

Out-of-Plane Deflections as a Diagnostic Tool and Application to PEP-II*

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Abstract

We derive simplified formulae that relate the ‘out-of-plane’ beam-beam deflections of colliding e^+ and e^- beams, to the off-diagonal entry of the Σ -matrix expressing the convoluted beam sizes. These formulae can be used to fit measured deflection curves and extract information about the relative tilt angle of two colliding bunches in the transverse plane.

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1 Introduction

Measurement of the beam-beam deflections at the interaction point (IP) of e^+e^- colliders has long been recognized as a useful tool for beam diagnostics [1, 2, 3]. Usually the deflection is measured ‘in-plane’, *i.e.* in the same plane (horizontal or vertical) as the one in which the two colliding bunches move when one beam is scanned across the other. These measurements are useful to determine the transverse convoluted beam sizes. In contrast, information about the relative rotation of the two bunches in the transverse plane is contained in the ‘out-of-plane’ deflections. In this paper, we report practical fitting formulae that can be used to extract such information. They are derived from the Bassetti-Erskine equations (Sec. II) in the limit of small tilt angles, in the presence of small bunch centroid offsets, and for flat beams. The formalism of in-plane deflections is recalled in Sec. III, and that of out-of-plane deflections presented in Sec. IV. As an illustration, in Sec. V we apply these formulas to the analysis of beam-beam deflection measurements carried out in PEP-II at SLAC.

2 The Bassetti-Erskine Formula

Consider 2 two-dimensional gaussian beams (Beam 1 and Beam 2) containing respectively N_1 and N_2 particles. Each beam is described by the following sigma matrix ($i=1,2$):

$$\sigma_i = \begin{bmatrix} \sigma_{xi}^2 \cos^2 \phi_i + \sigma_{yi}^2 \sin^2 \phi_i & (\sigma_{xi}^2 - \sigma_{yi}^2) \cos \phi_i \sin \phi_i \\ (\sigma_{xi}^2 - \sigma_{yi}^2) \cos \phi_i \sin \phi_i & \sigma_{yi}^2 \cos^2 \phi_i + \sigma_{xi}^2 \sin^2 \phi_i \end{bmatrix},$$

where ϕ_1 and ϕ_2 are the tilt angles, σ_{xi} and σ_{yi} the rms widths of the two beams.

As the two beams collide, one is interested in the deflection angles of the two beam *centroids*. The expression for the deflection angles is reported in [4] as a generalization of the Bassetti-Erskine formula [5] (the original Bassetti-Erskine formula in [5] applies to the deflection of a *single* particle against a colliding upright gaussian beam). In particular, the deflections θ_x and θ_y experienced by Beam 2 are given by

$$\theta_y + i\theta_x = N_1 K F_0(x, y, \Sigma), \quad (1)$$

where $K = -2r_e/\gamma$ (with r_e being the classical radius of the beam particle and γ the relativistic factor), and $F_0(x, y, \Sigma)$ is a complex function that can be expressed in terms of the so-called error function of complex argument $w(z) = \exp(-z^2)\text{erfc}(-iz)$:

$$F_0(x, y, \Sigma) = \frac{\sqrt{\pi}}{\sqrt{2(\Sigma_{11} - \Sigma_{22} + 2i\Sigma_{12})}} \left\{ w(\alpha_1) - w(\alpha_2) \exp \left[-\frac{1}{2} (\Sigma_{11}^{-1} x^2 + 2\Sigma_{12}^{-1} xy + \Sigma_{22}^{-1} y^2) \right] \right\},$$

with

$$\alpha_1 = \frac{x + iy}{\sqrt{2(\Sigma_{11} - \Sigma_{22} + 2i\Sigma_{12})}},$$

and

$$\alpha_2 = \frac{(\Sigma_{22} - i\Sigma_{12})x + i(\Sigma_{11} + i\Sigma_{12})y}{\sqrt{\Sigma_{11}\Sigma_{22} - \Sigma_{12}^2}\sqrt{2(\Sigma_{11} - \Sigma_{22} + 2i\Sigma_{12})}}.$$

In the expressions above Σ is the convolution matrix given by the sum of the sigma matrices of the two beams $\Sigma = \sigma_1 + \sigma_2$, while (x, y) are the components of the vector giving the position of the centroid of Beam 2 relative to that of Beam 1.

3 In-Plane Deflections

When doing a beam-beam scan along x (ideally with no vertical offset, *i.e.* $y = 0$) the in-plane deflection angle is by definition θ_x . Similarly, when scanning along y (ideally with no horizontal offset, *i.e.* $x = 0$) the in-plane deflection angle is by definition θ_y . If the transverse distributions of the two beams are exactly upright (vanishing tilt angles) and the beam centroids have no undesired offsets, one can rewrite the Bassetti-Erskine formula in a more compact form. Specifically, after introducing the scaled quantities $\bar{x} = x/\sqrt{2(\Sigma_{11} - \Sigma_{22})}$ and $\bar{y} = y/\sqrt{2(\Sigma_{11} - \Sigma_{22})}$ for the horizontal in-plane deflections one has ($\Sigma_{11} \neq \Sigma_{22}$) [2]:

$$\theta_x = N_1 K \frac{\sqrt{\pi}}{\sqrt{2(\Sigma_{11} - \Sigma_{22})}} e^{-\bar{x}^2} \left[\operatorname{erfi}(\bar{x}) - \operatorname{erfi}\left(\frac{\bar{x}\Sigma_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right) \right], \quad (2)$$

where $\operatorname{erfi}(t)$ is related to the imaginary part of the error function $\operatorname{erfi}(t) = \operatorname{erf}(it)/i = (2/\sqrt{\pi}) \int_0^t d\tau \exp \tau^2$. In turn, the vertical in-plane deflection angle reads

$$\theta_y = -N_1 K \frac{\sqrt{\pi}}{\sqrt{2(\Sigma_{11} - \Sigma_{22})}} e^{-\bar{y}^2} \left[\operatorname{erf}(\bar{y}) - \operatorname{erf}\left(\frac{\bar{y}\Sigma_{11}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right) \right]. \quad (3)$$

In the two equations above the diagonal elements of the convoluted Σ -matrix are simply given by $\Sigma_{11} = \sigma_{x1}^2 + \sigma_{x2}^2$ and $\Sigma_{22} = \sigma_{y1}^2 + \sigma_{y2}^2$.

The formulae above have been used in several e^+e^- colliders [1, 2, 3, 6, 7] to extract beam-size information from beam-beam deflections. In-plane deflections, however, are not very sensitive to, and therefore not particularly suitable for detecting the presence of, small tilt angles in the transverse beam distributions, *i.e.* the presence of a non-vanishing Σ_{12} [8]. In fact, by expanding the Bassetti-Erskine formula for the in-plane deflections as Taylor series in Σ_{12} around $\Sigma_{12} = 0$, one finds that the first-order terms vanish. Of course Eqs. (2) and (3) are the zero-order terms of such expansions. Incidentally, it should also be mentioned that there are no first-order contributions from beam offsets either.

In the limit $\Sigma_{11} \gg \Sigma_{22}$ (flat beams) the two expressions (2) and (3) become:

$$\theta_x \simeq N_1 K \frac{\sqrt{\pi}}{\sqrt{2\Sigma_{11}}} \exp\left(-\frac{x^2}{2\Sigma_{11}}\right) \operatorname{erfi}\left(\frac{x}{\sqrt{2\Sigma_{11}}}\right), \quad (4)$$

and

$$\theta_y \simeq -N_1 K \frac{\sqrt{\pi}}{\sqrt{2\Sigma_{11}}} \exp\left(\frac{y^2}{2\Sigma_{11}}\right) \left[\operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{11}}}\right) - \operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{22}}}\right) \right]. \quad (5)$$

4 Out-of-Plane Deflections

Out-of-plane deflections are those that occur in the plane orthogonal to that in which the two beam centroids lie during a beam-beam scan. In this Section we report the expressions for the first-order dependence of the out-of-plane deflection angles, on the off-diagonal Σ -matrix element Σ_{12} , and on the beam centroid offsets. It turns out that the zero-order term in the Taylor series of the two deflection angles vanishes; in other words, there are no out-of-plane deflections if the bunches are not tilted and if there is no relative transverse beam offset in the plane orthogonal to that in which the beam scan is carried out.

First, let us consider the out-of-plane deflection angle θ_x when doing a y -scan. Through first order in the horizontal offset Δx and in Σ_{12} we have:

$$\theta_x(y) = \frac{\partial\theta_x}{\partial\Delta x}\Delta x + \frac{\partial\theta_x}{\partial\Sigma_{12}}\Sigma_{12}. \quad (6)$$

The general expression for the first-order corrections for arbitrary Σ_{11} and Σ_{22} is quite lengthy. However, if one assumes that $\Sigma_{11} \gg \Sigma_{22}$, (flat beams), after introducing the notation

$$g(y) = \operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{11}}}\right) - \operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{22}}}\right), \quad (7)$$

one finds the following more manageable expressions:

$$\begin{aligned} \frac{\partial\theta_x}{\partial\Delta x} &= \frac{N_1 K}{\Sigma_{11}} \left\{ \left[1 - \sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}} \exp\left(-\frac{y^2}{2\Sigma_{22}}\right) \right] + \right. \\ &\quad \left. \sqrt{\frac{\pi}{2}} \frac{y}{\sqrt{\Sigma_{11}}} \exp\left(\frac{y^2}{2\Sigma_{11}}\right) g(y) \right\}, \end{aligned} \quad (8)$$

$$\frac{\partial\theta_x}{\partial\Sigma_{12}} = \frac{N_1 K}{\sqrt{\Sigma_{11}^3}} \left[\frac{y}{\sqrt{\Sigma_{11}}} + \sqrt{\frac{\pi}{2}} \left(1 + \frac{y^2}{\Sigma_{11}} \right) g(y) \right]. \quad (9)$$

We can introduce further simplifications by restricting ourselves to small deviations in y , *i.e.* $y \ll \sqrt{\Sigma_{11}}$. In this case we have

$$\begin{aligned} \theta_x(y) &\simeq \frac{N_1 K}{\Sigma_{11}} \left\{ \left[1 - \sqrt{\frac{\pi}{2}} \frac{y}{\sqrt{\Sigma_{11}}} \operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{22}}}\right) \right] \Delta x \right. \\ &\quad \left. - \sqrt{\frac{\pi}{2}} \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}}} \operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{22}}}\right) \right\}. \end{aligned} \quad (10)$$

An equation similar to (10) can also be found in [9]. In the same way, we can derive the out-of-plane deflection formula in the vertical plane, and obtain the following result, which is also valid through first order in the vertical beam offset Δy and in Σ_{12} :

$$\theta_y(x) \simeq \frac{N_1 K}{\sqrt{\Sigma_{11}\Sigma_{22}}} \left(\Delta y - x \frac{\Sigma_{12}}{\Sigma_{11}} \right) \exp\left(-\frac{x^2}{2\Sigma_{11}}\right). \quad (11)$$

Expressions (10) and (11) are fairly accurate for arbitrary x (provided that the flat-beam approximation holds).

Although both out-of-plane deflections (10) and (11) are dependent on Σ_{12} , one can easily see that the dependence is weaker in θ_x than in θ_y . In particular, we have $\theta_x \simeq \Sigma_{12}/\sqrt{\Sigma_{11}^3}$ and $\theta_y \simeq x\Sigma_{12}/(\sqrt{\Sigma_{11}^3}\sqrt{\Sigma_{22}})$. The peak value for θ_y is for $x \simeq \sqrt{\Sigma_{11}}$; therefore $\theta_x/\theta_y \simeq \sqrt{\Sigma_{22}/\Sigma_{11}} \ll 1$ for flat beams. As a consequence, for diagnostic purposes it may be preferable to measure the out-of-plane deflection angle θ_y associated with *horizontal* scans when $\Sigma_{11} \gg \Sigma_{22}$.

5 Application to PEP-II

Next we apply the formulae reported in the previous Sections to the analysis of the beam-beam deflection measurements carried out in PEP-II [10], where the on-line data analysis has so far been limited to in-plane deflections.

The purpose here is to illustrate the application of the formulae we have derived, rather than to extract accurate estimates of the bunch parameters. In particular, we simplified the error analysis. We assume that there is no error in the setting of the distance between the bunch centroids. Also, we evaluate the statistical error on the fitted beam parameters, by using the standard deviation of the measured points from the fitting curve as an estimate of the measurement errors.

We consider four sets of data: in-plane and out-of-plane deflections, for both horizontal and vertical beam-beam scans. In all cases the deflection measured is that of the *electron* beam, and is therefore proportional to N_+ , the number of particles contained in the opposing positron bunch. The fitting has been performed using the statistics package of *Mathematica* [11].

5.1 In-plane horizontal deflections

The first dataset contains the measurement of the in-plane deflections during a horizontal scan. The normalized deflection angle θ_x measured for several values of the relative distance of the two beam centroids along x , is shown as dots in Fig. 1. The fitting model is based on the simplified formula (4):

$$\theta_x = a_1 \exp\left(-\frac{(x-x_0)^2}{b_1^2}\right) \operatorname{erfi}\left(\frac{x-x_0}{b_1}\right) + c_1 \quad (12)$$

As only relative changes in position and angle matter, this expression allows for arbitrary offsets in the origin of the horizontal position (x_0) and angle (c_1); a_1 and b_1 are related to the IP beam parameters by $b_1 = \sqrt{2\Sigma_{11}}$ and $a_1 = N_+K\sqrt{\pi}/\sqrt{2\Sigma_{11}}$. The fit yields the following results:

$$\begin{aligned} a_1 &= 14.1 \pm 0.1, \\ b_1 &= 365.9 \pm 4.3, \end{aligned}$$

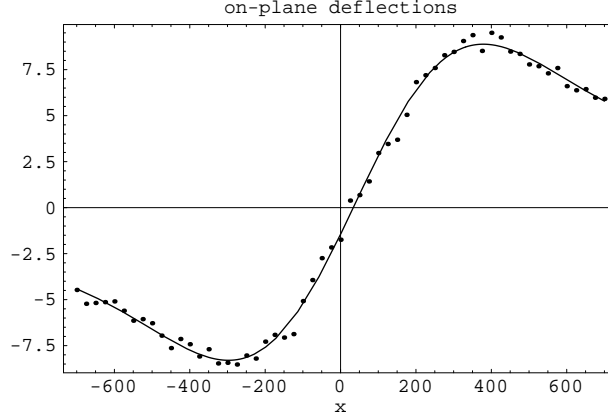


Figure 1: Normalized in-plane deflection angle θ_x of the electron beam, vs. the relative horizontal separation between the two beams. The vertical scale is in units of μrad per $10^{10} e^+$ /bunch; the horizontal scale is in μm .

$$\begin{aligned} c_1 &= 0.29 \pm 0.06, \\ x_0 &= 40.5 \pm 2.6, \end{aligned}$$

from which one extracts $\Sigma_x \equiv \sqrt{\Sigma_{11}} = 259 \pm 3 \mu\text{m}$ and $N_+K = a_1 b_1 / \sqrt{\pi} = 2900 \mu\text{m} \times \mu\text{rad} / 10^{10} e^+$. Note that in the flat-beam limit invoked here, the convoluted horizontal beam size Σ_x can be extracted using only the shape of the deflection curve (b_1 parameter), while the absolute scale of deflection angles is left floating: fitting a_1 provides only a consistency check. If the horizontal to vertical aspect ratio of the beams is not sufficiently large (as was for instance the case at SLC), then the horizontal deflection depends on both Σ_{11} and Σ_{22} . In this case, the absolute scale of the normalized deflection angles (and hence that of the beam currents) must be known, and the fitted parameters are instead expressed in terms of Σ_{11} and Σ_{22} (plus the arbitrary position and angle offsets) [2].

5.2 Out-of-plane vertical deflections

The second dataset is presented in Fig. 2, and is relative to the out-of-plane deflection angle θ_y measured during the same horizontal scan as in dataset # 1. The formula describing the data is Eq. (11), leading to the fitting model

$$\theta_y = \left(a_2 - x \frac{c_2}{\Sigma_{11}} \right) \exp \left(-\frac{x^2}{2\Sigma_{11}} \right) + b_2, \quad (13)$$

with $a_2 = N_+K \Delta y / \sqrt{\Sigma_{11}\Sigma_{22}}$ and $c_2 = N_+K \Sigma_{12} / \sqrt{\Sigma_{11}\Sigma_{22}}$. Because an accurate estimate of Σ_{11} is already available from the analysis of dataset # 1, its value in (13) is a constant rather than a fitted variable. The fit yields:

$$\begin{aligned} a_2 &= 4.9 \pm 0.6, \\ b_2 &= -2.22 \pm 0.35, \\ c_2 &= -1761 \pm 137. \end{aligned}$$

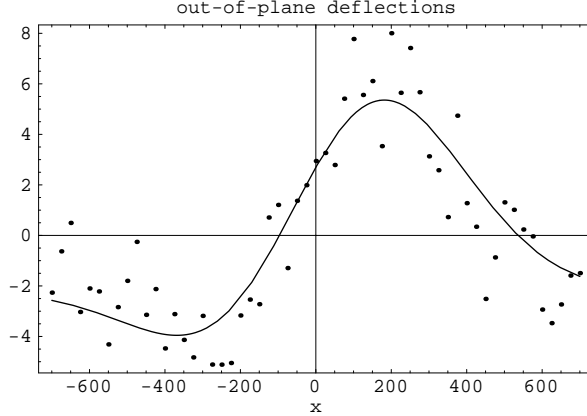


Figure 2: Normalized out-of-plane deflection angle θ_y of the electron beam, vs. the relative horizontal separation between the two beams. The vertical scale is in units of $\mu\text{rad per } 10^{10} e^+/\text{bunch}$; the horizontal scale is in μm .

Note that datasets # 1 and 2 are insufficient to determine Σ_{12} and Σ_{22} separately; only their ratio is measurable at this stage. To obtain an estimate of Σ_{22} one has to perform a beam-beam scan in the vertical direction. This is done with the third dataset.

5.3 In-plane vertical deflections

Dataset # 3 is displayed in Fig. 3. The fitting model in this case is [see Eq. (5)]

$$\theta_y = a_3 \exp\left(\frac{(y - y_0)^2}{2\Sigma_{11}}\right) \left[\text{erf}\left(\frac{y - y_0}{\sqrt{2\Sigma_{11}}}\right) - \text{erf}\left(\frac{y - y_0}{c_3}\right) \right] + b_3. \quad (14)$$

The value of Σ_{11} is again fixed to the value extracted from dataset # 1; $a_3 = -N_+ K \sqrt{\pi} / \sqrt{2\Sigma_{11}}$, and $c_3 = \sqrt{2\Sigma_{22}}$. The fit yields:

$$\begin{aligned} a_3 &= -15.45 \pm 0.12, \\ b_3 &= -0.22 \pm 0.09, \\ c_3 &= 10.71 \pm 0.23, \\ y_0 &= -0.62 \pm 0.12. \end{aligned}$$

The comments made in Sec. 5.1 apply here as well: the flat-beam approximation allows to float the scale normalization a_3 , and to extract Σ_{22} from the shape of the deflection curve alone. In this limit, $\Sigma_y \equiv \sqrt{\Sigma_{22}} = 7.6 \pm 1.6 \mu\text{m}$ [12]. Combining this value with the results of the analysis of datasets # 1 and # 2, one can finally determine the off-diagonal entry $\Sigma_{12} = -1188 \pm 190 \mu\text{m}^2$.

5.4 Out-of-plane horizontal deflections

The fourth and final dataset (Fig. 4) contains the out-of-plane deflection measured during a vertical scan. The fitting model is in this case based on Eq. (10) (neglecting the y -dependence

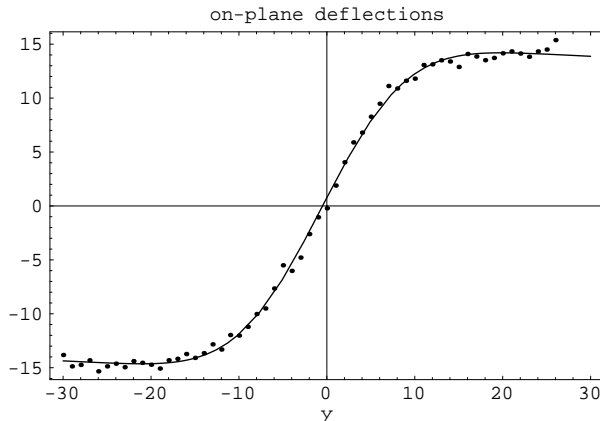


Figure 3: Normalized in-plane deflection angle θ_y of the electron beam (in μrad per $10^{10}e^+$ /bunch), vs. the relative vertical separation of the two beams (in μm).

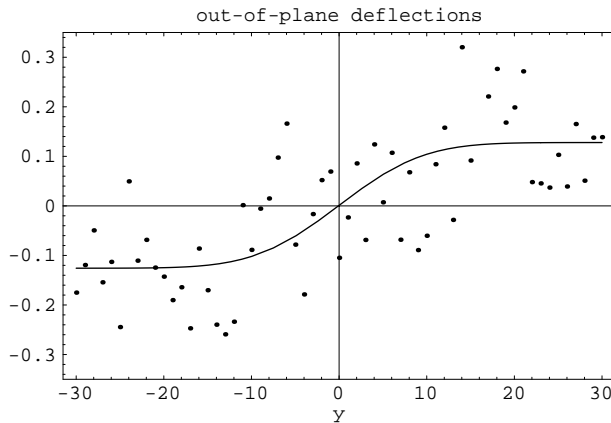


Figure 4: Normalized out-of-plane deflection angle θ_x of the electron beam (in μrad per $10^{10}e^+$ /bunch), vs. the relative vertical separation of the two beams (in μm).

of the term proportional to the Δx offset):

$$\theta_x = b_4 - a_4 \operatorname{erf}\left(\frac{y}{\sqrt{2\Sigma_{22}}}\right). \quad (15)$$

Here we used for Σ_{22} the result from the analysis of dataset # 3, and $a_4 = N_+ K \sqrt{\pi} \Sigma_{12} / \sqrt{2\Sigma_{11}^3}$. One finds:

$$\begin{aligned} a_4 &= -0.127 \pm 0.016, \\ b_4 &= 0 \pm 0.01. \end{aligned}$$

One can use this result (together with the value of Σ_{11} from dataset # 1) to obtain an independent estimate of the off-diagonal term $\Sigma_{12} = -1380 \pm 207 \mu\text{m}^2$, consistent with the previous result within the estimated error.

6 Conclusions

Simplified versions of the Bassetti-Erskine equation can be used as fitting formulae to determine the full convoluted Σ -matrix from measured beam-beam deflections. In particular we have shown that out-of-plane deflection data can be used to determine the off-diagonal entries $\Sigma_{12} = \Sigma_{21}$ of the Σ matrix. If the horizontal/vertical aspect ratios of each of the two beams are known (from separate assumptions or measurements), Σ_{12} can be used to estimate the relative tilt angle between the two colliding bunches. Detecting and correcting such a tilt angle would help improve the luminosity.

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