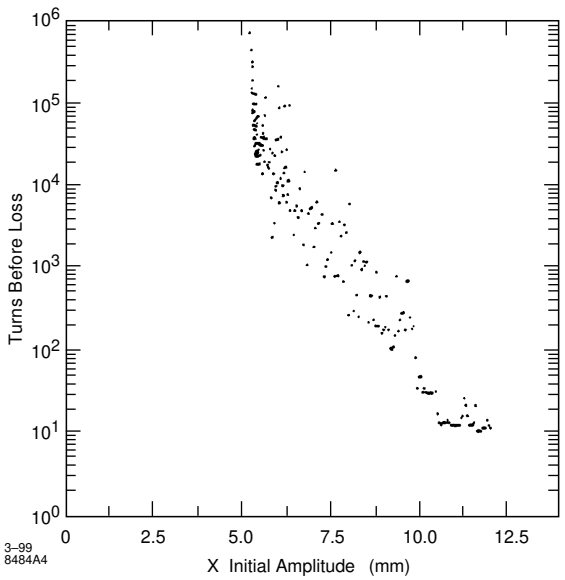

Figure 3. Survival plot for an SSC model.

severely truncated to a sufficiently low order (e.g. 10-th order). On the other hand, once truncated, particle tracking can be sped up greatly. How to appropriately use one-turn map to calculate the dynamic aperture is an important area of study.

6 Near the Dynamic Aperture

It should not be a surprise that the underlying dynamics is extremely complex and there are many curious observations to be made, especially when one approaches the dynamic aperture. One of the most striking can be observed on the particle in Fig.3 that was lost near the 10^6-th turn. As shown in Fig.4,[7] this particle seems to stay stable happily for all of its 10^6-turn lifetime, but is suddenly lost in the last 30 turns without any apparent warning. Fig.4 rules out any "diffusion" effect as the instability mechanism. Ideas have been proposed by accelerator physicists of methods to search for early warning signals of a long-term particle loss. Fig.4 is an indication that the task is not an easy one.

A large-scale chaos (e.g. the type predicted by Chirikov) can be detected relatively easily by tracking a single particle. To detect more subtle effects (e.g. local chaos, Arnold diffusion, etc.), one often uses the technique of

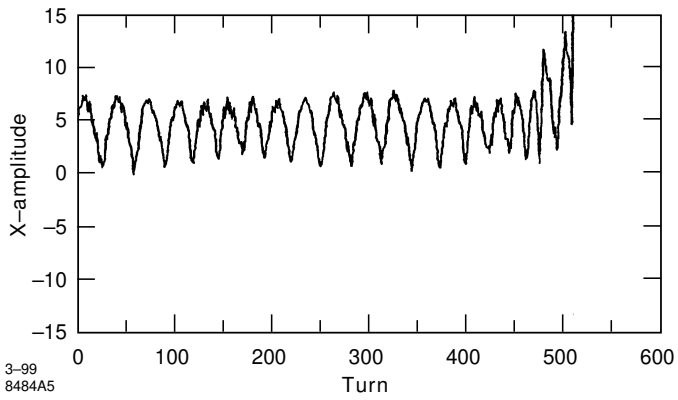

Figure 4. The last 512 turns of the particle that survives 10^6 turns in Fig.3. The particle seems to take off suddenly in its last 30 turns of life without any warning.

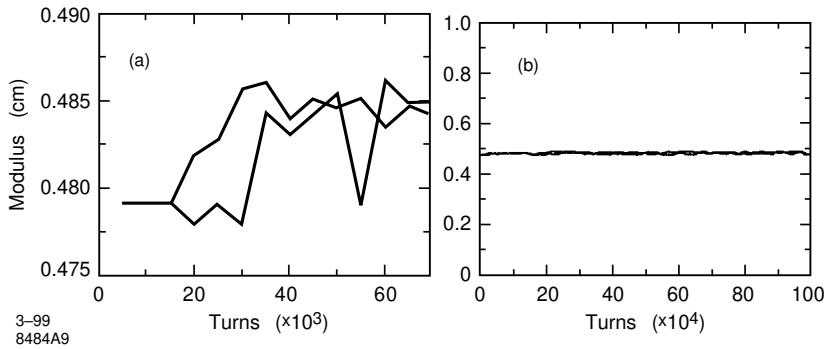

Figure 5. Two particles for the SSC initially very close to each other become separated and follow distinctly different trajectories after some number of turns. This is an indication of chaos – although a small-scaled one because both particles seem to be stable up to the number of turns tracked. (a) fine scale, (b) gross scale.

Lyapunov in which two particles initially very close to each other in phase space are tracked, and their distance of separation registered as a function of time. A local chaos is identified when this distance grows exponentially. Local chaos however may or may not lead to true instability. Fig.5 shows an example of local chaos which is apparently stable for a long time.[8]

A slight variation of the Lyapunov method is shown in Fig.6 for the collider LHC.[9] The unperturbed tunes are indicated by a star in Fig.6. The

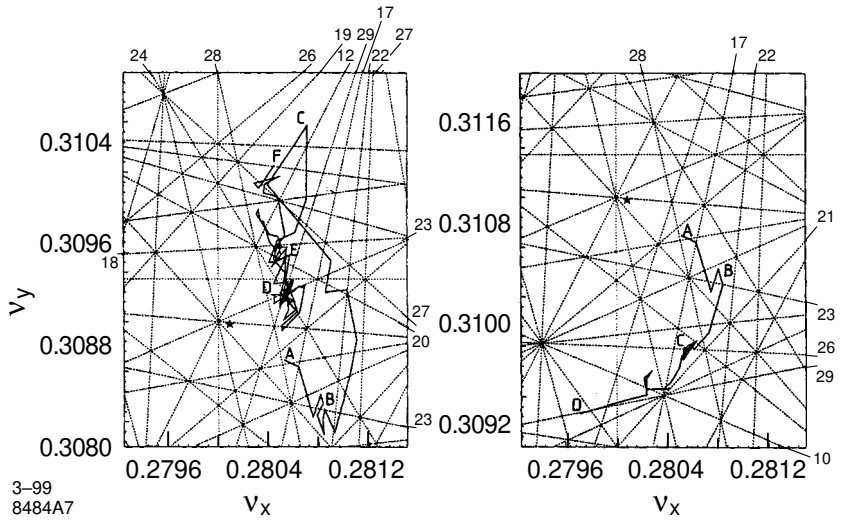

Figure 6. The wondering paths of the instantaneous tunes of two neighboring particles in the (ν_x, ν_y) plane for the LHC. Resonances up to 29-th order are indicated by dotted lines.

instantaneous tunes are obtained by sampling 1000 turns for each reading on the tune path. The two neighboring particles start with the same tunes, follow each other closely for 4000 turns and then depart to take on their own destiny, encountering very different sets of resonances. Eventually both particles are lost, one in 65000 turns, the other in 15000 turns.

7 Nonlinear Dynamics Experiments

As mentioned, accelerators are ideal tools to study nonlinear dynamics. In the Indiana Univeristy Cyclotron, for example, an electron-cooled beam has a small transverse emittance and a small energy spread, which makes it ideal for probing the phase space. To explore the phase space, the cooled beam is kicked and its subsequent turn-by-turn motion detected by two position monitors. Fig.7 shows one such results when the horizontal tune ν_x is close to a 4-th order resonance.[10] From such measurements, one obtains detailed information on the nonlinear Hamiltonian of the accelerator system. In particular, it was measured that the "island tune" (the tune value corresponding to motion around the islands instead of around the origin) is 0.0013.

The IU Cyclotron has also been used to study the nonlinear dynamics of a dissipative parametric resonance system.[11] The beam dynamics is observed

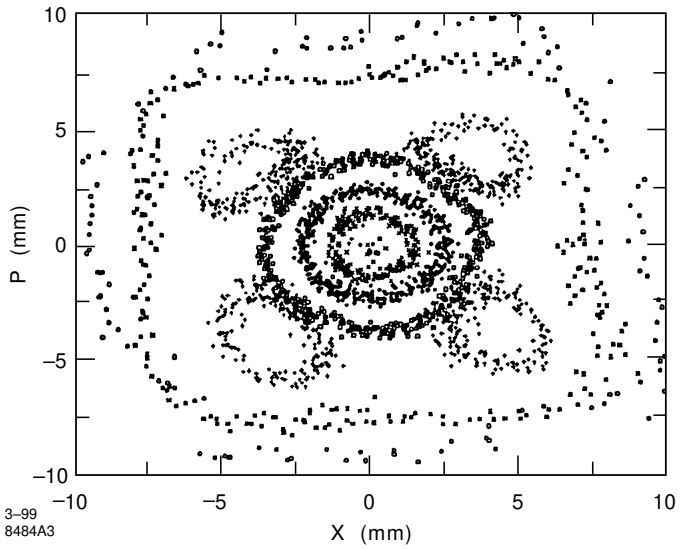

Figure 7. Measurements at the IUCF showing detailed phase space topology when $\nu_x \approx 15/4$.

in the longitudinal motion of the stored proton beam. The dissipation was provided by electron cooling. The parametric driving was done by kicking the beam transversely by a kicker whose strength is modulated at a frequency close to the longitudinal synchrotron frequency of the stored beam. The longitudinal beam distribution was found to split into two beamlets, each following one stable *attractor*. These observations were in agreement with expectation. Fig.8 shows the numerical simulation of what is expected. The complexity as well as the chaotic nature (especially outside the rf bucket) of the dynamics are clearly seen.

In the effort to compare detailed tracking simulation with the measured dynamic apertures in storage rings, it is often found important to include any *rippling* effects on the otherwise-static accelerator parameters even to very low levels. In particular, the beam lifetime (and dynamic aperture) is often found to depend sensitively on a small ripple in the tune. Long-term diffusion effects in the presence of intentional nonlinearities (eight sextupoles) and tune ripple were studied experimentally at SPS. Fig.9 shows some of the results. In the absence of tune ripple, the beam had a long lifetime. When a tune ripple with an amplitude of 1.65×10^{-3} and frequency of 9 Hz was introduced, the lifetime dropped to 7 hrs. When the same rippling amplitude was divided

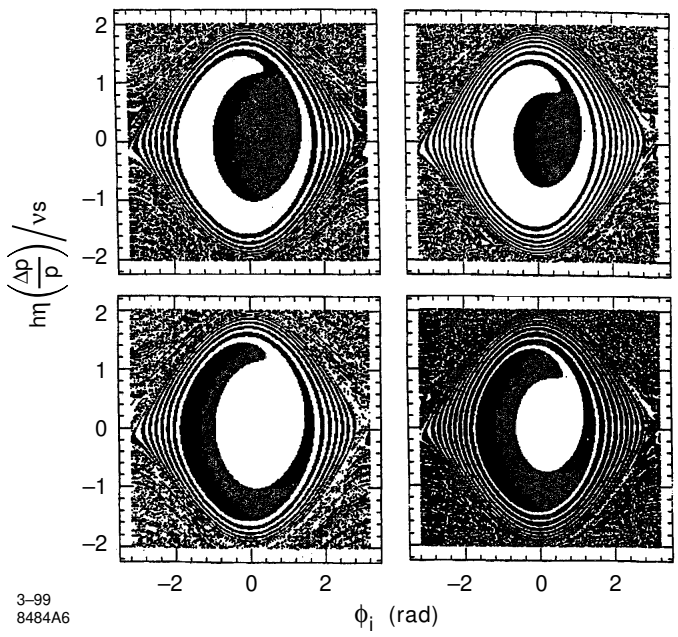

Figure 8. Black dots show the initial phase-space points which damp to either the inner (upper graphs) or outer (lower graphs) attractors at modulation frequencies 230 Hz (left frames) and 240 Hz (right frames) obtained from numerical simulations.

between two rippling frequencies at 9 Hz and 180 Hz, the lifetime dropped further to 2 hrs. The observations that a small tune ripple of 10^{-3} causes significant diffusion and that richness in tune rippling frequency enhances the diffusion agree qualitatively with simulation results.

8 Summary

1. Nonlinear dynamics and chaos effects play a very important role in accelerator physics. Choice of critical design parameters of the modern, high-performance accelerators depends on a clear understanding of the dynamics involved.

2. Once constructed, an accelerator can serve as a particularly powerful and versatile experimental tool to study various intricate aspects of nonlinear dynamics.

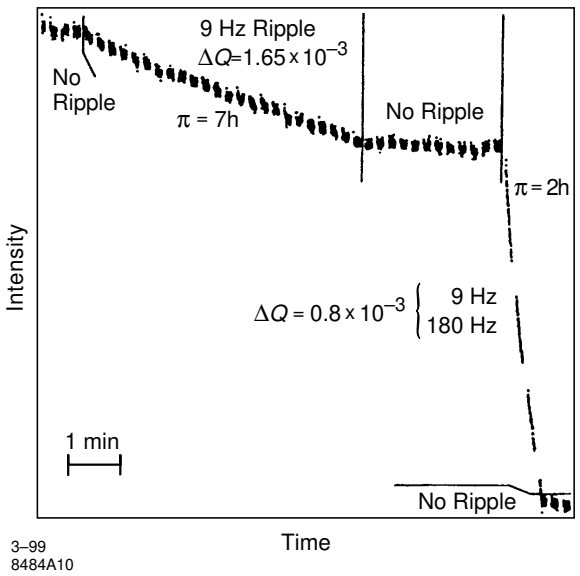

Figure 9. Diffusion measurements at the SPS in the presence of nonlinearities and tune ripple.

References

1. B.V. Chirikov, Report 267, Nucl. Phys. Inst. Siberian Sec. USSR Acad. Sci. Novosibirsk (1969); Trans. by A.T. Sanders, CERN Trans. 71-40 (1971).
2. A.W. Chao, AIP Proc. 127, Phys. of High Energy Part. Acc., BNL/SUNY, 1983, p.201.
3. V.I. Arnold, Rusian Math. Surveys 18, 85 (1964).
4. L.C. Teng, IEEE Trans. Nucl. Sci. NS-20, 843 (1973).
5. A.J. Dragt, AIP Proc. 87, Phys. of High Energy Part. Acc., Fermilab, 1981, p.147.
6. Y. Yan et al., SSCL Report 303 (1990).
7. G. Bourianoff and Y. Yan, private communication (1990).
8. T. Garavaglia et al., SSCL Report 268 (1990).
9. F. Gallucio and F. Schmidt, AIP Proc. 225, Workshop on Effects of Errors, Corpus Christi, 1991, p.86.
10. S.Y. Lee et al., Phys. Rev. Lett. 67, 3768 (1991).
11. Y. Wang et al., Phys. Rev. E 49, 1610 (1994).