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Supersymmetric Yang-Mills Theories from Domain Wall Fermions

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Abstract

We present work in progress on employing domain wall fermions to simulate $N = 1$ supersymmetric Yang-Mills theories on the lattice in $d = 4$ and $d = 3$ dimensions. The geometrical nature of domain wall fermions gives simple insights into how to construct these theories. We also discuss the obstacles associated with simulating the $N = 2$ theory in $d = 4$.

1 Chirality and accidental supersymmetry

There has been intense interest in supersymmetry (SUSY) in the past two decades. The past several years have witnessed many interesting and compelling speculations about strongly coupled SUSY theories. It would be

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interesting to test these conjectures on the lattice. However, since Poincaré symmetry does not exist on the lattice, supersymmetry does not either. In principle, one could tune lattice theories to the SUSY critical point. However, just as Poincaré symmetry is recovered without fine tuning, one might hope that SUSY could be similarly obtained in the continuum limit.

The secret to why the Poincaré symmetric point takes no work to find is that the imposition of hypercubic symmetry and gauge symmetry ensures that Poincaré symmetry is an *accidental* symmetry. All allowed operators that violate the symmetry are irrelevant. Therefore the continuum limit of the theory automatically exhibits more symmetry than it possesses at finite lattice spacing. If supersymmetry could arise as an accidental symmetry as well, then simulation of such theories would not entail fine-tuning of parameters, and would be relatively simple.

The outlook for this approach in SUSY theories with scalar fields is poor...scalar mass terms violate SUSY, are relevant, and cannot be forbidden by any symmetry, unless the scalars are Goldstone bosons. I will return to this issue later, when we talk about $N = 2$ super Yang-Mills (SYM) theories.

However, there are some SUSY theories which do not entail scalars. Of particular interest is $N = 1$ SYM theory in $d = 4$ dimensions. In this theory, the only relevant SUSY violating parameter is the gaugino mass, which can be forbidden by a discrete chiral symmetry. Thus if one can realize the chiral symmetry on the lattice, $N = 1$ SUSY can arise as an accidental symmetry in the continuum limit. This is where domain wall fermions [1, 2] come in, for which chiral symmetry violation (for weak coupling) tends to zero exponentially fast in the domain wall separation. In this talk we clarify how domain wall fermions may be used to study SYM theories [1]. Throughout this talk we will actually discuss only the continuum version, as it is simpler to formulate, if less rigorous, and there are no technical or conceptual obstacles to translating this work to the lattice. We address in turn $N = 1$ in $d = 4$, $N = 1$ in $d = 3$, and $N = 2$ in $d = 4$.

[1] It has long been recognized that SYM theory can arise accidentally as the low energy limit of a theory with gauge and chiral symmetry, and the correct fermion representation [3]. Lattice implementation of $N = 1$ SYM with Wilson fermions and fine-tuning is discussed in [4, 5, 6, 7]. Using domain wall fermions for simulation of $N = 1$ SYM was suggested in [8, 9, 10], but here we follow a different approach. Our results parallel prior work on $N = 1$ SYM theories in the overlap formulation [11, 12], which is equivalent to domain walls with infinite separation.

2 $N = 1$ SUSY Yang-Mills theory in $d = 4$ dimensions

$N = 1$ SUSY Yang-Mills theory in $d = 4$ Minkowski space consists of a gauge group with a massless adjoint Majorana fermion, the gaugino. It is expected to exhibit all sorts of fascinating features, such as confinement, discrete vacua, domain walls, and excitations on these domain walls which transform as fundamentals under the gauge group [13].

The gaugino should arise as an edge state in a 5-d theory of domain wall fermions. The only subtlety in using the machinery of domain wall fermions is how to obtain a single Majorana fermion in Minkowski space, since without modification, the theory gives rise to massless Dirac fermions in Euclidian space.

Let us first review how a massless Dirac fermion arises in the domain wall approach. Consider a Dirac fermion in a 5-dimensional Euclidian continuum, where the fifth dimension is compact: $x_5 = R\theta$, $\theta \in (-\pi, \pi]$. The mass of the fermion is given by a periodic step function

$$m(x_5) = M\epsilon(\theta) = \begin{cases} +M & -\pi/2 < \theta \leq \pi/2 \\ -M & \text{otherwise} \end{cases} \tag{1}$$

We introduce gauge fields independent of the coordinate x_5, so that the Euclidian action is given by

$$S_5 = \int d^5x \; i\overline{\Psi} D(x_5)\Psi \;, \quad D(x_5) = [\not{D}_4 + \partial_5\gamma_5 + m(x_5)] \;, \qquad \gamma_\mu = \gamma_\mu^\dagger \;. \tag{2}$$

Here $\not{D}_4$ is the usual $d = 4$ gauge covariant derivative for a Dirac fermion in the adjoint representation. It is convenient to expand Psi and $\bar{\Psi}$ as [14]

$$\begin{aligned} \Psi(x_\mu, x_5) &= \textstyle\sum_n [b_n(x_5)P_+ + f_n(x_5)P_-]\,\psi_n(x_\mu) \;, \\ \overline{\Psi}(x_\mu, x_5) &= \textstyle\sum_n \bar{\psi}_n(x_\mu)\,[b_n(x_5)P_- + f_n(x_5)P_+] \;. \end{aligned}$$

Here $P_\pm = (1 \pm \gamma_5)/2$ are the chiral projection operators, ψ_n and $\bar{\psi}_n$ are ordinary 4-d Dirac spinors, and b_n, f_n form a complete basis of periodic functions satisfying the eigenvalue equations

$$[\partial_5 + m(x_5)]b_n = \mu_n f_n \;, \qquad [-\partial_5 + m(x_5)]f_n = \mu_n b_n \;. \tag{3}$$

With this expansion, the action S_5 may be rewritten as a theory of an infinite number of 4-d flavors with masses μ_n,

$$S_5 = \sum_n \int d^4x \; \bar{\psi}_n(x)\,[i\not{D}_4 + i\mu_n]\,\psi_n(x) \;. \tag{4}$$

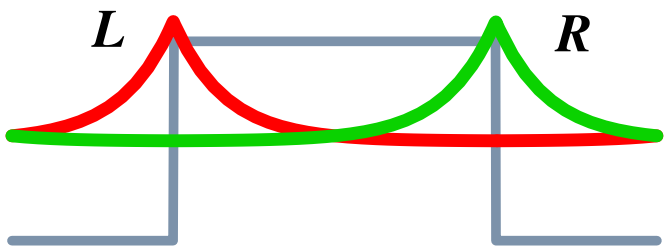

Figure 1: *The left- and right-handed zeromode components, localized at the mass kinks.*

It is straight forward to solve the above equations for μ_n. First of all, one finds zero modes

$$\mu_0 = 0 \ , \qquad b_0(x_5) = e^{-\int^{x_5} m(y)\, dy} \ , \qquad f_0(x_5) = e^{+\int^{x_5} m(y)\, dy} \ . \tag{5}$$

Note that b_0 is localized at $\theta = -\pi/2$, while f_0 is localized at $\theta = +\pi/2$. Nonzero modes have wave functions which are linear combinations of sines and cosines appropriately matched at the locations of the domain walls. The corresponding eigenvalues are doubly degenerate

$$\mu_n = \sqrt{M^2 + n^2/R^2} \ , \quad n = \pm 1, \pm 2, \ldots \tag{6}$$

If instead of having a kink-like mass profile for the Ψ fermions we had a constant mass M (again with periodic boundary conditions), the corresponding eigenvalues $\bar{\mu}$ would be

$$\bar{\mu}_0 = M \ , \qquad \bar{\mu}_n = \sqrt{M^2 + n^2/R^2} \ , \quad n = \pm 1, \pm 2, \ldots \tag{7}$$

Note that for $n \neq 0$, the eigenvalues μ_n and $\bar{\mu}_n$ are equal. It follows that the ratio of fermion determinants for a kink and a constant mass is given by (assuming appropriate regularization)

$$\frac{\det\left[i(\not{\partial}_4 + \gamma_5 \partial_5 + M\epsilon(\theta))\right]}{\det\left[i(\not{\partial}_4 + \gamma_5 \partial_5 + M)\right]} = \frac{\det\left[i\not{\partial}_4\right]}{\det\left[i(\not{\partial}_4 + M)\right]} \tag{8}$$

Note that the right hand side of the above equation corresponds to a massless Dirac fermion and an uninteresting Pauli-Villars field. The left and right handed components of the massless Dirac fermion correspond to the edge states b_0 and f_0 (see Fig. 1.). This method for obtaining a single massless Dirac fermion is robust when transcribed on the lattice [15]: the beauty of the method is that there is no chirality in 5-d, and if one shifts or renormalizes the fermion mass term in the 5-d theory by δm (with $|\delta m| < M$), the effective 4-d theory still has a massless mode. That is because there is a gap in the bulk, so that b_0 and f_0 fall off exponentially, while a chiral symmetry breaking fermion mass must be proportional to the (exponentially small) overlap of b_0 and f_0.

In order to simulate $N = 1$ SYM theory, we need to impose a Majorana condition on ψ_0. Note that in 4-d Minkowski space, the Majorana condition is $\psi = C\bar{\psi}^T$, where C is the charge conjugation matrix, satisfying $C^{-1}\gamma^\mu C = -\gamma^{\mu\,T}$ and $C^{-1}T_a C = -T_a^T$ for generators T_a of real or pseudo-real representations of the gauge group. In Minkowski space charge conjugation interchanges left- and right-handed particles. In our Euclidian domain wall theory, the left- and right-handed modes live on the two different kinks. This suggests that the correct "Majorana" condition for the 5-d Euclidian theory is to define a 5-d reflection which interchanges the two chiral zeromodes, $\mathcal{R}_5 : \theta \to -\theta$, and to impose the constraint on the 5-d Dirac fermions

$$\Psi = \mathcal{R}_5 C \bar{\Psi}^T \tag{9}$$

The 5-d path integral then results in a fermion pfaffian, rather than a fermion determinant:

$$Z_5 = \mathrm{Pf}\left[i\mathcal{R}_5 C\left(\not{D}_4 + \gamma_5\partial_5 + m(x_5)\right)\right] \ . \tag{10}$$

It is straightforward to check that we are taking the pfaffian of an antisymmetric operator, as is required [2]. In terms of the mode expansion in 4-d fields, note that $\mathcal{R}_5$ interchanges $b_n(x_5) \leftrightarrow f_n(x_5)$, so the constraint yields

$$\begin{aligned} \Psi &= \textstyle\sum_n \left[b_n(x_5)P_+ + f_n(x_5)P_-\right]\psi_n(x_\mu) \\ &= \textstyle\sum_n \left[b_n(x_5)P_+ + f_n(x_5)P_-\right] C\bar{\psi}_n^T(x_\mu) = \mathcal{R}_5 C\bar{\Psi}^T \end{aligned} \tag{11}$$

[2] Actually, the operator is only antisymmetric if the fermion is in a real representation, such as an adjoint, instead of a pseudoreal representation. Thus our method is consistent with Witten's result [16] that a theory of a single Weyl pseudoreal fermion is sick.

which implies the conventional (Euclidian) Majorana constraint on the 4-d fermion fields:

$$\psi_n(x_\mu) = C\bar{\psi}_n^T(x_\mu) \ . \tag{12}$$

Using the same technique as in the Dirac case to remove bulk modes, we arrive at a formula for the pfaffian of a massless Majorana fermion:

$$\frac{\mathrm{Pf}\left[i\mathcal{R}_5 C(\not{D}_4 + \gamma_5\partial_5 + M\epsilon(\theta))\right]}{\mathrm{Pf}\left[i\mathcal{R}_5 C(\not{D}_4 + \gamma_5\partial_5 + M)\right]} = \frac{\mathrm{Pf}\left[iC\not{D}_4\right]}{\mathrm{Pf}\left[iC(\not{D}_4 + M)\right]} \tag{13}$$

This formula is easily extended to the lattice by replacing the Dirac action by the Wilson action in all five dimensions [1]. As mentioned before, this leads to an answer identical to that derived by Neuberger [12], although derived in a somewhat different way.

By using Neuberger's closed expression for the domain wall determinant, it is possible to show that the lattice version of the above pfaffian is positive definite, and hence can be computed unambiguously as the square root of the Dirac determinant [3]. Thus the domain wall approach has an added advantage over the Wilson fermion strategy, which suffers from a pfaffian which is not positive definite [17]. it is therefore feasible with present technology to begin exploring this interesting theory.

3 $N = 1$ SUSY Yang-Mills theory in $d = 3$ dimensions

$N = 1$ SYM in $d = 3$ is an interesting theory as well, especially in light of Witten's recent discussion of dynamical SUSY breaking [18]. Once again the spectrum consists of a gauge symmetry and a Majorana fermion, the gaugino. There are two independent relevant operators that break SUSY: the gaugino mass and the (quantized) Chern-Simons term, with one linear combination of the two being supersymmetric. In what follows we will assume that form some gauge groups it is possible to formulate the lattice theory such that the coefficient of the Chern-Simons term in the effective 3-d continuum theory vanishes (work in progress here!). In that case, the only relevant SUSY breaking operator is once again the gaugino mass. If we can realize

[3] This observation was made to DK by Y. Kikukawa.

chiral symmetry and gauge symmetry with a Majorana fermion, SUSY will once again arise in the continuum as an accidental symmetry, modulo the unresolved issue of the Chern-Simons term.

We saw above that without constraints, a 5-d domain wall theory led to a massless Dirac fermion in 4-d; to end up with a Majorana fermion we had to impose a generalization of the Majorana constraint, which effectively took the square root of the 5-d domain wall determinant. However, following the same procedure in one fewer dimensions, a 4-d domain wall system with a Dirac fermion gives rise to two massless Dirac fermions in 3-d, four times as many degrees of freedom as we wish! In particular,

$$\frac{\det\left[i(D_i\gamma_i + \gamma_4\partial_4 + M\epsilon(\theta))\right]}{\det\left[i(D_i\gamma_i + \gamma_4\partial_4 + M)\right]} = \frac{\left[\det i\not{D}_3\right]^2}{\left[\det i(\not{D}_3 + M)\right]^2} \tag{14}$$

where on the left hand side, the index i runs from 1 to 3 and the γ matrices are 4×4; on the right hand side, $\not{D}_3$ is the 3-d Dirac operator (2×2 dimensional in spinor space). Therefore it is clear we need to impose two binary constraints on the system.

First of all, instead of using 4-d Dirac domain wall fermions, we can impose the 4-d Euclidian Majorana constraint, $\psi = C_4\bar{\psi}^T$, where C_4 is a 4-d charge conjugation matrix. This naturally gives rise to a 3-d theory with two Majorana fermions localized at the two kinks. To reduce the spectrum to a single Majorana fermion in 3-d we use the trick of the previous section and constrain the field further to be Majorana under the 3-d charge conjugation matrix C_3, and a simultaneous reflection $\mathcal{R}_4$ in the compact fourth dimension. Thus the simultaneous constraints are:

1. $\Psi(x_i, x_4) = C_4\bar{\Psi}^T(x_i, x_4)$,
2. $\Psi(x_i, x_4) = \mathcal{R}_4 C_3\bar{\Psi}^T(x_i, x_4)$

To be explicit, one can choose the γ matrix basis:

$$\gamma_i = \sigma_1 \otimes \sigma_i \ , \quad \gamma_4 = \sigma_3 \otimes 1 \ , \qquad C_3 = 1 \otimes \sigma_2 \ , \qquad C_4 = \sigma_1 \otimes \sigma_2 \ . \tag{15}$$

It isn't obvious how to simultaneously impose these two constraints until one uses constraint (1) to replace constraint (2) by

2'. $\Psi(x_i, x_4) = \mathcal{R}_4 C_3 C_4^{-1}\Psi(x_i, x_4)$

This last constraint, relating Ψ to its reflection, tells us that we are living on an orbifold — only half the world we were considering represents independent degrees of freedom. So what we do is impose constraint (1) and compute the path integral over half our original space, namely for $\theta \in (0, \pi]$ with suitable boundary conditions at the fixed points of $\mathcal{R}_4$:

$$\left[1 - C_3 C_4^{-1}\right] \Psi(x_i, x_4)\Big|_{x_4 = 0,\, \pi R} = 0 \tag{16}$$

Then one finds the desired result,

$$\frac{\mathrm{Pf}\left[iC_4(D_i\gamma_i + \gamma_4\partial_4 + M\epsilon(\theta))\right]}{\mathrm{Pf}\left[iC_4(D_i\gamma_i + \gamma_4\partial_4 + M)\right]} = \frac{\mathrm{Pf}\left[iC_3 \not{D}_3\right]}{\mathrm{Pf}\left[iC_3(\not{D}_3 + M)\right]}\ , \qquad \theta \in (0, \pi]\ . \tag{17}$$

We have not yet completed our analysis of the reality/positivity of the 4-d pfaffians on the lattice, and the related issue of the Chern-Simons term in the effective 3-d theory.

4 $N = 2$ SUSY Yang-Mills theory in $d = 4$ dimensions

$N = 2$ SYM in $d = 4$ would be fascinating to simulate on the lattice, since in the continuum it exhibits a vast array of interesting phenomena [19]. One might think that it impossible to do without fine tuning, however, because of the scalar fields in the $N = 2$ gauge multiplet. However, a promising idea is to formulate the theory first as an $N = 1$ SUSY theory in $d = 6$ (starting from a domain wall theory in $d = 7$) [11]. The light spectrum of the $d = 6$ theory, with UV cutoff Λ_6 would consist of gauge fields and a Weyl fermion. Then at a scale $\Lambda_4 \ll \Lambda_6$, one compactifies to $d = 4$: the extra two gauge boson polarizations become the complex scalar of the $d - 4$, $N = 2$ gauge multiplet, while the Weyl fermion in $d = 6$ becomes the required two Weyl fermions in $d = 4$. Furthermore, all gauge, ϕ^4 and Yukawa couplings in the $d = 4$ effective theory are derived from the $d = 6$ gauge coupling g_6.

This approach is made respectable by the fact that in the continuum, the $N = 1$ SUSY algebra in $d = 6$ reduces under compactification to the $N = 2$ SUSY algebra in $d = 4$ [20].

Of course, the idea is still to have the target $N = 2$ theory arise as an *accidental* symmetry in the effective theory. What one must try to do then

is to take Λ_4 sufficiently smaller than Λ_6 so that by the time one has scaled down to Λ_4 and passed over to the $d = 4$ effective theory, the theory is "supersymmetric enough" to ensure that the noxious scalar masses radiatively generated in the effective $d = 4$ theory are "small enough".

How small is "small enough"? To study the $N = 2$ theory in the strongly coupled region, where it is interesting, we need the scalar mass m_s to satisfy $m_s \ll \Lambda_{SQCD}$ where Λ_{SQCD} is the scale where the $N = 2$ gauge interactions get strong.

Unfortunately this is impossible to achieve. The $N = 1$ supersymmetry in the $d = 6$ theory is only a symmetry of the operators of leading dimension; SUSY is violated by higher dimension operators, suppressed by powers of Λ_6. Thus the SUSY violating radiatively generated scalar masses in the $d = 4$ effective theory will be suppressed by powers of Λ_4/Λ_6. We can suppress these terms as much as we want, by taking this ratio to be very small! However, the mass scale Λ_{SQCD} is always smaller as it is exponentially small in Λ_4/Λ_6.

To understand this, define the dimensionless gauge coupling $\hat{g}_6 = g_6\Lambda_6$ in the $d = 6$ theory. Since we begin with a weakly coupled domain wall fermion in $d = 7$ $\hat{g}_6 \lesssim 1$. The coupling of the $d = 4$ theory renormalized at the compactification scale Λ_4 is then given by $g_4 = g_6\Lambda_4 = \hat{g}_6\Lambda_4/\Lambda_6$. Therefore

$$\Lambda_{SQCD} \sim \Lambda_4 e^{-8\pi^2/g_4^2} \sim \Lambda_4 e^{-8\pi^2/\hat{g}_6^2(\Lambda_6/\Lambda_4)^2} \ll \Lambda_4 e^{-(\Lambda_6/\Lambda_4)^2} \ . \tag{18}$$

We see that while we obtain scalar masses suppressed by powers of Λ_4/Λ_6, the strong interaction scale Λ_{SQCD} is *exponentially* suppressed in the same ratio. It follows that one cannot study the $N = 2$ theory in the interesting strongly interacting regime starting from a weakly coupled domain wall in $d = 7$, without fine tuning.

The above argument does not rule out studying $N = 2$ SYM in $d = 3$ by compactifying a $d = 4$ theory with approximate $N = 1$ supersymmetry, since the gauge coupling in $d = 3$ does not run logarithmically. However, this $d = 3$ theory has no ground state in the continuum, and so it does not seem interesting to simulate.

5 Conclusions

Domain wall fermions offer a compelling advantage over Wilson fermions in simulating $N = 1$ supersymmetric Yang-Mills theories on the lattice in $d = 4$ and $d = 3$. In each case, supersymmetry arises as an accidental symmetry,

without fine-tuning. Both of these theories should be interesting to study in the near future.

As for SUSY theories with scalars: it is hard to imagine how one can evade fine-tuning — after all, if one did have such a method, it would provide an alternative to SUSY as a solution to the hierarchy problem!

It would be interesting to study perfect supersymmetric actions to try to extract the analogue of a Ginsparg-Wilson relation for supersymmetry, for then one might identify a clever approach to SUSY theories with scalars in the spectrum, one that minimizes the fine-tuning problems.

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