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Neutrino masses and mixings in non-factorizable geometry

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Abstract: We study bulk fermion fields in the localized gravity model with non-factorizable metric recently proposed by Randall and Sundrum, and Gogberashvili. In addition to a tower of weak-scale Kaluza–Klein states we find a zero mode for any value of the fundamental fermion mass. If the fermion mass is larger than half the curvature of the compact dimension, the zero mode can be localized on the "hidden" 3-brane in the Randall–Sundrum model. Identifying this mode with a right-handed neutrino provides a new way for obtaining small Dirac neutrino masses without invoking a see-saw mechanism. Cancellation of the parity anomaly requires introducing an even number of bulk fermions. This naturally leads to a strong hierarchy of neutrino masses and generically large mixing angles.

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1. Introduction

Theories with extra spatial dimensions have received great attention recently, when it was shown that they could provide a solution to the gauge-hierarchy problem. If space-time is a product of Minkowski space with n compact dimensions, with Standard Model fields localized in the three extended spatial dimensions (i.e., on a 3-brane) and gravity propagating in the extra space, then the strength of gravity on the 3-brane is governed by an effective Planck scale $M_{\rm Pl}^2 = M^{n+2}\, V_n$, where M is the fundamental scale of gravity and V_n the volume of the compact space [1]. If this space is sufficiently large, the fundamental scale M can be of order 1 TeV, thus removing the large disparity between the gravitational and the electroweak scales.

An intriguing alternative to the above scenario invokes a non-factorizable geometry with a metric that depends on the coordinates of the extra dimensions [2, 3]. In the simplest scenario due to Randall and Sundrum (RS) one considers a single extra dimension, taken to be a S^1/Z_2 orbifold parameterized by a coordinate $y = r_c\,\phi$, with r_c the radius of the compact dimension, $-\pi \le \phi \le \pi$, and the points (x, ϕ) and $(x, -\phi)$ identified [3]. There are two 3-branes located at the orbifold fixed points: a "visible" brane at $\phi = \pi$ containing the Standard Model fields, and a "hidden" brane at $\phi = 0$. The solution of Einstein's equations for this geometry leads to the non-factorizable metric

$$ {\rm d}s^2 = e^{-2kr_c|\phi|}\,\eta_{\mu\nu}\,{\rm d}x^\mu {\rm d}x^\nu - r_c^2\,{\rm d}\phi^2\,, \tag{1.1} $$

where x^μ are the coordinates on the four-dimensional surfaces of constant ϕ, and the parameter k is of order the fundamental Planck scale M. (This solution can only be trusted if $k < M$, so the bulk curvature is small compared with the fundamental Planck scale.) The two 3-branes carry vacuum energies tuned such that $V_{\rm vis} = -V_{\rm hid} = -24M^3k$, which is required to obtain a solution respecting four-dimensional Poincaré invariance. In between the two branes is a slice of AdS_5 space.

With this setup, the effective Planck scale seen by particles confined to four-dimensional space-time is $M_{\rm Pl}^2 = (M^3/k)(1 - e^{-2kr_c\pi})$, which is of order the fundamental scale M. Unlike the scenarios with large extra dimensions considered in [1], the scale M is therefore not of order the weak scale. However, the "warp factor" $e^{-2kr_c|\phi|}$ in the metric (1.1) has important implications for the masses of particles confined to the visible brane. The Lagrangian for these particles depends on the induced metric on the brane, $g_{\mu\nu}^{\rm vis} = e^{-2kr_c\pi}\,\eta_{\mu\nu}$, and after field renormalization any mass parameter m_0 in the fundamental theory is promoted into an effective mass parameter $m = e^{-kr_c\pi}\,m_0$ governing the physical properties of particles on the brane [3]. With $kr_c \approx 12$ this mechanism produces weak-scale physical masses and couplings from fundamental masses and couplings of order the Planck scale. As a consequence of the warp factor, the Kaluza–Klein excitations of gravitons have weak-scale mass splittings and couplings [4, 5]. This is in contrast with the Kaluza–Klein spectrum

of gravitons propagating in large extra dimensions, which consists of a large number of light modes (with splittings of order the compactification scale) with gravitational couplings. The same properties (i.e., weak-scale masses and couplings) are shared by bulk scalars and vector particles propagating in the extra dimension [6, 7, 8].

In order for this model to provide a viable solution to the hierarchy problem it is important to address the question of how to stabilize the radius r_c of the extra dimension, and the related question of the potentially disastrous cosmology of a visible universe confined to a brane with negative tension [9, 10, 11] (see also [12, 13]). A mechanism for radius stabilization utilizing a bulk scalar field has been proposed in [14]. In the presence of such a mechanism, one finds a conventional cosmological expansion for temperatures below the weak scale [15]. The couplings of the radion field to Standard Model particles may have interesting implications for collider searches [15, 16]. Other phenomenological consequences of the RS model have been discussed in [17], and an alternative model avoiding the negative-tension brane has been proposed in [5].

Resolving the hierarchy problem by introducing extra dimensions poses new challenges. For instance, operators mediating proton decay, lepton-number violation or flavor-changing neutral currents must be sufficiently suppressed. Likewise, the see-saw mechanism for generating small neutrino masses cannot be invoked if the highest energy scale governing physics on the visible brane is the weak scale. However, there is now increasing evidence that the atmospheric neutrino anomaly [18, 19, 20] and the solar neutrino problem [21, 22] are explained in terms of neutrino oscillations, which require small but non-vanishing neutrino masses. Several four-dimensional alternatives to the see-saw mechanism not requiring a high-energy scale have been proposed, such as radiatively generated neutrino masses [23] and composite models [24]. However, it would be interesting to find new mechanisms that are intrinsically higher dimensional. In the context of models with large extra dimensions ideas in this direction have been presented in [25, 26], and some concrete models have been worked out in [27, 28, 29]. They contain a massless Standard Model singlet propagating in the bulk of the extra compact space, which serves as a right-handed neutrino. Then the effective four-dimensional Yukawa coupling is suppressed by a volume factor $1/\sqrt{V_n}$, reflecting the small overlap between the right-handed neutrino in the bulk and the left-handed one on the 3-brane. By construction, this factor provides a suppression of neutrino masses of order $v/M_{\rm Pl}$, reminiscent of the see-saw mechanism. However, this idea does not work in a scenario with small extra dimensions such as the RS model, simply because of the lack of a volume suppression factor.

In this letter we investigate the possibility of incorporating bulk fermions in the RS model. As in the case of scalars [6] and vector fields [7, 8] propagating in the compact dimension we find that the Kaluza–Klein modes have weak-scale masses even though the fermion mass in the fundamental, five-dimensional theory is of order the Planck scale. The fermion case is more interesting, however, because

the extension of the Dirac algebra to five dimensions leads to a different propagation of left- and right-handed modes. After imposing the orbifold boundary conditions the geometry supports a left-handed or a right-handed zero mode for any value of the fundamental fermion mass, one of which can be localized on the hidden brane of the RS model. This is different from the scalar and vector cases, where zero modes exists only for vanishing mass in the fundamental theory. The localization of a right-handed zero mode on the hidden brane provides a new mechanism for obtaining small neutrino masses, which can be realized by coupling the Higgs and left-handed lepton fields of the Standard Model, localized on the visible brane, to a right-handed fermion in the bulk. The neutrino mass can be tuned over many orders of magnitude by a small change of the bulk fermion mass. Moreover, cancellation of the parity anomaly [30, 31] forces us to introduce an even number of bulk fermions. This naturally leads to a neutrino mass hierarchy and potentially large mixing angles.

2. Bulk fermions

Our starting point is the action for a Dirac fermion with mass m of order the fundamental scale M propagating in a five-dimensional space with the metric (1.1), which we write in the form [32][2]

$$S = \int \mathrm{d}^4x \int \mathrm{d}\phi \, \sqrt{G} \left\{ E_a^A \left[\frac{i}{2} \bar{\Psi} \gamma^a (\partial_A - \overleftarrow{\partial_A}) \Psi + \frac{\omega_{bcA}}{8} \bar{\Psi} \{\gamma^a, \sigma^{bc}\} \Psi \right] - m \, \mathrm{sgn}(\phi) \, \bar{\Psi} \Psi \right\} , \tag{2.1}$$

where $G = \det(G_{AB}) = r_c^2 \, e^{-8\sigma}$ with $\sigma = k r_c |\phi|$ is the determinant of the metric. We use capital indices $A, B, \ldots$ for objects defined in curved space, and lower-case indices $a, b, \ldots$ for objects defined in the tangent frame. The matrices $\gamma^a = (\gamma^\mu, i\gamma_5)$ provide a four-dimensional representation of the Dirac matrices in five-dimensional flat space. The quantity $E_a^A = \mathrm{diag}(e^\sigma, e^\sigma, e^\sigma, e^\sigma, 1/r_c)$ is the inverse vielbein, and ω_{bcA} is the spin connection. Because in our case the metric is diagonal, the only non-vanishing entries of the spin connection have $b = A$ or $c = A$, giving no contribution to the action in (2.1).

The sign change of the mass term under $\phi \to -\phi$ is necessary in order to conserve ϕ-parity, as required by the Z_2 orbifold symmetry of the RS model. Such a mass term can be obtained, e.g., by coupling the fermion to a pseudoscalar (under ϕ-parity) bulk Higgs field. For a single bulk fermion in five dimensions ϕ-parity is broken at the quantum level, giving rise to the so-called parity anomaly [30, 31]. To cancel this anomaly, we will later consider an even number of fermion fields.

[2] We do not include a five-dimensional Majorana mass term of the form $\Psi^T C \, \Psi$ in the action, because later we will assign lepton number to the bulk fermion.

Using an integration by parts, and defining left- and right-handed spinors $\Psi_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)\Psi$, the action can be written as

$$S = \int \mathrm{d}^4x \int \mathrm{d}\phi\, r_c \left\{ e^{-3\sigma} \left(\bar{\Psi}_L\, i\partial\!\!\!/\, \Psi_L + \bar{\Psi}_R\, i\partial\!\!\!/\, \Psi_R \right) - e^{-4\sigma}\, m\, \mathrm{sgn}(\phi) \left(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \right) \right.$$
$$\left. - \frac{1}{2r_c} \left[\bar{\Psi}_L \left(e^{-4\sigma} \partial_\phi + \partial_\phi\, e^{-4\sigma} \right) \Psi_R - \bar{\Psi}_R \left(e^{-4\sigma} \partial_\phi + \partial_\phi\, e^{-4\sigma} \right) \Psi_L \right] \right\}, \tag{2.2}$$

where we impose periodic boundary conditions $\Psi_{L,R}(x,\pi) = \Psi_{L,R}(x,-\pi)$ on the fields. The action is even under the Z_2 orbifold symmetry if $\Psi_L(x,\phi)$ is an odd function of ϕ and $\Psi_R(x,\phi)$ is even, or vice versa. To perform the Kaluza–Klein decomposition we write

$$\Psi_{L,R}(x,\phi) = \sum_n \psi_n^{L,R}(x)\, \frac{e^{2\sigma}}{\sqrt{r_c}}\, \hat{f}_n^{L,R}(\phi)\,. \tag{2.3}$$

Because of the Z_2 symmetry of the action it is sufficient to restrict the integration over ϕ from 0 to π. The behavior of the solutions for negative ϕ is then determined by their Z_2 parity. $\{\hat{f}_n^L(\phi)\}$ and $\{\hat{f}_n^R(\phi)\}$ are two complete, orthonormal (with a scalar product defined below) sets of functions on the interval $\phi \in [0,\pi]$, subject to certain boundary conditions. We will construct them as the eigenfunctions of hermitian operators on this interval. Inserting the ansatz (2.3) into the action and requiring that the result take the form of the usual Dirac action for massive fermions in four dimensions,

$$S = \sum_n \int \mathrm{d}^4x \left\{ \bar{\psi}_n(x)\, i\partial\!\!\!/\, \psi_n(x) - m_n\, \bar{\psi}_n(x)\, \psi_n(x) \right\}, \tag{2.4}$$

where $\psi \equiv \psi_L + \psi_R$ (except for possible chiral modes) and $m_n \geq 0$, we find that the functions $\hat{f}_n^{L,R}(\phi)$ must obey the conditions

$$\int_0^\pi \mathrm{d}\phi\, e^\sigma\, \hat{f}_m^{L*}(\phi)\, \hat{f}_n^L(\phi) = \int_0^\pi \mathrm{d}\phi\, e^\sigma\, \hat{f}_m^{R*}(\phi)\, \hat{f}_n^R(\phi) = \delta_{mn}\,,$$
$$\left(\pm \frac{1}{r_c}\, \partial_\phi - m \right) \hat{f}_n^{L,R}(\phi) = -m_n\, e^\sigma\, \hat{f}_n^{R,L}(\phi)\,. \tag{2.5}$$

The boundary conditions $\hat{f}_m^{L*}(0)\, \hat{f}_n^R(0) = \hat{f}_m^{L*}(\pi)\, \hat{f}_n^R(\pi) = 0$, which follow since either all left-handed or all right-handed functions are Z_2-odd, ensure that the differential operators $(\pm\frac{1}{r_c}\,\partial_\phi - m)$ are hermitian and their eigenvalues m_n real. (Since the equations are real, the functions $\hat{f}_n^{L,R}(\phi)$ could be chosen real without loss of generality.)

It is convenient to introduce the new variable $t = \epsilon\, e^\sigma \in [\epsilon, 1]$ with $\epsilon = e^{-kr_c\pi}$, rescale $\hat{f}_n^{L,R}(\phi) \to \sqrt{kr_c\epsilon}\, f_n^{L,R}(t)$, and define the quantities

$$\nu = \frac{m}{k}\,, \qquad x_n = \frac{m_n}{\epsilon k} = \frac{m_n}{k}\, e^{kr_c\pi}\,. \tag{2.6}$$

The small parameter $\epsilon \sim 10^{-16}$ sets the ratio between the electroweak and the gravitational scales. The two conditions in (2.5) now become

$$\int_\epsilon^1 \mathrm{d}t\, f_m^{L*}(t)\, f_n^L(t) = \int_\epsilon^1 \mathrm{d}t\, f_m^{R*}(t)\, f_n^R(t) = \delta_{mn}\,,$$
$$(\pm t\,\partial_t - \nu) f_n^{L,R}(t) = -x_n t\, f_n^{R,L}(t)\,, \tag{2.7}$$

and the boundary conditions are $f_m^{L*}(\epsilon)\, f_n^R(\epsilon) = f_m^{L*}(1)\, f_n^R(1) = 0$. The system of coupled, first-order differential equations for $f_n^{L,R}(t)$ implies the second-order equations

$$\left[t^2\partial_t^2 + x_n^2 t^2 - \nu(\nu \mp 1)\right] f_n^{L,R}(t) = 0\,. \tag{2.8}$$

Dimensional analysis shows that the eigenvalues x_n are of order unity, corresponding to weak-scale fermion masses m_n in the four-dimensional theory.

The solution of the differential equations is straightforward. We start by looking for zero modes, i.e., solutions with $x_n = 0$. In this case the first-order equations in (2.7) decouple. The properly normalized solutions are

$$f_0^{L,R}(t) = f_0^{L,R}(1)\, t^{\pm\nu} \propto e^{\pm m r_c |\phi|}\,, \qquad |f_0^{L,R}(1)|^2 = \frac{1 \pm 2\nu}{1 - \epsilon^{1\pm 2\nu}}\,. \tag{2.9}$$

Since these are even functions of ϕ, which do not vanish at the orbifold fixed points, only one of the zero modes is allowed by the orbifold symmetry. This mode exists irrespective of the value of the fermion mass m in the five-dimensional theory. Note that for $\nu > \frac{1}{2}$ the right-handed zero mode has a very small wave function on the visible brane: $f_0^R(1) \propto \epsilon^{\nu-\frac{1}{2}}$. This property will allow us to obtain small neutrino masses. The presence of fermion zero modes should not come as a surprise, since it is well known that in flat space-time they are associated with domain walls [33]. In our model the domain walls are provided by the 3-branes of the RS model, which separate the regions with a different sign of the fermion mass term. The functions $P_L\, f_n^L(t)$ and $P_R\, f_n^R(t)$, with $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$, can be associated with the "fermionic" and "bosonic" degrees of freedom of a supersymmetric, quantum-mechanical system [34]. The supersymmetry generators are $Q = (\partial_t - \nu/t)\gamma^0 P_L$ and $Q^\dagger = -(\partial_t + \nu/t)\gamma^0 P_R$, and the Kaluza–Klein modes are the eigenstates of the Hamiltonian $\{Q, Q^\dagger\}$. This explains why left- and right-handed modes have the same eigenvalues x_n. The two zero modes correspond to the ground-state solutions of the supersymmetric Hamiltonian. In our case, the requirement of orbifold symmetry allows only one of these solutions to be present.

The solutions of the differential equations (2.8) for the case $x_n > 0$ are Bessel functions. For convenience we assume that $\nu \neq \frac{1}{2} + N$ with an integer N. Then the most general solutions can be written in the form

$$f_n^{L,R}(t) = \sqrt{t}\left[a_n^{L,R}\, J_{\frac{1}{2}\mp\nu}(x_n t) + b_n^{L,R}\, J_{-\frac{1}{2}\pm\nu}(x_n t)\right]\,. \tag{2.10}$$

Option	Eigenvalues $x_n > 0$	a_n^L	a_n^R	Zero Mode
L, $\nu < \frac{1}{2}$	$J_{\frac{1}{2}-\nu}(x_n) = 0$	$\mathcal{N}_{\frac{3}{2}-\nu}(x_n)$	0	R
L, $\nu > \frac{1}{2}$	$J_{\nu-\frac{1}{2}}(x_n) = 0$	0	$\mathcal{N}_{\frac{1}{2}+\nu}(x_n)$	R
R	$J_{\frac{1}{2}+\nu}(x_n) = 0$	0	$\mathcal{N}_{\frac{3}{2}+\nu}(x_n)$	L

Table 1: Bulk fermion solutions for the two choices of boundary conditions, in the limit where $\epsilon = e^{-kr_c\pi} \to 0$. The wave functions $f_n^{L,R}(t)$ take the form (2.11) with coefficients $a_n^{L,R}$ given in the third and fourth columns.

For the special values $\nu = \frac{1}{2} + N$ the solutions are superpositions of Bessel functions of the first and second kind, which can obtained from our results using a limiting procedure. The two functions $f_n^L(t)$ and $f_n^R(t)$ are not independent, since they are coupled by the first-order differential equations in (2.7), which imply $b_n^L = a_n^R$ and $b_n^R = -a_n^L$. Hence, the solutions take the form

$$\begin{aligned} f_n^L(t) &= \sqrt{t}\left[a_n^L\, J_{\frac{1}{2}-\nu}(x_n t) + a_n^R\, J_{-\frac{1}{2}+\nu}(x_n t)\right] , \\ f_n^R(t) &= \sqrt{t}\left[a_n^R\, J_{\frac{1}{2}+\nu}(x_n t) - a_n^L\, J_{-\frac{1}{2}-\nu}(x_n t)\right] . \end{aligned} \tag{2.11}$$

To proceed we must specify the boundary conditions at the locations of the 3-branes. This will give rise to a discrete spectrum of Kaluza–Klein modes. Orbifold symmetry allows two choices of boundary conditions: either all left-handed fields are odd under ϕ-parity and all right-handed ones even ("option L"), or all right-handed fields are odd and all left-handed ones even ("option R"). In the first case the boundary conditions are $f_n^L(\epsilon) = f_n^L(1) = 0$, and in the second case $f_n^R(\epsilon) = f_n^R(1) = 0$. Which of these choices is realized in nature is a questions that cannot be answered without understanding the physics on the 3-branes, which is beyond the scope of the field-theory model suggested in [3]. The two cases are straightforward to analyze. Using the asymptotic behavior of the Bessel functions, $J_n(x) \sim x^n$ as $x \to 0$, it follows that in the limit $\epsilon \to 0$ only one of the two terms in the wave functions in (2.11) remains. Taking $\epsilon \to 0$ is an excellent approximation unless we were to consider integrals of the functions $f_n^{L,R}(t)$ with weight functions that are singular as $t \to 0$. Table 1 summarizes the results for the eigenvalues and eigenfunctions of the non-zero modes in that limit. The solutions shown correspond to positive ϕ and must be extended to negative ϕ in accordance with the orbifold symmetry. For option L the results take a different form depending on whether $\nu < \frac{1}{2}$ or $\nu > \frac{1}{2}$, as indicated. The second column in the table shows the equation that determines the eigenvalues x_n. In the next two columns we give the values of the coefficients $a_n^{L,R}$ of the properly normalized solutions. The normalization constants $\mathcal{N}_a(x)$ obey $|\mathcal{N}_a(x)|^2 = 2/[J_a(x)]^2$. The last column shows the chirality of the zero mode. The zero-mode wave functions can be recovered by taking the limit $x_n \to 0$; however, the normalization constants do not apply in this case. In figure 1 we show the first few

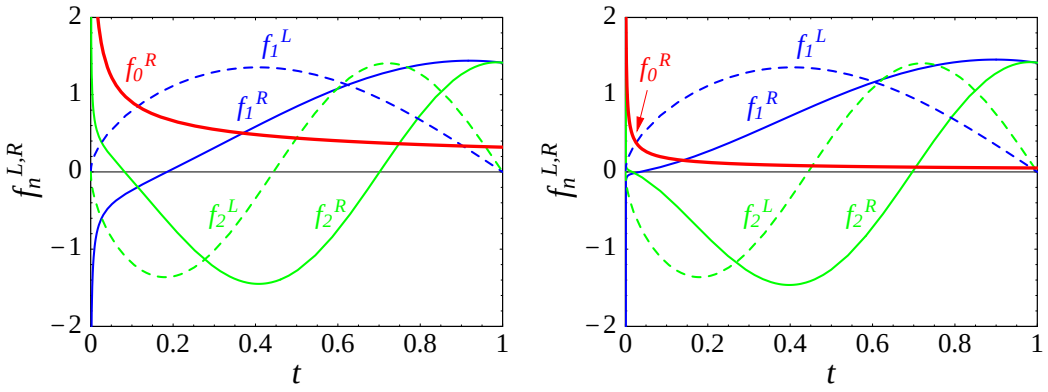

Figure 1: Right-handed (solid) and left-handed (dashed) Kaluza–Klein modes with $n \leq 2$ for $\nu = m/k = 0.45$ (left) and 0.55 (right), with boundary conditions such that all left-handed fields vanish at $t = \epsilon$ and $t = 1$ (option L). We show exact results obtained with $\epsilon = 10^{-16}$. In both cases, the corresponding eigenvalues are $x_1 \approx 2.49$ and $x_2 \approx 5.60$.

Kaluza–Klein modes for option L and two values of the parameter ν just below or above the critical value $\nu = \frac{1}{2}$. The important point to note is the localization of the right-handed zero mode $f_0^R(t)$ on the hidden brane (at $t = \epsilon$) for $\nu > \frac{1}{2}$.

For the special case of integer ν the exact solutions for the wave functions can be expressed in terms of trigonometric functions. As an example, we quote results for $\nu = 0$ and $\nu = 1$ choosing for the boundary conditions option L, which will be of special relevance to our discussion below. In both cases the non-zero eigenvalues are given by $x_n = n\pi/(1-\epsilon)$ with an integer $n \geq 1$, and the left-handed solutions are $f_n^L(t) = \mathcal{N} \sin[x_n(t-\epsilon)]$, where $|\mathcal{N}|^2 = 2/(1-\epsilon)$. For $\nu = 0$, the right-handed solutions are given by

$$f_0^R(t) = \frac{\mathcal{N}}{\sqrt{2}}\,, \qquad f_n^R(t) = -\mathcal{N} \cos[x_n(t-\epsilon)]\,, \tag{2.12}$$

whereas for $\nu = 1$ they take the form

$$f_0^R(t) = \frac{\mathcal{N}}{\sqrt{2}}\,\frac{\sqrt{\epsilon}}{t}\,, \qquad f_n^R(\tau) = \mathcal{N}\left(\frac{\sin[x_n(t-\epsilon)]}{x_n t} - \cos[x_n(t-\epsilon)]\right)\,. \tag{2.13}$$

3. Yukawa interactions and neutrino phenomenology

We will now show how including a sterile bulk fermion in the RS model can provide a mechanism for obtaining small Dirac neutrino masses, which is quite different from the see-saw mechanism. We focus first on a single fermion generation and consider a scenario where all matter and gauge fields charged under the Standard Model gauge group are confined to the visible brane at $\phi = \pi$, whereas a gauge-singlet fermion field propagates in the bulk. After integration over the compact extra dimension we obtain a tower of four-dimensional Kaluza–Klein fermions in the four-dimensional

theory, as shown in (2.4). We choose boundary conditions such that there is a right-handed zero mode (option L) with wave function $f_0^R(t)$ given in (2.9). Only this choice will lead to an interesting neutrino phenomenology.

Omitting gauge interactions, the action for a Higgs doublet $H = (\phi_1, \phi_2)$, a left-handed lepton doublet $L = (\nu_L, e_L)$ and a right-handed lepton e_R localized on the visible brane is

$$\begin{aligned} S = & \int \mathrm{d}x^4 \, \sqrt{-g_{\mathrm{vis}}} \left\{ g_{\mathrm{vis}}^{\mu\nu} \, \partial_\mu H_0^\dagger \, \partial_\nu H_0 - \lambda \left(|H_0|^2 - v_0^2 \right)^2 \right\} \\ & + \int \mathrm{d}x^4 \, \sqrt{-g_{\mathrm{vis}}} \left\{ \bar{L}_0 \hat{\gamma}^\mu \partial_\mu L_0 + \bar{e}_{R0} \hat{\gamma}^\mu \partial_\mu e_{R0} - \left(y_e \bar{L}_0 H_0 e_{R0} + \mathrm{h.c.} \right) \right\} , \end{aligned} \tag{3.1}$$

where $g_{\mathrm{vis}}^{\mu\nu} = e^{2kr_c\pi} \, \eta^{\mu\nu}$ is the induced metric on the brane, $\sqrt{-g_{\mathrm{vis}}} = \det(-g_{\mu\nu}^{\mathrm{vis}}) = e^{-4kr_c\pi}$, and $\hat{\gamma}^\mu = E_a^\mu(\phi = \pi) \, \gamma^a = e^{kr_c\pi} \, \gamma^\mu$. To restore a canonical normalization of the fields on the brane we must perform the rescalings $H_0 \to e^{kr_c\pi} \, H$, $L_0 \to e^{\frac{3}{2}kr_c\pi} \, L$ and $e_{R0} \to e^{\frac{3}{2}kr_c\pi} \, e_R$, upon which the action takes the form

$$\begin{aligned} S = & \int \mathrm{d}x^4 \left\{ \partial_\mu H^\dagger \partial^\mu H - \lambda \left(|H|^2 - v^2 \right)^2 \right\} \\ & + \int \mathrm{d}x^4 \left\{ \bar{L} \, i\partial\!\!\!/ \, L + \bar{e}_R \, i\partial\!\!\!/ \, e_R - \left(y_e \bar{L} H e_R + \mathrm{h.c.} \right) \right\} , \end{aligned} \tag{3.2}$$

where $v = e^{-kr_c\pi} \, v_0$. The remarkable feature noted in [3] is that all dimensionful parameters such as the Higgs vacuum expectation value get rescaled by the warp factor and turned from Planck-scale into weak-scale couplings, whereas dimensionless parameters such as λ and y_e remain unchanged.

We now introduce a Yukawa coupling of the bulk fermion with the Higgs and lepton fields. With our choice of boundary conditions all left-handed Kaluza–Klein modes vanish at the visible brane, so only the right-handed modes can couple to the Standard Model fields on the brane. However, in a more realistic scenario which takes into account a finite width of the 3-branes there will most likely be a non-zero (and indeed sizeable) overlap of the left-handed modes with the Standard Model fields. Hence, in order to avoid weak-scale neutrino masses and lepton-number violating interactions we assign lepton number $L = 1$ to the bulk fermion state. Then the only gauge-invariant coupling is of the form

$$S_Y = - \int \mathrm{d}x^4 \, \sqrt{-g_{\mathrm{vis}}} \left\{ \hat{Y}_5 \bar{L}_0(x) \widetilde{H}_0(x) \Psi_R(x, \pi) + \mathrm{h.c.} \right\} , \tag{3.3}$$

where $\widetilde{H} = i\sigma_2 H^*$, and the Yukawa coupling $\hat{Y}_5$ is naturally of order $M^{-1/2}$, with M the fundamental Planck scale of the theory. Rescaling the Standard Model fields in the way described above, and inserting for the bulk fermion the Kaluza–Klein ansatz (2.3), we find

$$S_Y = - \sum_{n \geq 0} \int \mathrm{d}x^4 \left\{ y_n \bar{L}(x) \widetilde{H}(x) \psi_n^R(x) + \mathrm{h.c.} \right\} \tag{3.4}$$

with the effective Yukawa couplings

$$y_n = \sqrt{k}\,\hat{Y}_5\,f_n^R(1) \equiv Y_5\,f_n^R(1)\,, \tag{3.5}$$

where Y_5 is naturally of order unity. After electroweak symmetry breaking, this Yukawa interaction gives rise to a neutrino mass term $\bar{\psi}_L^\nu\,\mathcal{M}\,\psi_R^\nu + \text{h.c.}$, which in the basis $\psi_L^\nu = (\nu_L, \psi_1^L, \ldots, \psi_n^L)$ and $\psi_R^\nu = (\psi_0^R, \psi_1^R, \ldots, \psi_n^R)$, with $n \to \infty$, takes the form

$$\mathcal{M} = \begin{pmatrix} vy_0 & vy_1 & \ldots & vy_n \\ 0 & m_1 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \ldots & m_n \end{pmatrix}. \tag{3.6}$$

As a consequence, there will be a mixing of the Standard Model neutrino ν_L with the heavy, sterile (with respect to the Standard Model gauge interactions) bulk neutrinos ψ_n^L. The Kaluza–Klein excitations of the bulk fermion have masses m_n of order the weak scale v. Thus, in order to obtain a light neutrino we need $|y_0| \ll 1$, which requires having a very small wave function of the zero mode on the visible brane, i.e., $|f_0^R(1)| \ll 1$. But this is precisely what happens if the fundamental fermion mass m satisfies the condition $m > k/2$. Since, as mentioned earlier, the curvature k must be smaller than the fundamental scale M, this is a natural requirement in the context of the RS model.

In order to study the properties of the physical neutrino states we diagonalize the squared mass matrix $\mathcal{M}\mathcal{M}^\dagger$. The eigenvalues of this matrix are the squares of the physical neutrino masses, and the unitary matrix U defined such that $U^\dagger\mathcal{M}\mathcal{M}^\dagger U$ is diagonal determines the physical left-handed neutrino mass eigenstates via $\psi_L^\nu = U\psi_L^{\rm phys}$. We denote by m_ν the mass of the lightest neutrino $\nu_L^{\rm phys}$ and define a mixing angle θ_ν such that $\nu_L = \cos\theta_\nu\,\nu_L^{\rm phys} + \ldots$, where the dots represent the admixture of heavy, sterile bulk states. To leading order in the small parameter $|y_0| \ll 1$ we obtain

$$m_\nu = v|y_0|\cos\theta_\nu\,, \qquad \tan^2\!\theta_\nu = \sum_{n\geq 1}\frac{v^2|y_n|^2}{m_n^2}\,. \tag{3.7}$$

Since the mixing angle is constrained by experiment to be very small (see below), it follows from (2.9) and (3.5) that

$$m_\nu \simeq \sqrt{2\nu-1}\,|Y_5|\,\epsilon^{\nu-\frac{1}{2}}\,v \sim M\left(\frac{v}{M}\right)^{\nu+\frac{1}{2}}\;; \quad \nu > \frac{1}{2}\,. \tag{3.8}$$

This result is remarkable, as it provides a parametric dependence of the neutrino mass on the ratio of the electroweak and Planck scales that is different from the see-saw relation $m_\nu \sim v^2/M$, except for the special case where $\nu = \frac{3}{2}$. This flexibility allows us to reproduce a wide range of neutrino masses without any fine tuning. For instance, taking $v/M = 10^{-16}$, the phenomenologically interesting range of m_ν between $10^{-5}\,\text{eV}$ and $10\,\text{eV}$ can be covered by varying ν between 1.1 and 1.5.

The measurement of the invisible width of the Z^0 boson, which yields $n_\nu = 2.985 \pm 0.008$ for the apparent number of light neutrinos [35], implies that the mixing angle θ_ν must be of order a few percent. For instance, assuming an equal admixture of sterile neutrinos for the three generations of light neutrinos, we obtain $n_\nu = 3\cos^2\theta_\nu$ and hence $\tan^2\theta_\nu = 0.005 \pm 0.003$. From table 1 it follows that with our choice of boundary conditions the wave functions of all excited right-handed Kaluza–Klein modes obey $|f_n^R(1)| = \sqrt{2}$ (for $\epsilon \to 0$). We thus obtain

$$\tan^2\theta_\nu = \frac{v^2|Y_5|^2}{(\epsilon k)^2} \sum_{n=1}^{\infty} \frac{2}{x_n^2} = \frac{1}{2\nu+1} \frac{v_0^2|Y_5|^2}{k^2}\,, \tag{3.9}$$

where x_n are the roots of $J_{\nu-\frac{1}{2}}(x_n) = 0$. The infinite sum can be performed exactly and yields $1/(2\nu+1)$. To satisfy the bound on the mixing angle for $\nu = O(1)$ requires that $v_0|Y_5|/k \lesssim 0.1$, which is possible without much fine tuning. We emphasize, however, that it would be unnatural to have the dimensionless combination $v_0|Y_5|/k$ much less than unity, so a mixing angle θ_ν not much smaller than the current experimental bound is a generic feature of our scenario, which can be tested by future precision measurements.

So far we have shown how a right-handed bulk fermion can give a small Dirac mass to a Standard Model neutrino. We now generalize this mechanism to three neutrino flavors and more than one bulk fermion. Interestingly, such a generalization is forced upon us by the requirement that the parity anomaly for fermions in an odd number of dimensions vanish. When an odd number of bulk fermions in five dimensions are coupled to a gauge field or gravity, the ϕ-parity of the action (2.1) is broken at the quantum level [30, 31]. To obtain a minimal model that is anomaly free we thus introduce two bulk fermions with boundary conditions option L, so there are two massless right-handed zero modes.[3] In order to explain the atmospheric and solar neutrino anomalies in terms of neutrino oscillations one needs two very different mass-squared differences: $\Delta m_{21}^2 \ll \Delta m_{32}^2$, where $\Delta m_{ij} = m_{\nu_i}^2 - m_{\nu_j}^2$, and by convention $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$. This requires a minimum of two massive neutrinos; however, the third neutrino can be massless. In our minimal model this is indeed what happens. Although it is perhaps unconventional to consider a scenario where the number of right-handed neutrinos does not match the number of left-handed ones, we will see that our model explains successfully the known features of the neutrino mass and mixing parameters.

In order to explore this minimal model in more detail we ignore, for simplicity, the heavy Kaluza–Klein excitations of the bulk fermions and focus only on the zero modes. As mentioned above, the admixture of weak-scale sterile neutrino states

[3] More complicated models with four or more bulk states are possible. These states could be subject to different boundary conditions. If we impose lepton number, only the right-handed modes can couple to the Standard Model fields. At least two right-handed zero modes are needed for a successful neutrino phenomenology.

must be strongly suppressed. It is natural to allow for the possibility that the two bulk fermions have different masses $m_1 > m_2$ (of order the Planck scale) in the fundamental theory, and that they couple with similar strength to the three left-handed neutrino flavors. According to (2.9) and (3.5), the effective Yukawa couplings of the two right-handed zero modes $\psi_0^{R,1}$ and $\psi_0^{R,2}$ can then be parameterized as $x_f\,\epsilon^{\nu_1-\frac{1}{2}}$ and $y_f\,\epsilon^{\nu_2-\frac{1}{2}}$ with $\nu_i = m_i/k$ (for $i = 1, 2$) and flavor-dependent couplings x_f, y_f (with $f = e, \mu, \tau$) of order unity. Note that the Yukawa couplings of the two zero modes have a very different magnitude: $x_f/y_f = O(\epsilon^{\nu_1-\nu_2})$. The resulting neutrino mass term $\bar\psi_L^\nu\,\mathcal{M}\,\psi_R^\nu$ + h.c. in the truncated basis $\psi_L^\nu = (\nu_e^L, \nu_\mu^L, \nu_\tau^L)$ and $\psi_R^\nu = (\psi_0^{R,1}, \psi_0^{R,2})$ is

$$\mathcal{M} = v\,\epsilon^{\nu_2-\frac{1}{2}} \begin{pmatrix} \epsilon^{\nu_1-\nu_2}\,x_e & y_e \\ \epsilon^{\nu_1-\nu_2}\,x_\mu & y_\mu \\ \epsilon^{\nu_1-\nu_2}\,x_\tau & y_\tau \end{pmatrix} . \tag{3.10}$$

Diagonalizing the matrix $\mathcal{M}\mathcal{M}^\dagger$ to leading order in ϵ we find that the physical neutrino mass eigenstates comprise a massless left-handed neutrino ν_1, a very light Dirac neutrino with mass squared

$$m_{\nu_2}^2 = v^2\,\epsilon^{2\nu_1-1}\,\frac{|[e\mu]|^2 + |[\mu\tau]|^2 + |[\tau e]|^2}{|y_e|^2 + |y_\mu|^2 + |y_\tau|^2} \sim M^2 \left(\frac{v}{M}\right)^{2\nu_1+1} , \tag{3.11}$$

and a light Dirac neutrino with mass squared

$$m_{\nu_3}^2 = v^2\,\epsilon^{2\nu_2-1}\left(|y_e|^2 + |y_\mu|^2 + |y_\tau|^2\right) \sim M^2 \left(\frac{v}{M}\right)^{2\nu_2+1} . \tag{3.12}$$

In (3.11) we use the short-hand notation $[ij] \equiv x_i y_j - x_j y_i$. Since the lightest neutrino is massless it follows that $\Delta m_{21}^2 = m_{\nu_2}^2$ and $\Delta m_{32}^2 \simeq m_{\nu_3}^2$, and the ratio $\Delta m_{21}^2/\Delta m_{32}^2 \sim (v/M)^{2(\nu_1-\nu_2)}$. An interpretation of the solar neutrino anomaly in terms of neutrino oscillations based on the MSW effect [36] yields values of Δm_{21}^2 in the range 10^{-6}–$10^{-5}\,\mathrm{eV}^2$, whereas oscillations in vacuum would require a smaller value of order $10^{-10}\,\mathrm{eV}^2$ [22]. Such masses can be reproduced in our model by setting $\nu_1 \approx 1.34$–1.37 and $\nu_1 \approx 1.5$, respectively. An explanation of the atmospheric neutrino anomaly in terms of neutrino oscillations yields Δm_{32}^2 in the range $5\cdot 10^{-4}$–$6\cdot 10^{-3}\,\mathrm{eV}^2$ [21], which we can reproduce by taking $\nu_2 \approx 1.27$–1.29. In other words, we can understand the observed hierarchy of the experimentally favored neutrino masses in terms of a 5–10% difference of the bulk fermion masses in the fundamental theory. Note that in our minimal model the lightest neutrino is massless. This can be changed by introducing four (or more) bulk fermion states with three (or more) right-handed zero modes, in which case also the lightest neutrino becomes massive, with $m_{\nu_1}^2 \ll m_{\nu_2}^2$.

Despite the fact that a strong neutrino mass hierarchy is a generic feature of our model, the mixing matrix U relating the neutrino flavor and mass eigenstates

does not contain any small parameter. Defining $\nu_f = \sum_{i=1}^3 U_{fi}\,\nu_i$ we find that all the entries U_{fi} are of order unity. In the limit $\epsilon \to 0$ we obtain

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} \frac{[\mu\tau]^*}{N_1} & \frac{y_\mu^*[e\mu]-y_\tau^*[\tau e]}{N_1 N_2} & \frac{y_e}{N_2} \\ \frac{[\tau e]^*}{N_1} & \frac{y_\tau^*[\mu\tau]-y_e^*[e\mu]}{N_1 N_2} & \frac{y_\mu}{N_2} \\ \frac{[e\mu]^*}{N_1} & \frac{y_e^*[\tau e]-y_\mu^*[\mu\tau]}{N_1 N_2} & \frac{y_\tau}{N_2} \end{pmatrix} , \tag{3.13}$$

where $N_1^2 = |[e\mu]|^2 + |[\mu\tau]|^2 + |[\tau e]|^2$ and $N_2^2 = |y_e|^2 + |y_\mu|^2 + |y_\tau|^2$. A mixing matrix of this type, which lacks the strong hierarchy of the quark mixing matrix, can account for the experimental constraints on the neutrino mixing angles. (In fact, it has been pointed out that a fair fraction of random Dirac mixing matrices is consistent with these constraints [37].) The precise form of these constraints depends on how the data are analyzed, and whether the solar and atmospheric neutrino anomalies individually are interpreted in terms of two-neutrino or three-neutrino mixing. Constraints from the CHOOZ reactor experiment [38] combined with the atmospheric neutrino data imply that $|U_{e3}|^2 \lesssim$ few % [39], which means that $|y_e|$ should be less than $|y_\mu|$ and $|y_\tau|$. In the limit where $|y_e|^2 \ll |y_\mu|^2 + |y_\tau|^2$, the mixing angles θ_{12} and θ_{23} responsible for $\nu_e \leftrightarrow \nu_\mu$ and $\nu_\mu \leftrightarrow \nu_\tau$ oscillations obey the approximate relations

$$\begin{aligned} \sin^2 2\theta_{12} &\simeq \frac{4|x_e|^2(|y_\mu|^2 + |y_\tau|^2)\,|[\mu\tau]|^2}{\left[|x_e|^2(|y_\mu|^2 + |y_\tau|^2) + |[\mu\tau]|^2\right]^2} , \\ \sin^2 2\theta_{23} &\simeq \frac{4|y_\mu|^2|y_\tau|^2}{\left(|y_\mu|^2 + |y_\tau|^2\right)^2} . \end{aligned} \tag{3.14}$$

The atmospheric neutrino anomaly is best explained by near-maximal $\nu_\mu \leftrightarrow \nu_\tau$ mixing, such that $\sin^2 2\theta_{23} > 0.82$ [20]. This implies $0.64 < |y_\mu/y_\tau| < 1.57$, which clearly does not pose a problem for our model. Likewise, a large-mixing-angle solution to the solar neutrino problem requires $\sin^2 2\theta_{12} > 0.75$ [22], which yields $0.58 < |x_e|\sqrt{|y_\mu|^2 + |y_\tau|^2}/|[\mu\tau]| < 1.73$. The small-mixing-angle MSW solution, on the other hand, prefers $\sin^2 2\theta_{12} \sim 10^{-2}$, which would require that the quantities $|x_e|\sqrt{|y_\mu|^2 + |y_\tau|^2}$ and $|x_\mu y_\tau - x_\tau y_\mu|$ differ by about a factor 20. This could either be achieved by having $|x_e| \ll |x_{\mu,\tau}|$, or via a near degeneracy of $x_\mu y_\tau$ and $x_\tau y_\mu$.

4. Conclusions

We have studied bulk fermion solutions in the localized gravity model with non-factorizable geometry introduced by Randall and Sundrum to solve the gauge-hierarchy problem. Similar to the case of scalar or vector fields propagating in the extra compact dimension, we have found that the Kaluza–Klein modes have weak-scale masses even if the fermion mass in the fundamental, five-dimensional theory is of order the Planck scale. However, a distinct feature of bulk fermion solutions is the

possible presence of zero modes due to the fact that the 3-branes in the Randall–Sundrum model act as domain walls.

Our most important finding is that, if the fundamental mass m in the five-dimensional theory is larger than half the curvature k of the compact space, an appropriate choice of the orbifold boundary conditions leads to a right-handed zero mode localized on the hidden brane, whose wave function on the visible brane is power-suppressed in the ratio of the weak scale to the fundamental Planck scale. Coupling the Higgs and left-handed lepton fields of the Standard Model, localized on the visible brane, with a bulk right-handed neutrino provides a new mechanism for obtaining small neutrino masses. Remarkably, this mechanism leads to a generalization of the see-saw formula with a different parametric dependence on the ratio v/M, which can easily reproduce neutrino masses in the range $10^{-5}\,\mathrm{eV}$ to $10\,\mathrm{eV}$. Without much fine tuning the mixing of the Standard Model left-handed neutrino with sterile, weak-scale Kaluza–Klein excitations of the bulk fermion can be made consistent with experimental bounds. However, a generic prediction of our model is that such a mixing occurs at a level not much below the present bound.

Finally, we have shown that with an even number of bulk fermions one can obtain viable models of neutrino flavor oscillations, which naturally predict a mass hierarchy and a neutrino mixing matrix not containing any small parameter. A minimal implementation of this scenario consists of two right-handed neutrinos, identified with the zero modes of two bulk fermions with slightly different masses in the five-dimensional theory, coupled to the three left-handed neutrinos of the Standard Model. In this model we obtain a massless left-handed neutrino and two massive Dirac neutrinos with a large mass hierarchy and generically large mixing angles.

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