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New Color Decompositions for Gauge Amplitudes at Tree and Loop Level

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Abstract

Recently, a color decomposition using structure constants was introduced for purely gluonic tree amplitudes, in a compact form involving only the linearly independent subamplitudes. We give two proofs that this decomposition holds for an arbitrary number of gluons. We also present and prove similar decompositions at one loop, both for pure gluon amplitudes and for amplitudes with an external quark-antiquark pair.

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1 Introduction

The computation of multi-parton scattering amplitudes in QCD is essential for quantitative predictions of multi-jet cross-sections at high-energy colliders. The Feynman diagrams for such an amplitude can generate a thicket of complicated color algebra, tangled together with expressions composed of kinematic invariants. An extremely useful way to disentangle the color and kinematic factors is via *color decompositions* [1, 2, 3, 4, 5] in terms of Chan-Paton factors [6], or traces of $SU(N_c)$ matrices in the fundamental representation, λ^a. The traces are multiplied by purely kinematical coefficients, called partial amplitudes or subamplitudes. Originally motivated by the representation of gluon amplitudes in open string theory, such decompositions have been widely studied and applied at both the tree level [4] and the loop level [7]. Two major advantages of the trace-based color decompositions are that (a) subamplitudes are gauge-invariant, and (b) an important 'color-ordered' class of them receive contributions only from diagrams with a particular cyclic ordering of the external partons; these subamplitudes therefore have much simpler kinematic properties than the full amplitude.

Although the standard color decompositions are very effective, in some cases they are not quite optimal. For example, the decomposition of the purely gluonic tree amplitude is overcomplete. Explicitly, the n-gluon amplitude is written in terms of $(n-1)!$ single-trace color structures, as

$$\mathcal{A}_n^{\rm tree}(g_1, g_2, \ldots, g_n) = g^{n-2} \sum_{\sigma \in S_{n-1}} {\rm tr}(\lambda^{a_1}\lambda^{a_{\sigma_2}} \cdots \lambda^{a_{\sigma_n}}) A_n^{\rm tree}(1, \sigma_2, \ldots, \sigma_n), \tag{1.1}$$

where $A_n^{\rm tree}$ are the subamplitudes. The cyclic invariance of the trace has been used to fix its first entry, and the sum is over the set of non-cyclic permutations of n elements, $\sigma \in S_{n-1} \equiv S_n/\mathbb{Z}_n$, where $\sigma(2, \ldots, n) = (\sigma_2, \ldots, \sigma_n)$. However, the $(n-1)!$ subamplitudes appearing in the equation are not all independent. Besides cyclic and reflection invariances, which they inherit from the traces, subamplitudes also obey a $U(1)$ decoupling identity (also called a dual Ward identity, or subcyclic identity) [2, 8],

$$A_n^{\rm tree}(1, 2, 3, \ldots, n) + A_n^{\rm tree}(1, 3, 2, \ldots, n) + \cdots + A_n^{\rm tree}(1, 3, \ldots, n, 2) = 0. \tag{1.2}$$

This identity can be derived by setting λ^{a_2} equal to the unit matrix and collecting terms containing the same trace; this gives the amplitude for a $U(1)$ 'photon' and $n-1$ $SU(N_c)$ gluons, which must vanish. More general identities can be derived by assigning the $SU(N_c)$ generators for the external gluons to commuting subgroups such as $SU(N_1) \times SU(N_2)$ [9]. Kleiss and Kuijf found a linear relation between subamplitudes [10] (see section 2), which is consistent with all of these identities, and which reduces the number of linearly independent subamplitudes from $(n-1)!$ to $(n-2)!$.

In contrast, the conventional color factors for gluonic Feynman diagrams are composed of structure constants f^{abc}. In this case, the $U(1)$ and generalized decoupling identities are all manifest; for example, the $U(1)$ generator commutes with all other generators, so all structure constants containing it are zero. On the other hand, a particular string of contracted structure constants will not appear *ab initio* with a gauge-invariant kinematic coefficient. To pass from the f^{abc}-based decomposition to the trace-based decomposition, one substitutes $f^{abc} = -i\,{\rm tr}\left(\lambda^a\lambda^b\lambda^c - \lambda^b\lambda^a\lambda^c\right)$ and

simplifies and collects the various traces. In the process, gauge invariance of the partial amplitudes is gained, but properties such as $U(1)$ decoupling are no longer manifest. Another disadvantage of trace-based decompositions at tree level is that the color structure of amplitudes in the multi-Regge kinematics is obscured [11].

The trace-based decomposition is also not optimal at the loop level. At one loop, the standard color decomposition [5] for n-gluon amplitudes includes double trace structures, $\mathrm{tr}(\lambda^{a_1}\cdots\lambda^{a_{c-1}}) \times \mathrm{tr}(\lambda^{a_c}\cdots\lambda^{a_n})$, in addition to single traces of the type that appear at tree level. The subamplitudes multiplying the single trace structures give the leading contributions in the large N_c limit, and they are color-ordered. The double-trace subamplitudes, corresponding to subleading-in-N_c contributions, can be written in terms of permutations of the leading (color-ordered) subamplitudes [12] in a formula reminiscent of the tree-level Kleiss-Kuijf relation (see section 3). Because the permutation formula is rather complicated, its implementation in a numerical program can be rather slow. Similar formulae hold for one-loop amplitudes with two external quarks and $n-2$ gluons [13].

Recently, a new color decomposition for the n-gluon tree amplitude, in terms of structure constants rather than traces, has been presented [14], whose kinematic coefficients are just color-ordered subamplitudes. Because of this property, the new decomposition retains all the advantages of trace-based decompositions, yet avoids the disadvantages mentioned above. In particular, the tree-level n-gluon decomposition is automatically given in terms of the $(n-2)!$ independent subamplitudes, in a form that is very convenient for analyzing the high-energy limit.

The purpose of this paper is to derive the new tree-level decomposition in two different ways, and to present similar f^{abc}-based color decompositions at one loop, whose kinematic coefficients are again color-ordered subamplitudes. The leading and subleading-in-N_c contributions combine neatly into one expression in the new decompositions. Also, gluons circulating in the loop are put on the same footing as fermions in the loop.

Parton-level cross-sections require amplitude squares or interferences which are summed over all external color labels. We provide general expressions for color-summed cross-sections in terms of the new color decompositions. In appendix A we give explicit evaluations for quantities encountered in cross-section computations through $\mathcal{O}(\alpha_s^4)$.

The paper is organized as follows. In section 2 we give two proofs of the new color decomposition for the n-gluon tree amplitude conjectured in ref. [14], and provide accordingly the square of the tree amplitude summed over colors. In section 3 we present (and prove) new color decompositions for one-loop n-gluon amplitudes and one-loop amplitudes with an external quark-antiquark pair plus $n-2$ gluons. In addition, we compute the color-summed interference terms between tree amplitudes and one-loop amplitudes, and the square of one-loop amplitudes, which are relevant respectively for next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) calculations of jet production rates. The color-summed interference terms and squares are given in terms of color matrices; appendix A provides many of those required for jet rate computations up to $\mathcal{O}(\alpha_s^4)$. Finally, we outline how a new color decomposition for one-loop amplitudes with external photons, gluons and a quark-antiquark pair can be obtained from the amplitudes with only gluons and a $q\bar{q}$ pair. In section 4 we summarize our present understanding of the color decomposition of tree-level

and one-loop amplitudes, and comment briefly on possible extensions to multi-loop amplitudes.

2 Tree Color Decomposition

The new color decomposition for the n-gluon tree amplitude is[1] [14]

$$\begin{aligned}\mathcal{A}_n^{\rm tree}(g_1,\ldots,g_n) &= (ig)^{n-2}\sum_{\sigma\in S_{n-2}} f^{a_1 a_{\sigma_2} x_1} f^{x_1 a_{\sigma_3} x_2}\cdots f^{x_{n-3} a_{\sigma_{n-1}} a_n} A_n^{\rm tree}(1,\sigma_2,\ldots,\sigma_{n-1},n)\,,\\ &= g^{n-2}\sum_{\sigma\in S_{n-2}} (F^{a_{\sigma_2}}\cdots F^{a_{\sigma_{n-1}}})_{a_1 a_n} A_n^{\rm tree}(1,\sigma_2,\ldots,\sigma_{n-1},n)\,, \qquad (2.1)\end{aligned}$$

where $(F^a)_{bc}\equiv if^{bac}$ is an $SU(N_c)$ generator in the adjoint representation. This color decomposition is analogous to the standard decomposition for the tree amplitude with a quark-antiquark pair and $n-2$ gluons [3, 4],

$$\mathcal{A}_n^{\rm tree}(q_1,g_2,\ldots,g_{n-1},\bar q_n) = g^{n-2}\sum_{\sigma\in S_{n-2}} (\lambda^{a_{\sigma_2}}\cdots\lambda^{a_{\sigma_{n-1}}})_{i_1}^{\ \bar i_n} A_n^{\rm tree}(1_q,\sigma_2,\ldots,\sigma_{n-1},n_{\bar q})\,. \qquad (2.2)$$

The only difference between the two is the representation used for the color matrices, namely the adjoint representation for the n-gluon amplitude and the fundamental representation for the amplitude containing a $\bar q q$ pair.

Eq. (2.1) is written in terms of the $(n-2)!$ subamplitudes where legs 1 and n are adjacent. This is precisely the basis of linearly independent subamplitudes which is singled out by the Kleiss-Kuijf relation. In fact, we shall show that eq. (2.1) is equivalent to the Kleiss-Kuijf relation [10], which can be written as,

$$A_n^{\rm tree}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta}\sum_{\sigma\in {\rm OP}\{\alpha\}\{\beta^T\}} A_n^{\rm tree}(1,\sigma(\{\alpha\}\{\beta^T\}),n). \qquad (2.3)$$

Here $\{\alpha\}\cup\{\beta\}=\{2,3,\ldots,n-1\}$, n_β is the number of elements in the set $\{\beta\}$, the set $\{\beta^T\}$ is $\{\beta\}$ with the ordering reversed, and ${\rm OP}\{\alpha\}\{\beta^T\}$ is the set of 'ordered permutations' (also called mergings [10]) of the $n-2$ elements of $\{\alpha\}\cup\{\beta^T\}$ that preserve the ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta^T\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i.

We wish to show that the new color decomposition (2.1) is equivalent to inserting the Kleiss-Kuijf relation (2.3) into the standard color decomposition (1.1). We first substitute for the structure constants appearing in eq. (2.1), $f^{abc}=-i\,{\rm tr}(\lambda^a[\lambda^b,\lambda^c])$, and use the identity

$$f^{a_1 a_2 x_1} f^{x_1 a_3 x_2}\cdots f^{x_{n-3} a_{n-1} a_n} = (-i)^{n-2}\,{\rm tr}\left(\lambda^{a_1}\ [\lambda^{a_2},[\lambda^{a_3},\ldots,[\lambda^{a_{n-1}},\lambda^{a_n}]\ldots]]\right). \qquad (2.4)$$

We want to identify all terms that contain a trace of the form

$${\rm tr}(\lambda^{a_1}\lambda^{a_{\alpha_1}}\cdots\lambda^{a_{\alpha_{n-2-q}}}\lambda^{a_n}\lambda^{a_{\beta_1}}\cdots\lambda^{a_{\beta_q}}), \qquad (2.5)$$

[1] We choose the normalization of the fundamental representation matrices as ${\rm tr}(\lambda^a\lambda^b)=\delta^{ab}$. Hence our structure constants f^{abc} are larger than the conventional ones by a factor of $\sqrt{2}$.

because these should give rise to $A_n^{\rm tree}(1,\{\alpha\},n,\{\beta\})$. Since there are $n-2$ commutators, there are 2^{n-2} terms on the right-hand side of eq. (2.4), but only $\binom{n-2}{q}$ of them have exactly q λ matrices appearing to the right of λ^{a_n}. The a_i indices on the matrices to the right of λ^{a_n} must come in reversed order compared to how they appear in the f^{abc} string, but they can appear in that string in any relative order with respect to the a_i that end up to the left of λ^{a_n}. Thus, for any ordered permutation $\sigma \in {\rm OP}\{\alpha\}\{\beta^T\}$, the subamplitude $A_n^{\rm tree}(1,\sigma(\{\alpha\}\{\beta^T\}),n)$ appears in the new decomposition (2.1) accompanied by the desired trace (2.5); a relative sign of $(-1)^q = (-1)^{n_\beta}$ comes from the commutators. Collecting all such ordered permutations, we obtain the Kleiss-Kuijf formula for $A_n^{\rm tree}(1,\{\alpha\},n,\{\beta\})$, eq. (2.3), thus establishing its equivalence to the new color decomposition (2.1).

Hence we can derive the new n-gluon color decomposition via its connection to the Kleiss-Kuijf relation. The latter relation was checked up to $n=8$ in ref. [10]. It can be proved for all n using the same techniques that were used to prove an analogous one-loop formula [12], eq. (3.2) below. Consider first the set of color-ordered Feynman diagrams with only three-point vertices, which are 'multi-peripheral' with respect to 1 and n, by which we mean that all other external legs connect directly to the line extending from 1 to n; i.e. there are no non-trivial trees branching off of this line. Label the legs on one side of the 1–n line by α_i, and those on the other side by β_j. These diagrams contribute to both sides of the Kleiss-Kuijf relation (2.3) in the proper way: There is no relative ordering requirement on the α_i with respect to the β_j on the left-hand side of the relation, and the ordering is summed over on the right-hand side; the sign factor $(-1)^{n_\beta}$ comes from the antisymmetry of color-ordered three-point vertices (in Feynman gauge).

Next consider diagrams with non-trivial trees attached to the 1–n line. They work in exactly the same way as the multi-peripheral diagrams if the leaves (external legs) of each tree all belong to the same set, either $\{\alpha\}$ or $\{\beta\}$. However, if a tree contains leaves from both sets, then the diagram does not appear on the left-hand side of eq. (2.3), so one must show that it cancels out of the permutation sum on the right-hand side. For trees with only three-point vertices, this can be done using the antisymmetry of the vertices. For other cases the cancellations are slightly more complicated; one way of establishing them is via the generalized decoupling identities [9] mentioned in the introduction [12].

Figure 1: Graphical representation of a multi-peripheral color factor. A vertex stands for f^{abc}, and a bond for δ^{ab}.

There is another way to prove eq. (2.1). For this argument it is convenient to use a graphical notation for color factors made out of structure constants, in which f^{abc} is represented by a three-vertex and δ^{ab} (an index contraction) is represented by a line [1]. Then the color factors appearing

in eq. (2.1) are associated with the multi-peripheral color diagrams shown in fig. 1. The color factor for a generic n-gluon Feynman diagram is not of this form. However, we can use the Jacobi identity

$$f^{dac}f^{cbe} - f^{dbc}f^{cae} = f^{abc}f^{dce}, \tag{2.6}$$

shown graphically in fig. 2, to put it into this form [1]. Fig. 3 represents the color factor for a generic Feynman diagram graphically, as a tree structure with only three-point vertices. It also shows how it can be simplified into multi-peripheral form by repeated application of the Jacobi identity. (If the line running from 1 to n is in the fundamental representation, then the same steps, but using $\lambda^a\lambda^b - \lambda^b\lambda^a = if^{abc}\lambda^c$, lead directly to the color decomposition (2.2) for a quark-antiquark pair and $n-2$ gluons.)

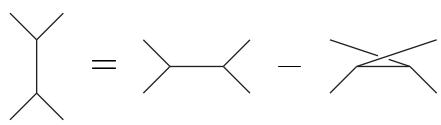

Figure 2: The Jacobi identity for structure constants in graphical notation.

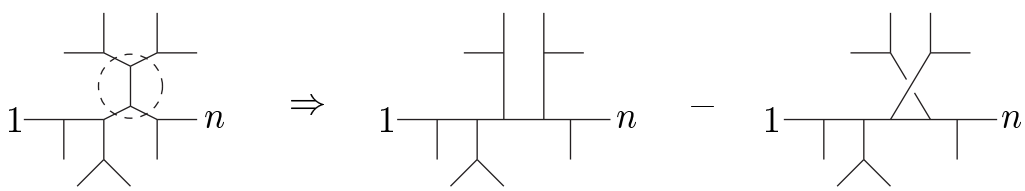

Figure 3: Graphical representation of the color factor for a generic tree-level Feynman diagram. Also shown is a step in its conversion to multi-peripheral form, by using the Jacobi identity in the dashed region.

So far we have established that a color decomposition of the form (2.1) exists, but we have not yet shown that the kinematic coefficients are equal to the partial amplitudes $A_n^{\rm tree}(1,\sigma_2,\ldots,\sigma_{n-1},n)$. However, we can do this rather simply by equating the new decomposition (but with unknown kinematic coefficients) to the standard one, eq. (1.1), and contracting both sides with $\mathrm{tr}(\lambda^{a_1}\lambda^{a_{\sigma_2}}\cdots\lambda^{a_{\sigma_{n-1}}}\lambda^{a_n})$, i.e. summing over all a_i, $i=1,\ldots,n$, for some permutation $\sigma\in S_{n-2}$. Because the kinematic coefficients contain no N_c dependence, we may retain only the leading contraction terms as $N_c\to\infty$. But it is well known that single traces for different cyclic orderings are orthogonal at large N_c [4]. Similarly, fig. 4 shows that only one of the multi-peripheral color factors in eq. (2.1) survives the contraction, because the other S_{n-2} permutations give rise to nonplanar, and hence N_c-suppressed, color topologies. Thus the contraction selects a unique term from either side of the equation, corresponding to the same ordering of the color indices a_i, and with the same weight at large N_c, namely N_c^n. This proves that the coefficients in the new color decomposition (2.1) are indeed the subamplitudes $A_n^{\rm tree}(1,\sigma_2,\ldots,\sigma_{n-1},n)$.

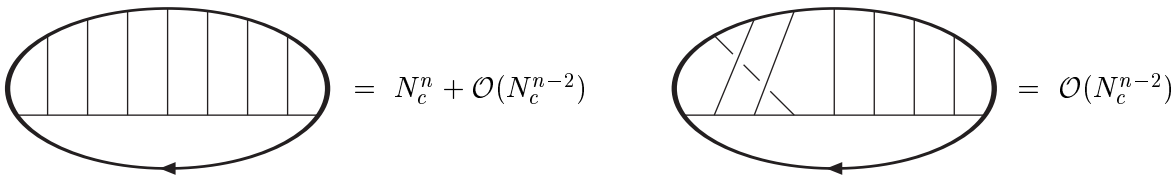

Figure 4: Contraction of different multi-peripheral color factors with a single λ trace. The heavy line with an arrow denotes the fundamental representation. The left diagram shows the contraction where a_i, $i = 2, 3, \ldots, n-1$, appear in the same order in the λ trace as in the multi-peripheral color factor. Every other contraction gives rise to a nonplanar (and hence color-suppressed) diagram of the type shown on the right.

Because we showed that the new color decomposition is equivalent to the Kleiss-Kuijf relation, our second proof of the new decomposition can alternatively be viewed as a proof of the Kleiss-Kuijf relation. Interestingly, this proof does not rely on any kinematic properties of color-ordered Feynman diagrams; only group-theoretic properties such as the Jacobi identity and the large-N_c behavior of color traces were used.

The decomposition (2.1) has a color-ladder structure which naturally arises in the configurations where the gluons are strongly ordered in rapidity, i.e. in the multi-Regge kinematics [15]. (Indeed, this is how eq. (2.1) was first discovered [14].) In contrast, some laborious work is necessary to obtain the color-ladder structure from the trace-based decomposition (1.1) in the multi-Regge kinematics [11]. Let gluons $2, \ldots, n-1$ be the particles produced in a scattering where gluons 1 and n are the incoming particles. The available final-state phase space may be divided into $(n-2)!$ simplices, in each of which the produced gluons are ordered in rapidity. Let us take the simplex defined by the rapidity ordering $y_2 > y_3 > \cdots > y_{n-2} > y_{n-1}$, and consider its sub-simplex with strong rapidity ordering $y_2 \gg y_3 \gg \cdots \gg y_{n-2} \gg y_{n-1}$. Then the crucial point to note is that the dominant subamplitudes in eq. (1.1) in the multi-Regge limit are $A_n^{\text{tree}}(1, \{\alpha\}, n, \{\beta^T\})$, where $\{\alpha\}$ and $\{\beta\}$ are both increasing sequences, whose union is $\{2, 3, \ldots, n-1\}$ ($\{\beta^T\}$ is $\{\beta\}$ in reversed order). In other words, OP$\{\alpha\}\{\beta\}$ contains the sequence $\{2, 3, \ldots, n-1\}$. If q is the number of elements in $\{\beta\}$, then for each q there are $\binom{n-2}{q}$ such choices of $\{\alpha\}$, $\{\beta\}$. Summing over $q = 0, 1, 2, \ldots, n-2$, there are 2^{n-2} orderings in total [16]. Using the identity (2.4) and the fact that in the multi-Regge kinematics

$$A_n^{\text{tree}}(1, \{\alpha\}, n, \{\beta^T\}) = (-1)^q A_n^{\text{tree}}(1, 2, \ldots, n), \tag{2.7}$$

one obtains the color-ladder structure of eq. (2.1), with just one allowed string of structure constants f^{abc}. The procedure can be repeated for all of the $(n-2)!$ simplices, thus generating the $(n-2)!$ strings of f^{abc}'s of eq. (2.1).

The tree-level partonic scattering cross-section is given by the square of the tree amplitude $\mathcal{A}_n^{\text{tree}}$, summed over all colors. This expression can be written either of two ways,

$$\sum_{\text{colors}} |\mathcal{A}_n^{\text{tree}}(1, \ldots, n)|^2 \quad = \quad (g^2)^{n-2} \sum_{i,j=1}^{(n-1)!} c_{ij} A_i^{\text{tree}} (A_j^{\text{tree}})^* \tag{2.8}$$

$$= \quad (g^2)^{n-2} \sum_{i,j=1}^{(n-2)!} \tilde{c}_{ij} A_i^{\rm tree} (A_j^{\rm tree})^* \,, \tag{2.9}$$

where eq. (2.8) is obtained from eq. (1.1), while eq. (2.9) is obtained from eq. (2.1), and the subscript i on $A_i^{\rm tree}$ now refers to the subamplitude $A_n^{\rm tree}$ evaluated for the $i^{\rm th}$ permutation P_i in S_{n-1} or S_{n-2}, respectively. In eq. (2.8), the color matrix c_{ij} is

$$c_{ij} = \sum_{\rm colors} {\rm tr}(\lambda^{a_1} P_i\{\lambda^{a_2} \cdots \lambda^{a_n}\}) \, [{\rm tr}(\lambda^{a_1} P_j\{\lambda^{a_2} \cdots \lambda^{a_n}\})]^* \,, \tag{2.10}$$

whereas in eq. (2.9) the matrix is

$$\tilde{c}_{ij} = \sum_{\rm colors} (P_i\{F^{a_2} \cdots F^{a_{n-1}}\})_{a_1 a_n} \left[(P_j\{F^{a_2} \cdots F^{a_{n-1}}\})_{a_1 a_n} \right]^* . \tag{2.11}$$

In refs. [10, 17], the Kleiss-Kuijf relation was used to calculate $\tilde{c}_{ij}$ from c_{ij}. From eq. (2.11) we see that it can be calculated directly in terms of structure constants. In appendix A we give $\tilde{c}_{ij}$ for $n = 4, 5, 6$.

Although the expression (2.9) seems to have fewer terms than eq. (2.8), the sparseness of the matrices is also important for determining which expression has the fastest numerical evaluation [17]. For an evaluation that is good only to leading order in N_c — the Leading Color Approximation, or LCA — it is best to use eq. (2.8) and the large-N_c orthogonality of different single traces, to obtain

$$\sum_{\rm colors} |\mathcal{A}_n^{\rm tree}(1, \ldots, n)|^2 \quad = \quad (g^2)^{n-2} \mathcal{C}_n(N_c) \sum_{\sigma \in S_{n-1}} \left[|A^{\rm tree}(1, \sigma_2, \ldots, \sigma_n)|^2 + \mathcal{O}\left(\frac{1}{N_c^2}\right) \right] , \tag{2.12}$$

where

$$\mathcal{C}_n(N_c) = N_c^{n-2}(N_c^2 - 1) \,. \tag{2.13}$$

Up to $n = 5$, the LCA is exact, and so eq. (2.8) is also superior to eq. (2.9) for an exact evaluation. For $n \geq 6$, where the LCA has subleading corrections, it is best to take the leading terms from eq. (2.8), and the subleading terms from eq. (2.9) (these are the terms proportional to a in eq. (A.3)) [17]. For $n = 7$, ref. [17] employed the linear dependences in the Kleiss-Kuijf relation to find an even more compact form for the subleading terms than using eq. (2.9). (See ref. [18] for another approach to numerically evaluating multi-parton amplitudes beyond the LCA.)

3 One-Loop Color Decompositions

The standard color decomposition for one-loop n-gluon amplitudes in $SU(N_c)$ gauge theory with n_f flavors of quarks is [5]

$$\mathcal{A}_n^{\text{1-loop}} = g^n \quad \Bigg[\quad N_c \sum_{\sigma \in S_n/\mathbb{Z}_n} {\rm tr}(\lambda^{a_{\sigma_1}} \cdots \lambda^{a_{\sigma_n}}) A_{n;1}^{[1]}(\sigma_1, \ldots, \sigma_n)$$
$$+ \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} {\rm tr}(\lambda^{a_{\sigma_1}} \cdots \lambda^{a_{\sigma_{c-1}}}) \, {\rm tr}(\lambda^{a_{\sigma_c}} \cdots \lambda^{a_{\sigma_n}}) A_{n;c}(\sigma_1, \ldots, \sigma_n)$$

$$+ \quad n_f \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{tr}(\lambda^{a_{\sigma_1}} \cdots \lambda^{a_{\sigma_n}}) A^{[1/2]}_{n;1}(\sigma_1, \ldots, \sigma_n) \Big] , \tag{3.1}$$

where $A_{n;c}$ are the subamplitudes, $S_{n;c}$ is the subset of S_n that leaves the double trace structure invariant, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. The superscript $[j]$ denotes the spin of the particle circulating in the loop. The subamplitudes $A^{[j]}_{n;1}$ are color-ordered, and give rise to the leading contributions to the cross-section in the large-N_c limit.

The subleading subamplitudes $A_{n;c>1}$ can be obtained from the leading ones $A^{[1]}_{n;1}$ through the permutation sum [12]

$$A_{n;c>1}(1, 2, \ldots, c-1; c, c+1, \ldots, n) \;=\; (-1)^{c-1} \sum_{\sigma \in \mathrm{COP}\{\alpha\}\{\beta\}} A^{[1]}_{n;1}(\sigma_1, \ldots, \sigma_n) , \tag{3.2}$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1, c-2, \ldots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{c, c+1, \ldots, n-1, n\}$, and $\mathrm{COP}\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1, 2, \ldots, n\}$ with n held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i.

In a dual representation of one-loop amplitudes, the subleading subamplitudes $A_{n;c>1}(\{\alpha^T\}; \{\beta\})$ come from the annulus diagram where gluons belonging to $\{\alpha\}$ and $\{\beta\}$ are emitted from the inner and outer boundaries of the annulus, respectively. Thus their color properties are very similar to those of the tree subamplitudes $A^{\mathrm{tree}}_n(1, \{\alpha\}, n, \{\beta\})$ if legs 1 and n are 'sewn together', i.e. contracted with $\delta^{a_1 a_n}$. Indeed, with this interpretation, eq. (3.2) has a very similar structure to the Kleiss-Kuijf relation (2.3). These remarks suggest that the new tree-level color decomposition should have a one-loop analog, where the strings of structure constants that appear are 'ring diagrams' instead of multi-peripheral diagrams, as depicted in fig. 5a.

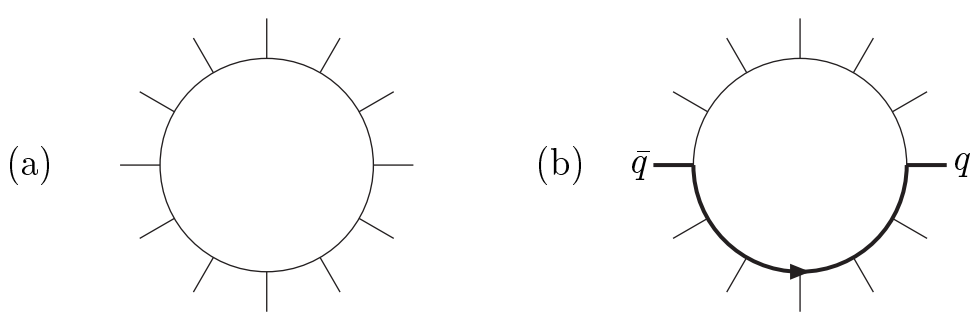

Figure 5: (a) Ring color factors for one-loop n-gluon amplitudes. (b) The corresponding color factors for one-loop amplitudes with an external quark-antiquark pair.

The new one-loop n-gluon color decomposition, expressed in terms of adjoint generator matrices $(F^a)_{bc}$, is

$$\mathcal{A}^{\text{1-loop}}_n = g^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \Big[\mathrm{tr}(F^{a_{\sigma_1}} \cdots F^{a_{\sigma_n}}) A^{[1]}_{n;1}(\sigma_1, \ldots, \sigma_n) + 2 n_f \, \mathrm{tr}(\lambda^{a_{\sigma_1}} \cdots \lambda^{a_{\sigma_n}}) A^{[1/2]}_{n;1}(\sigma_1, \ldots, \sigma_n) \Big] . \tag{3.3}$$

Here $S_{n-1} \equiv S_n/\mathbb{Z}_n$ is the group of non-cyclic permutations, and $\mathcal{R}$ is the reflection: $\mathcal{R}(1,2,\ldots,n) = (n,\ldots,2,1)$. Thus the number of linearly independent subamplitudes is $(n-1)!/2$.

The terms proportional to n_f in eq. (3.3) follow from eq. (3.1) using only the reflection identity satisfied by the $A_{n;1}$: $A^{[j]}_{n;1}(n,\ldots,2,1) = (-1)^n\, A^{[j]}_{n;1}(1,2,\ldots,n)$. To prove that the remaining terms in eq. (3.3) are correct, we follow closely the second proof of eq. (2.1). Using the Jacobi identity, the generic one-loop f^{abc} color factor can be turned into ring diagrams. The manipulations are exactly the same as those shown in fig. 3, once legs 1 and n are joined together to form a loop. This establishes eq. (3.3), but with an unknown coefficient for $\mathrm{tr}(F^{a_{\sigma_1}}\cdots F^{a_{\sigma_n}})$. Next we equate this expression to the standard color decomposition (3.1), and contract both sides with $\mathrm{tr}(\lambda^{a_{\sigma_1}}\cdots\lambda^{a_{\sigma_n}})$. Again a unique term survives on each side of the equation, at leading order in N_c, and this establishes that the coefficient is correctly given by $A^{[1]}_{n;1}(\sigma_1,\ldots,\sigma_n)$. (If we had not removed the reflections $\mathcal{R}$ from eq. (3.3), then both a ring diagram and its reflection would have contributed to the contraction; this would have led to a factor of $\frac{1}{2}$ in the coefficient of $\mathrm{tr}(F^{a_{\sigma_1}}\cdots F^{a_{\sigma_n}})$.)

Eq. (3.3) puts the gluon and fermion loops manifestly on the same footing. The contributions of the subleading-color subamplitudes $A_{n;c>1}$ are neatly packaged into the ring-diagram color structures. In addition, all of the generalized $U(1)$ decoupling identities are automatically incorporated. Finally, eq. (3.3) shows explicitly the difference between the tree-level and the one-loop color decomposition of the gluon sector for any number of external legs. For instance, if we take the amplitude for a Higgs boson plus three gluons, both eq. (1.1) and the gluon-loop contribution to eq. (3.1) have as a color factor $\mathrm{tr}(\lambda^{a_{\sigma_1}}\lambda^{a_{\sigma_2}}\lambda^{a_{\sigma_3}})$; the actual difference in the color structure is concealed in the different properties of the subamplitudes A^{tree}_3 and $A^{[1]}_{3;1}$. In contrast, eq. (2.1) and eq. (3.3) show explicitly the difference in color structure.

One-loop amplitudes contribute to NLO QCD cross-sections through their interference with tree amplitudes. Carrying out the color-summed NLO interference terms for amplitudes color-decomposed as in eqs. (2.1) and (3.3) is straightforward:

$$\begin{aligned}
&\sum_{\text{colors}} \mathcal{A}(1,\ldots,n)[\mathcal{A}(1,\ldots,n)]^*|_{\mathrm{NLO}} \\
&\quad = 2\sum_{\text{colors}} \mathrm{Re}\left(\mathcal{A}^{\mathrm{tree}}(\mathcal{A}^{\text{1-loop}})^*\right) \\
&\quad = 2(g^2)^{n-1}\,\mathrm{Re}\sum_{i=1}^{(n-2)!}\sum_{j=1}^{(n-1)!/2} A^{\mathrm{tree}}_i\left[\hat{c}_{ij}(A^{[1]}_j)^* + 2\,n_f\,\hat{d}_{ij}(A^{[1/2]}_j)^*\right],
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\hat{c}_{ij} &= \sum_{\text{colors}} (P_i\{F^{a_2}\cdots F^{a_{n-1}}\})_{a_1a_n}\left[\mathrm{tr}\left(F^{a_1}P_j\{F^{a_2}\cdots F^{a_n}\}\right)\right]^*, \\
\hat{d}_{ij} &= \sum_{\text{colors}} (P_i\{F^{a_2}\cdots F^{a_{n-1}}\})_{a_1a_n}\left[\mathrm{tr}(\lambda^{a_1}P_j\{\lambda^{a_2}\cdots\lambda^{a_n}\})\right]^*,
\end{aligned} \tag{3.5}$$

with P_i the ith permutation in S_{n-2} and P_j the jth permutation in $S_{n-1}/\mathcal{R}$.

NNLO production rates include the virtual contributions,

$$\sum_{\text{colors}} \mathcal{A}(1,\ldots,n)[\mathcal{A}(1,\ldots,n)]^*|_{\mathrm{NNLO}} = \sum_{\text{colors}}\left[2\,\mathrm{Re}\left(\mathcal{A}^{\mathrm{tree}}(\mathcal{A}^{\text{2-loop}})^*\right) + |\mathcal{A}^{\text{1-loop}}|^2\right]. \tag{3.6}$$

For the square of one-loop n-gluon amplitudes on the right-hand side of eq. (3.6), we obtain

$$\sum_{\text{colors}} |\mathcal{A}^{\text{1-loop}}|^2 = (g^2)^n \sum_{i,j=1}^{(n-1)!/2} \Bigg[\bar{c}_{ij} A_i^{[1]} (A_j^{[1]})^* + 4\, n_f \,\mathrm{Re}\, \Big[\bar{d}_{ij} A_i^{[1]} (A_j^{[1/2]})^* \Big] + 4\, n_f^2\, c_{ij} A_i^{[1/2]} (A_j^{[1/2]})^* \Bigg], \tag{3.7}$$

where

$$\begin{aligned} \bar{c}_{ij} &= \sum_{\text{colors}} \mathrm{tr}\,(F^{a_1} P_i \{F^{a_2} \cdots F^{a_n}\})\, [\mathrm{tr}\,(F^{a_1} P_j \{F^{a_2} \cdots F^{a_n}\})]^*, \\ \bar{d}_{ij} &= \sum_{\text{colors}} \mathrm{tr}\,(F^{a_1} P_i \{F^{a_2} \cdots F^{a_n}\})\, [\mathrm{tr}(\lambda^{a_1} P_j \{\lambda^{a_2} \cdots \lambda^{a_n}\})]^*, \end{aligned} \tag{3.8}$$

c_{ij} is given in eq. (2.10), and P_i is the ith permutation in $S_{n-1}/\mathcal{R}$. In appendix A we give the explicit values for the color matrices $\hat{c}_{ij}$, $\hat{d}_{ij}$, $\bar{c}_{ij}$, $\bar{d}_{ij}$ and c_{ij} that are required for cross-section computations up to $\mathcal{O}(\alpha_s^4)$.

A similar analysis can be applied to one-loop amplitudes with an external quark-antiquark pair plus $n-2$ gluons. These amplitudes have the standard color decomposition [13],

$$\mathcal{A}_n^{\text{1-loop}}(\bar{q}_1, q_2, g_3, \ldots, g_n) = g^n \sum_{j=1}^{n-1} \sum_{\sigma \in S_{n-2}/S_{n;j}} \mathrm{Gr}_{n;j}^{(\bar{q}q)}(\sigma_3, \ldots, \sigma_n) A_{n;j}(1_{\bar{q}}, 2_q; \sigma_3, \ldots, \sigma_n), \tag{3.9}$$

where the color structures $\mathrm{Gr}_{n;j}^{(\bar{q}q)}$ are defined by

$$\begin{aligned} \mathrm{Gr}_{n;1}^{(\bar{q}q)}(3, \ldots, n) &= N_c (\lambda^{a_3} \cdots \lambda^{a_n})_{i_2}^{\bar{i}_1}, \\ \mathrm{Gr}_{n;2}^{(\bar{q}q)}(3; 4, \ldots, n) &= 0, \\ \mathrm{Gr}_{n;j}^{(\bar{q}q)}(3, \ldots, j+1; j+2, \ldots, n) &= \mathrm{tr}(\lambda^{a_3} \cdots \lambda^{a_{j+1}})(\lambda^{a_{j+2}} \cdots \lambda^{a_n})_{i_2}^{\bar{i}_1}, \qquad j = 3, \ldots, n-2, \\ \mathrm{Gr}_{n;n-1}^{(\bar{q}q)}(3, \ldots, n) &= \mathrm{tr}(\lambda^{a_3} \cdots \lambda^{a_n})\, \delta_{i_2}^{\bar{i}_1}, \end{aligned} \tag{3.10}$$

and $S_{n;j} = \mathbb{Z}_{j-1}$ is the subgroup of S_{n-2} that leaves $\mathrm{Gr}_{n;j}^{(\bar{q}q)}$ invariant.

The leading-color subamplitudes in eq. (3.9) can be expressed in terms of color-ordered ('primitive') amplitudes as

$$\mathcal{A}_{n;1}(1_{\bar{q}}, 2_q, 3, \ldots, n) = A_n^{L,[1]}(1_{\bar{q}}, 2_q, 3, \ldots, n) - \frac{1}{N_c^2} A_n^{R,[1]}(1_{\bar{q}}, 2_q, 3, \ldots, n) + \frac{n_f}{N_c} A_n^{L,[1/2]}(1_{\bar{q}}, 2_q, 3, \ldots, n), \tag{3.11}$$

whereas the subleading contributions are given by the permutation formula [13],

$$\begin{aligned} &A_{n;j}(1_{\bar{q}}, 2_q; 3, \ldots, j+1; j+2, j+3, \ldots, n) \\ &\quad = (-1)^{j-1} \sum_{\sigma \in \mathrm{COP}\{\alpha\}\{\beta\}} \Big[A_n^{L,[1]}(\sigma(1_{\bar{q}}, 2_q, 3, \ldots, n)) - \frac{n_f}{N_c} A_n^{R,[1/2]}(\sigma(1_{\bar{q}}, 2_q, 3, \ldots, n)) \Big], \end{aligned} \tag{3.12}$$

where $\{\alpha\} = \{j+1, j, \ldots, 4, 3\}$, $\{\beta\} = \{1, 2, j+2, j+3, \ldots, n-1, n\}$, with $1_{\bar{q}}$ held fixed. In eqs. (3.11) and (3.12), the quantities $A_n^{L,[j]}$ and $A_n^{R,[j]}$ are defined in terms of color-ordered Feynman diagrams, much like the n-gluon subamplitudes $A_{n;1}^{[j]}$. However, because an external $\bar{q}q$ pair is present now, one also has to specify which way the line running from the antiquark to the quark turns when it enters the loop, left or right; this accounts for the additional index, L or R. Again the index $[j]$ denotes the spin j of a particle circulating around the loop, for those graphs in the primitive amplitude where the external $\bar{q}q$ line does not enter the loop. The reflection identity,

$$A_n^{R,[j]}(1_{\bar{q}}, 3, 4, \ldots, 2_q, \ldots, n-1, n) = (-1)^n \, A_n^{L,[j]}(1_{\bar{q}}, n, n-1, \ldots, 2_q, \ldots, 4, 3), \tag{3.13}$$

relates the left and right subamplitudes.

There are two types of subleading-color contributions in eq. (3.12), distinguished by whether they contain a factor of n_f or not. Although both types are generated by cyclicly ordered permutations, their origins are quite different. In particular, the subleading n_f terms arise from the $1/N_c$ term in the $SU(N_c)$ Fierz identity,

$$\sum_a (X_1 \lambda^a X_2)(Y_1 \lambda^a Y_2) = (X_1 Y_2)(Y_1 X_2) - \frac{1}{N_c}(X_1 X_2)(Y_1 Y_2), \tag{3.14}$$

where X_i, Y_i are strings of generator matrices λ^{a_i}. Such manifest $1/N_c$ terms are not generated by strings of structure constants. (In fact, $1/N_c$ terms are generally suppressed when f^{abc}s are present, because $1/N_c$ factors come from explicitly projecting out the $U(1)$ factor in $U(N_c) = SU(N_c) \times U(1)$, while structure constants accomplish this projection automatically.) Because of this fact, the freedom to include structure constants in a color decomposition does not seem to lead to any simpler representation of the subleading n_f terms in eq. (3.12), and we shall be content to simplify the no-n_f terms[2].

The new color decomposition for one-loop amplitudes with an external $\bar{q}q$ pair is

$$\begin{aligned} \mathcal{A}_n^{1\text{-loop}}(\bar{q}_1, q_2, g_3, \ldots, g_n) &= g^n \Bigg[\sum_{p=2}^{n} \sum_{\sigma \in S_{n-2}} (\lambda^{x_2} \lambda^{a_{\sigma_3}} \cdots \lambda^{a_{\sigma_p}} \lambda^{x_1})_{i_2}^{\bar{i}_1} (F^{a_{\sigma_{p+1}}} \cdots F^{a_{\sigma_n}})_{x_1 x_2} \\ &\qquad \times A_n^{R,[1]}(1_{\bar{q}}, \sigma_{p+1}, \ldots, \sigma_n, 2_q, \sigma_3, \ldots, \sigma_p) \qquad (3.15) \\ &+ \frac{n_f}{N_c} \sum_{j=1}^{n-1} \sum_{\sigma \in S_{n-2}/S_{n;j}} \mathrm{Gr}_{n;j}^{(\bar{q}q)}(\sigma_3, \ldots, \sigma_n) A_{n;j}^{[1/2]}(1_{\bar{q}}, 2_q; \sigma_3, \ldots, \sigma_n) \Bigg], \end{aligned}$$

where for $p = n$ the product of generators in the adjoint representation reduces to the identity, $(F \cdots F)_{x_1 x_2} \to \delta_{x_1 x_2}$; the color structures $\mathrm{Gr}_{n;j}^{(\bar{q}q)}$ are defined in eq. (3.10); and

$$\begin{aligned} A_{n;1}^{[1/2]}(1_{\bar{q}}, 2_q, 3, \ldots, n) &= A_n^{L,[1/2]}(1_{\bar{q}}, 2_q, 3, \ldots, n), \qquad (3.16) \\ A_{n;j}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \ldots, j+1; j+2, j+3, \ldots, n) &= (-1)^j \sum_{\sigma \in \mathrm{COP}\{\alpha\}\{\beta\}} A_n^{R,[1/2]}(\sigma(1_{\bar{q}}, 2_q, 3, \ldots, n)), \end{aligned}$$

[2]Since the n_f terms in the $A_{n;j>1}$ subamplitudes arise from projecting out a $U(1)$ factor, they can also be viewed as subamplitudes for a fictitious $SU(N_1) \times SU(N_2) \times U(1)$ theory containing two fermion representations, say $(\mathbf{N_1}, \mathbf{1})_{+1}$ (the external $\bar{q}q$ pair) and $(\mathbf{1}, \mathbf{N_2})_{+1}$ (the fermion in the loop). This description makes it a bit more transparent how the color structure $\mathrm{tr}(\lambda^{a_3} \cdots \lambda^{a_{j+1}}) \times (\lambda^{a_{j+2}} \cdots \lambda^{a_n})_{i_2}^{\bar{i}_1}$ multiplying these subamplitudes corresponds to their definition. But it does not lead to a distinct color decomposition.

with $\{\alpha\}$ and $\{\beta\}$ as in eq. (3.12). The new types of color structures appearing in eq. (3.15) are shown in fig. 5b.

Most of the subamplitudes $A_n^{R,[1]}$ do not contribute at leading order in the large N_c limit. (Only those of the form $A_n^{R,[1]}(1_{\bar{q}},\ldots,2_q)$ do.) Therefore the technique of contracting with a suitable color structure at large N_c cannot be used to prove eq. (3.15). Instead we follow the first line of reasoning that we used at tree-level; that is, we substitute for the structure constants in terms of λ matrices and collect terms. The result agrees with eq. (3.9), after substituting in eqs. (3.11) and (3.12) for the leading and subleading subamplitudes and making use of the reflection identity (3.13).

The contribution of one-loop amplitudes with an external quark-antiquark pair to NLO QCD cross sections parallels the n-gluon case, eq. (3.4). Carrying out the color-summed NLO interference terms for amplitudes color-decomposed as in eqs. (2.2) and (3.15), and omitting the n_f terms, we obtain,

$$\begin{aligned}
&\sum_{\text{colors}} \mathcal{A}(\bar{q}_1,q_2,g_3,\ldots,g_n)_{\text{no}-n_f}[\mathcal{A}(\bar{q}_1,q_2,g_3,\ldots,g_n)_{\text{no}-n_f}]^*|_{\text{NLO}} \\
&\quad = 2\sum_{\text{colors}} \text{Re}\left(\mathcal{A}^{\text{tree}}(\mathcal{A}^{\text{1-loop}})^*_{\text{no}-n_f}\right) \qquad (3.17)\\
&\quad = 2(g^2)^{n-1}\,\text{Re}\sum_{p=2}^{n}\sum_{i,j=1}^{(n-2)!}\hat{m}_{ij}^{(p)}A_i^{\text{tree}}(1_{\bar{q}},2_q,i_3,\ldots,i_n)\left[A_j^{R,[1]}(1_{\bar{q}},j_{p+1},\ldots,j_n,2_q,j_3,\ldots,j_p)\right]^*,
\end{aligned}$$

where

$$\hat{m}_{ij}^{(p)} = \sum_{\text{colors}} (P_i\{\lambda^{a_3}\cdots\lambda^{a_n}\})_{i_2}^{\bar{i}_1}\left[P_j\{(\lambda^{x_2}\lambda^{a_3}\cdots\lambda^{a_p}\lambda^{x_1})_{i_2}^{\bar{i}_1}(F^{a_{p+1}}\cdots F^{a_n})_{x_1x_2}\}\right]^\dagger, \qquad (3.18)$$

with P_i the ith permutation in S_{n-2}, and P_j the jth permutation in S_{n-2} at fixed p. The partitions p span a matrix-valued vector $\hat{m}_{ij}^{(p)}$, which we give in appendix A for $n=4,5$, as required for cross-section computations up to $\mathcal{O}(\alpha_s^4)$. For $n=5$, we have checked that eq. (3.17), with $\hat{m}_{ij}^{(p)}$ given in eq. (A.16), agrees with the no-n_f terms of ref. [13]. The n_f terms for $n=4,5$ have been computed in refs. [19] and [13], respectively.

Finally, we note that by converting gluons into photons we can obtain multi-photon amplitudes at tree and loop level. In order to convert a gluon into a photon we make use of the embedding of $U(1)$ in $U(N_c)$. We replace the quark-gluon vertex factor $g\lambda^a$ with the quark-photon vertex factor $\sqrt{2}\,eQ_q\,I$, where Q_q is the quark electric charge and the factor $\sqrt{2}$ is due to our choice of normalization of the λ matrices. Since the identity matrix I commutes with the λ matrices, all possible attachments of the photon to the quark line contribute to the same color structure. Thus subamplitudes involving a photon can be obtained by summing over permutations of gluon subamplitudes in which the photon assumes all possible places in the ordering.

Photons do not couple to gluons. In a f^{abc}-based decomposition this is apparent: When the index a corresponds to the identity matrix I, the $U(N_c)$ generator F^a in the adjoint representation vanishes. In this paper we presented new color decompositions for gluon tree amplitudes, eq. (2.1), and for the gluon-loop sector of one-loop amplitudes, eqs. (3.3) and (3.15); we still use the λ-based decomposition for quark-gluon tree amplitudes, eq. (2.2), and for the quark-loop sector of one-loop

amplitudes. Thus it might seem that we can bring no new insight to the calculation of multi-photon amplitudes. However, that is not the case for one-loop amplitudes with an external $\bar{q}q$ pair: In the no-n_f terms in eq. (3.15) a photon can still attach to the external quark line, i.e. to the string of λ matrices; however, it manifestly cannot attach to the string of F matrices. We immediately see that subamplitudes of type $A_n^{R,[1]}(1_{\bar{q}}, \sigma_3, \ldots, \sigma_n, 2_q)$, corresponding in eq. (3.15) to partitions with $p = 2$, do not contribute, because there is no λ^a matrix to replace by the identity. (The λ^{x_i} matrices correspond to internal gluons and so should not be replaced.) In the λ-based decomposition [20], the vanishing contribution of these subamplitudes is realized only after summing over all permutations. Thus the new color decomposition we obtain for $\bar{q}qg\cdots g\gamma$ one-loop amplitudes, by converting a gluon into a photon in eq. (3.15), and summing over all the possible attachments of the photon to the quark line, is more compact than the one obtained from the λ-based decomposition. For $n = 5$ we verified that explicitly by computing the $\bar{q}qgg\gamma$ one-loop amplitude from eq. (3.15), and comparing it to (and checking its equivalence with) the same amplitude, but λ-decomposed [20].

4 Discussion

In ref. [14] a new color decomposition was presented for the n-gluon tree amplitude; it was shown to be equivalent to the standard one up to $n = 7$ and was conjectured to be valid for arbitrary n. In the present paper we have proven this conjecture. We have also presented, and proven, analogous color decompositions for one-loop n-gluon amplitudes and one-loop amplitudes with an external quark-antiquark pair plus $n - 2$ gluons.

The new color decompositions have the advantage of being written in terms of just the independent subamplitudes. This makes the decomposition of an amplitude more compact. It is particularly useful for computing the color-summed interference terms between tree amplitudes and one-loop amplitudes, and the square of one-loop amplitudes, which are relevant respectively for NLO and NNLO calculations of jet production rates. The color-summed interference terms and squares are given in terms of color matrices; we have provided in the appendix explicit values required for cross-section computations up to $\mathcal{O}(\alpha_s^4)$.

We expect that structure-constant-based decompositions will also be useful for general multi-loop scattering amplitude calculations, once they have been derived. In a particular case they have already proven useful: Two-loop four-gluon amplitudes in $N = 4$ supersymmetric Yang-Mills theory were computed via their unitarity cuts and were presented in a trace-based formalism [21], but they can be more compactly expressed in an f^{abc} basis [22]. In the latter basis only two types of color structures appear; these are represented graphically in fig. 6a by the planar and nonplanar double box diagrams. The kinematic object multiplying each structure is simply the scalar integral (i.e., the ϕ^3 Feynman diagram) for the graph with the same topology.

Some of this simplicity is undoubtedly due to $N = 4$ supersymmetry. However, group theory alone restricts the possible multi-loop color structures considerably. At one loop, we used the Jacobi identity to remove nontrivial trees from the set of n-gluon color structures. We can similarly remove all such trees from multi-loop n-gluon color structures. The remaining color factors are illustrated in

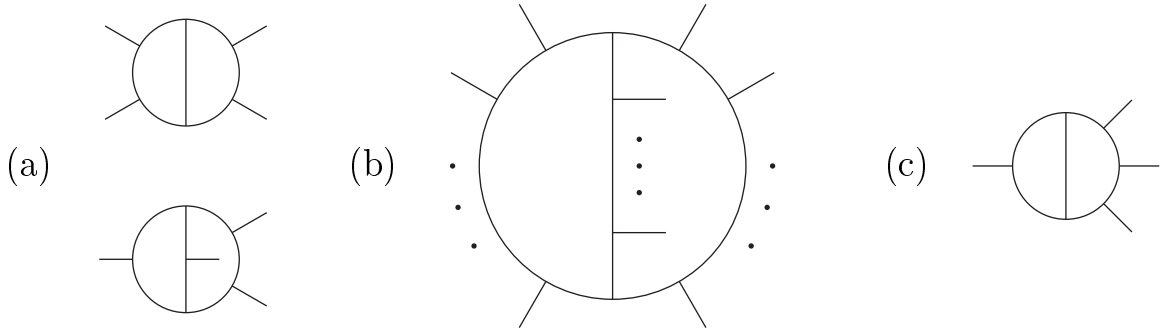

Figure 6: (a) The only color factors appearing in the two-loop 4-gluon amplitude in $N = 4$ super-Yang-Mills theory. (b) The types of color factors for multi-loop n-gluon amplitudes that remain after removing all nontrivial trees. (c) Example of an 'unwanted' 4-gluon planar color factor.

fig. 6b (for the two-loop case). In the one-loop case, this reduction exhausted the use of the Jacobi identity; in the multi-loop case this is no longer true. For example, in the two-loop four-gluon amplitude the nonplanar double box diagram in fig. 6a could be removed in favor of the planar double box diagram in fig. 6a and another planar diagram, shown in fig. 6c. Yet this would be an unwise move, at least in the $N = 4$ supersymmetric case, in the sense that the nonplanar scalar double box integral would then have to be associated with planar color factors. Thus the Jacobi identity should be wielded more judiciously in the multi-loop case, and we leave this to future work.

Finally, we remark that even if a general, multi-loop f^{abc}-based color decomposition is not yet available, the results in this paper can still be employed in the context of specific multi-loop gluonic calculations based on unitarity cuts [21, 7, 22]. For example, the unitarity cuts of two-loop amplitudes are given in terms of tree-level and one-loop amplitudes. Color decomposing these amplitudes as in this paper leads directly, after sewing the color factors on each side of the cut, to an f^{abc}-based decomposition for the cut of the two-loop amplitude. This decomposition is gauge invariant on the cut, by virtue of the gauge invariance of the tree and one-loop amplitudes. The only potential difficulty is that to reconstruct the amplitude from its cuts, one has to find a single function that consistently matches all the cuts, and the color factors may have to be manipulated at this point. Still, this seems likely to be a more efficient than a corresponding approach using a trace-based decomposition of tree and one-loop amplitudes, at least for computing terms that are subleading in the number of colors.

At first sight it may seem like a step backward to abandon trace-based color decompositions of gluonic scattering amplitudes in favor of ones based on structure constants. However, we have seen in this paper that with an appropriate choice of color factors, one does not really have to give up any desirable properties of the trace-based formalism (such as gauge invariance), at least at the tree and one-loop level. We look forward to multi-loop extensions of this work, and their application to

more precise jet rate computations.

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A The color matrices

We first give the color matrices appearing in the color-summed tree-level cross-section, defined in eq. (2.11). For $n = 4$ they are

$$\tilde{c}_{ij} = \mathcal{C}_4(N_c) \cdot \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \qquad A_i^{\text{tree}} = \begin{pmatrix} (1,2,3,4) \\ (1,3,2,4) \end{pmatrix}, \tag{A.1}$$

and for $n = 5$,

$$\tilde{c}_{ij} = \mathcal{C}_5(N_c) \cdot \begin{pmatrix} 8 & 4 & 4 & 2 & 2 & 0 \\ 4 & 8 & 2 & 0 & 4 & 2 \\ 4 & 2 & 8 & 4 & 0 & 2 \\ 2 & 0 & 4 & 8 & 2 & 4 \\ 2 & 4 & 0 & 2 & 8 & 4 \\ 0 & 2 & 2 & 4 & 4 & 8 \end{pmatrix}, \qquad A_i^{\text{tree}} = \begin{pmatrix} (1,2,3,4,5) \\ (1,2,4,3,5) \\ (1,3,2,4,5) \\ (1,3,4,2,5) \\ (1,4,2,3,5) \\ (1,4,3,2,5) \end{pmatrix}, \tag{A.2}$$

where $\mathcal{C}_n(N_c)$ is defined in eq. (2.13). The above results agree with ref. [17], up to an overall factor due to the different choice of normalization of the matrices in the fundamental representation. The color matrix $\tilde{c}_{ij}$ for $n > 5$ shows the $\mathcal{O}(1/N_c^2)$ dependence outlined in eq. (2.9). For $n = 6$, $\tilde{c}_{ij}$ is a 24×24 matrix. Rather than give the entire matrix, we note that its essential information is contained in its first row, i.e. in the matrix elements of $(1,2,3,4,5,6)$ with the 24 permutations. Defining $\tilde{c}_{(123456)(1\sigma(2345)6)} \equiv \mathcal{C}_6(N_c)\,\gamma(\sigma(2345))$, we have

$$\begin{array}{llll} \gamma(2345) = 16, & \gamma(3245) = 8, & \gamma(4235) = 4, & \gamma(5234) = 2, \\ \gamma(2354) = 8, & \gamma(3254) = 4, & \gamma(4253) = 2, & \gamma(5243) = 0, \\ \gamma(2435) = 8, & \gamma(3425) = 4, & \gamma(4325) = 0, & \gamma(5324) = 0, \\ \gamma(2453) = 4, & \gamma(3452) = 2, & \gamma(4352) = 0, & \gamma(5342) = a, \\ \gamma(2534) = 4, & \gamma(3524) = 2, & \gamma(4523) = 2 + a, & \gamma(5423) = a, \\ \gamma(2543) = 0, & \gamma(3542) = 0, & \gamma(4532) = a, & \gamma(5432) = a, \end{array} \tag{A.3}$$

where $a = 24/N_c^2$. The terms proportional to a agree with eq. (A.10) of ref. [17].

The color matrices for the tree-level one-loop interference, defined in eq. (3.5), are for $n = 4$

$$\hat{c}_{ij} = \mathcal{C}_5(N_c) \cdot \begin{pmatrix} 2 & -2 & 0 \\ 0 & -2 & 2 \end{pmatrix}, \qquad \hat{d}_{ij} = \mathcal{C}_4(N_c) \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \tag{A.4}$$

$$(A_{4;1}^{[1]})_j = \begin{pmatrix} (1,2,3,4) \\ (1,2,4,3) \\ (1,3,2,4) \end{pmatrix}, \tag{A.5}$$

and for $n = 5$,

$$\hat{c}_{ij} = \mathcal{C}_6(N_c) \cdot \begin{pmatrix} 2 & -2 & 0 & -2 & 0 & 2 & 0 & 0 & a & -a & a & a \\ 0 & -2 & 2 & -2 & 2 & 0 & a & a & a & 0 & 0 & a \\ 0 & 0 & a & -2 & -a & 2 & 2 & -2 & 0 & 0 & a & a \\ a & a & a & 0 & 0 & 2 & 0 & -2 & 2 & 2 & a & 0 \\ a & -2 & 0 & 0 & 2 & -a & a & 0 & a & 2 & 2 & 0 \\ a & 0 & a & a & 2 & 0 & a & -2 & 0 & 2 & 0 & 2 \end{pmatrix}, \tag{A.6}$$

$$\hat{d}_{ij} = \mathcal{C}_5(N_c) \cdot \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & b & -b & b & b \\ 0 & -1 & 1 & -1 & 1 & 0 & b & b & b & 0 & 0 & b \\ 0 & 0 & b & -1 & -b & 1 & 1 & -1 & 0 & 0 & b & b \\ b & b & b & 0 & 0 & 1 & 0 & -1 & 1 & 1 & b & 0 \\ b & -1 & 0 & 0 & 1 & -b & b & 0 & b & 1 & 1 & 0 \\ b & 0 & b & b & 1 & 0 & b & -1 & 0 & 1 & 0 & 1 \end{pmatrix}, \tag{A.7}$$

$$(A^{[1]}_{5;1})_j = \begin{pmatrix} (1,2,3,4,5) \\ (1,2,3,5,4) \\ (1,2,4,3,5) \\ (1,2,4,5,3) \\ (1,2,5,3,4) \\ (1,2,5,4,3) \\ (1,3,2,4,5) \\ (1,3,2,5,4) \\ (1,3,4,2,5) \\ (1,3,5,2,4) \\ (1,4,2,3,5) \\ (1,4,3,2,5) \end{pmatrix}, \tag{A.8}$$

where $a = 24/N_c^2$, $b = 2/N_c^2$. Although the matrices $\hat{c}_{ij}$ and $\hat{d}_{ij}$ have a relatively small dimension, they are not particularly sparse. We can improve the sparseness considerably by using the tree-level $U(1)$ decoupling identity, eq. (1.2), in every column where three identical entries appear, thus trading three terms for one. Of course, more than $(n-2)!$ different tree subamplitudes will now appear. Using also the tree-level reflection identity, we arrive at the formula

$$\sum_{\text{colors}} \mathcal{A}(1,\ldots,5)[\mathcal{A}(1,\ldots,5)]^*|_{\text{NLO}} \tag{A.9}$$
$$= 4g^8 \mathcal{C}_6(N_c)\,\text{Re} \sum_{i=1}^{(n-1)!/2} \left[A_i^{\text{tree}} \Big(A_i^{[1]} + \frac{n_f}{N_c} A_i^{[1/2]} \Big)^* + \frac{1}{N_c^2} A^{\text{tree}}_{(24135)\cdot i} \Big(12 A_i^{[1]} + 2\,\frac{n_f}{N_c} A_i^{[1/2]} \Big)^* \right],$$

where $(24135)\cdot i$ is the permutation (24135) composed with the ith permutation. In comparing this formula to eq. (11) of ref. [23], we see that the $A_{5;3}$ terms can be removed, at the price of multiplying the remaining $(1/N^2)$-suppressed pure-glue (no n_f) terms by a factor of six. This form for the NLO five-gluon cross-section has been noted independently by W. Kilgore [24].

For the NNLO terms in eq. (3.7) with $n = 4$ we have

$$\bar{c}_{ij} = \mathcal{C}_6(N_c) \cdot \begin{pmatrix} 2+a & a & a \\ a & 2+a & a \\ a & a & 2+a \end{pmatrix}, \tag{A.10}$$

$$\bar{d}_{ij} = \mathcal{C}_5(N_c) \cdot \begin{pmatrix} 1+b & b & b \\ b & 1+b & b \\ b & b & 1+b \end{pmatrix} , \tag{A.11}$$

$$c_{ij} = \mathcal{C}_4(N_c) \cdot \begin{pmatrix} 1-b+d & d & d \\ d & 1-b+d & d \\ d & d & 1-b+d \end{pmatrix} , \qquad d = -\frac{1}{N_c^2} + \frac{3}{N_c^4} , \tag{A.12}$$

where a, b are defined below eq. (A.8). Equivalently,

$$\begin{aligned} &\sum_{\text{colors}} |\mathcal{A}^{\text{1-loop}}(1,2,3,4)|^2 \\ &\quad = g^8\, \mathcal{C}_6(N_c) \Bigg[\sum_{i=1}^{3} \Big(2\,|A_i^{[1]}|^2 + 4\frac{n_f}{N_c} \operatorname{Re} A_i^{[1]} (A_i^{[1/2]})^* + 4\frac{n_f^2}{N_c^2}(1-b)\,|A_i^{[1/2]}|^2 \Big) \\ &\qquad + a \Big| \sum_{i=1}^{3} A_i^{[1]} \Big|^2 + 4\frac{n_f}{N_c} b \operatorname{Re} \Big(\sum_{i=1}^{3} A_i^{[1]} \Big) \Big(\sum_{i=1}^{3} A_i^{[1/2]} \Big)^* + 4\frac{n_f^2}{N_c^2}\, d \Big| \sum_{i=1}^{3} A_i^{[1/2]} \Big|^2 \Bigg] . \end{aligned} \tag{A.13}$$

The color matrix-valued vector $\hat{m}_{ij}^{(p)}$ for the tree-level one-loop interference for NLO processes with an external quark-antiquark pair, defined in eq. (3.18), is for $n = 4$,

$$\hat{m}_{ij}^{(2,3,4)} = \mathcal{C}_4(N_c) \cdot \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \frac{1}{N_c^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \frac{1}{N_c^2} \begin{pmatrix} e & 1+e \\ 1+e & e \end{pmatrix} \end{pmatrix} \tag{A.14}$$

$$A_i^{\text{tree}} = \begin{pmatrix} (1,2,3,4) & (1,2,4,3) \end{pmatrix} , \tag{A.15}$$

where $e = 1/N_c^2$. For $n = 5$ we obtain,

$$\hat{m}_{ij}^{(2,3,4,5)} = -\mathcal{C}_5(N_c) \tag{A.16}$$

$$\times \left(\begin{array}{c} \begin{pmatrix} b & b & b & 0 & 0 & 1 \\ b & b & 0 & 1 & b & 0 \\ b & 0 & b & b & 1 & 0 \\ 0 & 1 & b & b & 0 & b \\ 0 & b & 1 & 0 & b & b \\ 1 & 0 & 0 & b & b & b \end{pmatrix} \\ \frac{1}{N_c^2} \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 \end{pmatrix} \\ \frac{1}{N_c^2} \begin{pmatrix} e & e & 1+e & e & -1+e & 1+e \\ e & e & -1+e & 1+e & 1+e & e \\ 1+e & e & e & e & 1+e & -1+e \\ -1+e & 1+e & e & e & e & 1+e \\ e & 1+e & 1+e & -1+e & e & e \\ 1+e & -1+e & e & 1+e & e & e \end{pmatrix} \\ \frac{1}{N_c^2} \begin{pmatrix} f & e+f & e+f & -1+2e+f & -1+2e+f & 3e+f \\ e+f & f & -1+2e+f & 3e+f & e+f & -1+2e+f \\ e+f & -1+2e+f & f & e+f & 3e+f & -1+2e+f \\ -1+2e+f & 3e+f & e+f & f & -1+2e+f & e+f \\ -1+2e+f & e+f & 3e+f & -1+2e+f & f & e+f \\ 3e+f & -1+2e+f & -1+2e+f & e+f & e+f & f \end{pmatrix} \end{array} \right)$$

$$A_i^{\text{tree}} = \Big((1,2,3,4,5)\ (1,2,3,5,4)\ (1,2,4,3,5)\ (1,2,4,5,3)\ (1,2,5,3,4)\ (1,2,5,4,3) \Big)\ , \tag{A.17}$$

where $f = 1/N_c^4$. In both eqs. (A.14) and (A.16), for each value of p, the permutations P_j of the gluons in the loop amplitude are understood to appear in the same order as the tree amplitude permutations A_i^{tree}.

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