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Abstract

Self-consistent treatment of a space-charge-dominated beam is generalized for the case of relativistic bunch. Analytical derivations are performed in the limit of a high brightness beam. Shape of the stationary bunch profile as well as expression for space charge limited beam current are derived. Applicability of well-known ellipsoidal model to bunched beam in RF field is discussed.

1 INTRODUCTION

Emittance conservation and prevention of halo formation in a high brightness particle beam in RF accelerator are issues for existing and future high intensity accelerator projects. If the beam is matched with external focusing and accelerating field, its distribution function as well as beam emittance are conserved. Matched stationary beam does not exhibit halo formation. Finding matched conditions for the beam requires solutions of the self-consistent problem for beam distribution function in 6-dimensional phase space, which is typically possible only by numerical methods. In Ref. [1] an approximate analytical solution for self-consistent distribution of a bright bunched beam was found. In this paper the solution is generalized for the case of a relativistic bunch.

2 SELF-CONSISTENT SPACE CHARGE POTENTIAL OF THE BEAM

General approach to find a stationary self-consistent beam distribution function is to represent it as a function of time-independent Hamiltonian, f = f(H), and then to solve the Poisson's equation. The Hamiltonian for particle motion in an RF field with continuous focusing is given by [2]

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, U_{ext} + q\, \frac{U_b}{\gamma^2}, \qquad (1)$$

$$U_{ext} = \frac{E}{k_z}\, [I_o(\frac{k_z r}{\gamma}) \sin(\varphi_s - k_z \zeta) - \sin\varphi_s + k_z \zeta \cos\varphi_s] + \frac{G_t r^2}{2}, \qquad (2)$$

where p_x and p_y are transverse particle momentum, $p_z = p - p_s$ and $\zeta = z - z_s$ are deviations from longitudinal momentum and position of synchronous particle, respectively, U_{ext} is the potential of an external field, U_b is the space charge potential of the beam, E is the amplitude of the accelerating field, φ_s is the synchronous phase, G_t is the gradient of the focusing field, r is the particle radius, $k_z = 2\pi/(\beta\lambda)$ is the wave number and λ is the wavelength. In Ref. [1] the first approximation for a self-consistent potential of the beam was found:

$$U_b = - \frac{\gamma^2}{1 + \delta}\, U_{ext}\ , \qquad (3)$$

where $\delta \approx (k b_\varphi)^{-1}$ is a small parameter, inversely proportional to the dimensionless beam brightness, $b_\varphi = 2I\, a^2/(\beta\gamma B I_c\, \varepsilon_t^2)$, I is the beam current, k=1..3 is the form factor, a is the aperture radius, $I_c = 4\pi\varepsilon_o m c^3/q$ is the characteristic value of the beam current, B is the bunching factor, and ε_t is the transverse beam emittance. Equation (3) indicates that the stationary particle distribution of the bright beam has such a shape that the space charge potential is opposite to the external potential. This phenomenon is known from plasma physics as Debye shielding for nonneutral plasmas.

3 SELF-CONSISTENT BEAM PROFILE

A self consistent space charge distribution of a matched beam in a channel is attained from the Poisson's equation:

$$\rho(r,\zeta) = - \varepsilon_o\, [\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial U_b}{\partial r}) + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2}] = 2\varepsilon_o\, \frac{\gamma^2}{1+\delta}\, G_t\ . \qquad (4)$$

The space charge density of a high brightness beam is nearly constant within the bunch. From Eq. (3) it follows, that, in the first approximation, space charge potential of the beam is the same function of coordinates, as the external potential, with opposite sign. Therefore, equation U_{ext} (r, ζ)= const gives the family of equipotential lines of space charge field of the beam:

$$I_o(\frac{k_z r}{\gamma}) \sin(\varphi_s - k_z \zeta) - \sin\varphi_s + k_z \zeta \cos\varphi_s + \frac{G_t k_z}{2E}\, r^2 = \text{const.} \qquad (5)$$

However, in general case, bunch boundary does not create an equipotential surface, therefore Eq. (5) does not coincide with bunch profile. Instead of Eq. (5) we use the following equation for beam boundary:

$$I_o(\frac{k_z r}{\gamma}) \sin(\varphi_s - k_z \zeta) - \sin\varphi_s + k_z \zeta \cos\varphi_s + C\, (k_z r)^2 = \text{const.} \qquad (6)$$

Equation (6) differs from Eq. (5) by inserted parameter C, which is used to adjust bunch shape in such a way, that self field of the bunch is approximately opposite to external field. Constant in right side of Eq. (6) is determined from the condition, that longitudinal bunch size is, in the first approximation, the same as for zero - current mode, const = $2\varphi_s \cos\varphi_s - 2 \sin\varphi_s$ [1]. Bunch profile described by Eq.(6) reminds separatrix shape in phase space (ζ, p_z). Longitudinal space charge field of this bunch repeats (with negative sign) the RF field inside the bunch. In transverse direction, the space charge forces are close to linear function of coordinate and compensate for external focusing forces. For a long bunch, $\beta\lambda$ >> R_{max}, the Bessel function can be

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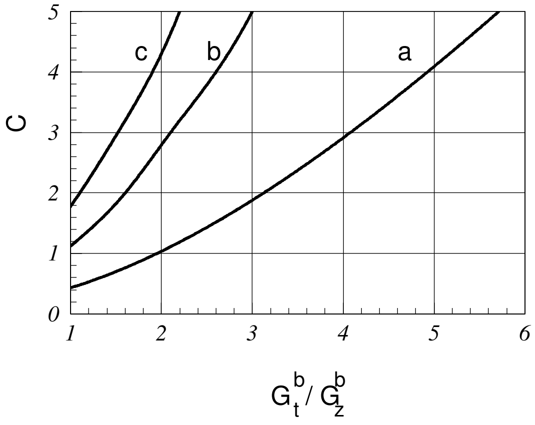

Fig. 1. Coefficient C in bunch shape for $\varphi_s = -30^o$ as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) $\gamma = 1$, b) $\gamma = 3$, c) $\gamma = 6$.

approximated as $I_o(\chi) \approx 1 + \chi^2/4$, and equation (6) for bunch boundary becomes:

$$R(\zeta)=\frac{\beta\lambda}{2\pi}\sqrt{\frac{(2\varphi_s - k_z\zeta)\cos\varphi_s - \sin\varphi_s - \sin(\varphi_s - k_z\zeta)}{C + \frac{1}{4\gamma^2}\sin(\varphi_s - k_z\zeta)}} . \quad (7)$$

Parameter C can be expressed as a function of ratio of transverse, G_t^b, and longitudinal, G_z^b, gradients of space charge forces inside the bunch, see Fig. 1.

According to Eq. (3), if space charge forces are known, the opposite field defines required external field. Gradients of external field are calculated from Eq. (2) in the vicinity of synchronous phase, $k_z\zeta << 1$, $k_z r << 1$. Utilizing expansions $I_o(\chi) \approx 1 + \chi^2/4$ and $\sin(\varphi_s - \xi) \approx \sin\varphi_s - \xi\cos\varphi_s - (1/2)\xi^2\sin\varphi_s$, the external potential is:

$$U_{ext}=\frac{G_z\zeta^2}{2} + \frac{G_t r^2}{2}\left[1- \frac{G_z\sin(\varphi_s - k_z\zeta)}{2\gamma^2 G_t\sin\varphi_s}\right] \approx G_z\frac{\zeta^2}{2} + G_{t,\,eff}\frac{r^2}{2}, \quad (8)$$

where G_z is a longitudinal gradient of external field

$$G_z = 2\pi\,\frac{E\,|\sin\,\varphi_s|}{\beta\lambda}, \quad (9)$$

and $G_{t,\,eff}$ is an effective transverse gradient of external field, depressed due to RF defocusing:

$$G_{t,\,eff} = G_t\left(1 - \frac{G_z}{2\,\gamma^2 G_t}\right) . \quad (10)$$

Taking into account Eq. (3), the relationships between gradients of space charge field and that of external field are

$$G_z^b = - \left(\frac{1}{1+\delta}\right)\frac{2\pi E\,|\sin\,\varphi_s|}{\beta\lambda},\; G_t^b = - \left(\frac{1}{1+\delta}\right)\left[G_t - \frac{\pi E\,|\sin\varphi_s|}{\gamma^2\beta\lambda}\right] . \quad (11)$$

Eqs. (11) together with dependencies, presented in Fig. 1, uniquely define the shape of the stationary bunch for given values of accelerating field, E, focusing gradient, G_t, synchronous phase, φ_s, wavelength, λ, and beam energy, γ.

4 MAXIMUM BEAM CURRENT

Performed study allows us to determine the maximum beam current of bunched beam. The volume of the bunch is calculated from

$$V = \pi\int_{z_{min}}^{z_{max}} R^2(\zeta)\,d\zeta = \frac{(\beta\lambda)^3}{8\pi^2 C} f(\varphi_s) , \quad (12)$$

where function $f(\varphi_s)$ is given by

$$f(\varphi_s) = 3\varphi_s\sin\varphi_s - \frac{9}{2}\varphi_s^2\cos\varphi_s + \cos\varphi_s - \cos2\varphi_s . \quad (13)$$

Total charge of the bunch is $Q = \rho\cdot V$ and the beam current, $I = Q\,c/\,\lambda$, is

$$I = \frac{I_{max}}{1+\delta}, \qquad I_{max} = I_c\left(\frac{\beta^3\gamma^2}{16\pi^3 C}\right)\left(\frac{G_t\,q\,\lambda^2}{m\,c^2}\right) f(\varphi_s) , \quad (14)$$

where I_{max} is a maximum beam current for infinitely high brightness beam. Function $f(\varphi_s)$, Eq.(13), is close to the cubic function of the synchronous phase, φ_s^3. Therefore, maximum beam current is proportional to the cube of the synchronous phase. It is in qualitative agreement with analysis, based on the well-known ellipsoidal approximation to bunched beam [2].

Substitution of parameter δ into Eq. (14) gives an explicit expression for beam current:

$$I = I_{max}\left(1 - \frac{\varepsilon_t^2}{\alpha^2}\right), \quad (15)$$

where α defines normalized acceptance of the channel in presence of transverse focusing and RF field:

$$\alpha = a\sqrt{\frac{\beta^2\gamma}{8\pi^3 BC}\left(\frac{G_t\,q\,\lambda^2}{m\,c^2}\right) f(\varphi_s)} . \quad (16)$$

Eq. (14) gives a unique expression for the beam current limit (without separate transverse and longitudinal limits) for every combination of E, G_t, φ_s and λ.

Fig. 2 illustrates particle-in-cell simulation results of proton bunched beam dynamics with energy of $\gamma = 3$ and maximum possible current of I_{max} = 12A in the field with E = 20 kV/cm, $\varphi_s = -30^o$, G_t = 20 kV/cm^2, $\delta = 0.1$, λ = 3.7 cm. Gradients of space charge forces of the beam obtained from Eqs. (11) are G_z^b = 16.4 kV/cm^2, G_t^b = 17.3 kV/cm^2. The ratio of gradient is $G_t^b / G_z^b = 1.05$, which corresponds to the bunch with parameter C = 1.2 (see Fig. 1b). Beam dynamics simulations show that bunch shape is approximately kept constant.

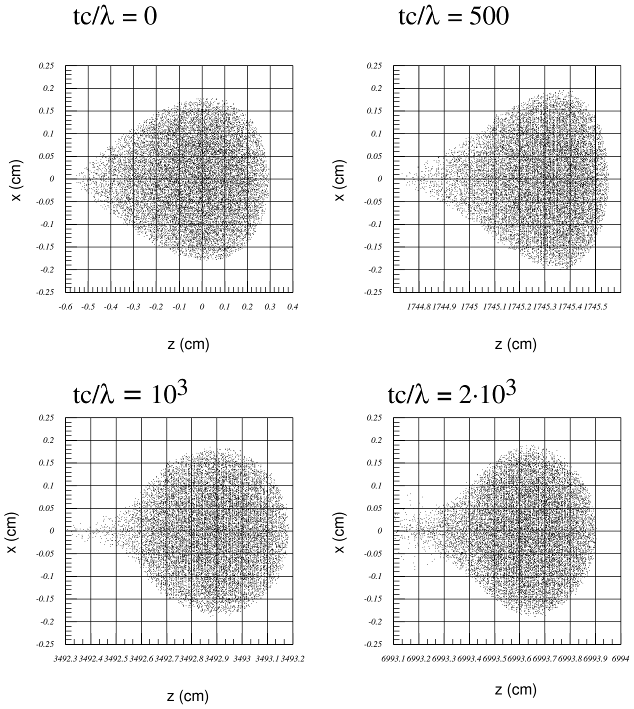

Fig. 2. Dynamics of relativistic proton bunched beam.

5 APPLICABILITY OF ELLIPSOID MODEL

Let us discuss applicability of the well known approximation of the bunch by uniformly populated ellipsoid. In the vicinity of synchronous particle, where external forces are approximately linear functions of coordinates, external potential is given by Eq. (8). Substitution of Eq. (8) into Eq. (3) gives for potential of stationary bunch:

$$U_b = -\frac{\rho}{2\varepsilon_o G_t}\left(G_z \frac{\zeta^2}{2} + \frac{G_{t,\,eff}}{2} r^2\right), \qquad (17)$$

where ρ is defined by Eq. (4). Potential, Eq. (17), corresponds to uniformly populated ellipsoid. In a moving system of coordinates, potential of ellipsoid, U_b', with space charge density $\rho' = \rho/\gamma$ is

$$U_b' = -\frac{\rho'}{2\varepsilon_o}\left(M\zeta'^2 + \frac{1-M}{2} r^2\right), \qquad (18)$$

where $\zeta' = \zeta\,\gamma$ is a longitudinal deviation from the center of ellipsoid and M is a function of ratio of ellipsoid semi-axes:

$$M(R, \gamma l) = \frac{R^2\gamma l}{2}\int_o^\infty \frac{ds}{(R^2+s)\,(\gamma^2 l^2+s)^{3/2}}. \qquad (19)$$

After transformation to laboratory system, the beam potential, $U_b = \gamma U_b'$, is

$$U_b = -\frac{\rho}{2\varepsilon_o}\left[M\gamma^2\zeta^2 + \frac{1-M}{2} r^2\right]. \qquad (20)$$

Comparison of Eq. (17) and Eq. (20) gives

$$M(R, \gamma l) = \frac{G_z}{2\,\gamma^2 G_t}. \qquad (21)$$

Taking into account, that volume of ellipsoid with semi-axes R and l is $V = (4/3)\pi R^2 l$, the maximum bunched beam current, which can be carried by an ellipsoid is

$$I_{max} = I_c \frac{2}{3}\gamma^2\left(\frac{R^2 l}{\lambda^3}\right)\left(\frac{G_t\, q\, \lambda^2}{m\, c^2}\right). \qquad (22)$$

Since bunch with current, Eq. (22), completely cancels for external field, expression (22) gives both transverse and longitudinal current limit. Let us substitute gradient of focusing field, G_t, by the value of zero-current phase advance, $\sigma_o = S/(\beta c)[q\,G_t/(m\gamma)]^{1/2}$, of betatron oscillations per period $S = N\beta\lambda$ of a pure focusing structure (without RF field). In presence of RF field effective focusing gradient is $G_{t,\,eff} = G_t(1 - M)$, see Eqs. (10), (21). Therefore, zero-current phase advance per period, $\sigma_{o,t}$, including both focusing and RF defocusing terms is defined by $\sigma_{o,t}^2 = \sigma_o^2\,(1 - M)$. Phase width of the bunch can be approximately taken as $2\varphi_s$ and, therefore, half of bunch length is $l = \beta\lambda\varphi_s/(2\pi)$. With introduced values, Eq. (22) gives for the current limit

$$I_{max} = \frac{4}{3}\,\frac{mc^2}{Z_o\, q}\,\beta\gamma^3\,\frac{\varphi_s\,\sigma_{ot}^2}{(1-M)\,N^2}\left(\frac{R}{\lambda}\right)^2, \qquad (23)$$

where $Z_o = (c\varepsilon_o)^{-1} = 376.73\Omega$ is the impedance of the free space. Expression (23) is the well known transverse current limit. Let us show that Eq. (22) gives also longitudinal current limit. Substitution of parameter M, Eq. (21), and amplitude of accelerating field E from Eq. (9) into Eq. (22) gives for current limit:

$$I_{max} = \frac{8\pi^2}{3Z_o}\,\frac{E\sin\varphi_s}{\beta M}\,\frac{R^2 l}{\lambda^2}, \qquad (24)$$

which is well-known expression for longitudinal current limit in RF filed. Performed analysis shows that approximation of stationary self-consistent bunched beam by uniformly populated ellipsoid is valid for small bunches, $R \ll \beta_s\lambda$, $l \ll \beta_s\lambda$, while more general analysis results in bunch shape, described by Eq. (6).

6 REFERENCES

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