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On the Kinematics of Undulator Girder Motion *

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The theory of rigid body kinematics is used to derive equations that govern the control and measurement of the position and orientation of undulator girders. The equations form the basis of the girder matlab software on the LCLS control system. The equations are linear for small motion and easily inverted as desired. For reference, some relevant girder geometrical data is also given.

1 Rigid Body Motion

An ideal, perfectly rigid body, is one that can be moved and rotated in space but does not deform in any way. "Kinematics" refers the study of the motion of such bodies. It deals with the issues of what motions are possible, how to describe them, and the effects of constraints on the motion. It is an old discipline and not much in vogue any more. Only one surviving 'theorem' is relevant to this paper, Chasle's theorem [1], and it is hardly a theorem. It states that you can completely describe any motion of any rigid body by a single pure translation and a single pure rotation. That means all you can know about the motion can be boiled down to answering the questions, how did it translate, and how did it rotate.

The theorem is generally true for three dimensional objects with three translational degrees of freedom and three rotational degrees of freedom — six degrees of freedom overall. It turns out that in the case of the undulator girders, it is simpler to derive the overall 3D motion by first analyzing the 2D transverse motion of the girder in two planes of different longitudinal coordinate z. So we will only develop equations for 2D motion. Such motion involves three degrees of freedom, two spatial and one rotational, and is illustrated in Figure 1.

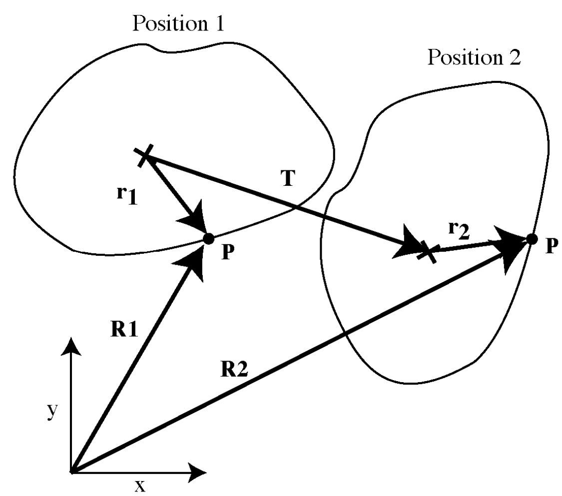

Figure 1: Two dimensional representation of rigid body motion.

Figure 1 shows a planar body in two possible positions: 1 and 2. The net motion of an arbitrary point P fixed in the body can be described by the difference of vectors from the origin of a fixed coordinate system to the point P: $\mathbf{R_2} - \mathbf{R_1}$. If you know where P is in position 1, and you also know $\mathbf{R_2} - \mathbf{R_1}$, you can calculate its location in the position 2. To ap-

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ply Chasle's theorem we express $\mathbf{R_2} - \mathbf{R_1}$ in terms of another translation vector $\mathbf{T}$ and a rotation about and arbitrarily chosen point fixed in the body. In the figure, $\mathbf{r_1}$ and $\mathbf{r_2}$ represent vectors from this origin to P, and the vector $\mathbf{T}$ represents the translation of the origin. Clearly, for a rigid body, $\mathbf{r_1}$ and $\mathbf{r_2}$ must have the same length, but they may have different directions if the body has rotated. By inspection we have

$$\mathbf{R_2} - \mathbf{R_1} = \mathbf{T} + \mathbf{r_2} - \mathbf{r_1}, \tag{1}$$

and we know that $\mathbf{r_2}$ can be obtained by a rotation of $\mathbf{r_1}$.

To simplify the discussion, we will only calculate differential motion such that all body rotation angles are much less than 1. Because the interesting motions of the girders are very small compared to the girder size, all changes in rotational angles involved are also very small compared to 1, and differential motion is an appropriate description. If higher accuracy is desired, then the differential motion can be integrated as a series of small steps.

If the $d\theta$ is defined as the differential rotation angle, and $\mathbf{e_z}$ is a unit vector in the z direction, then

$$\mathbf{r_2} - \mathbf{r_1} = d\theta \mathbf{e_z} \times \mathbf{r_1}.$$

If we apply this to equation 1 we have

$$\mathbf{dR} = \mathbf{R_2} - \mathbf{R_1} = d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_1}. \tag{2}$$

We are essentially done with the theory. Equation 2 expresses the differential change of position of any point in the body as a single translation vector $d\mathbf{T}$ and a single differential rotation angle $d\theta$ (the same for all points in the body) and the rotation of a relative position vector $\mathbf{r_1}$. A 3-D version of this equation would have in place of $d\theta$ a rotation vector of three angles, but otherwise would look about the same.

The trick now is to apply the theory.

2 Girder Motion—2D

Using the above theory, simple as it is, we can calculate the position of the girder from either, (1) the readings of potentiometers that measure linear displacement at arbitrary points on the girder, or (2) the shaft rotation angles of the five cams that support the girder. Inversely, this theory allows us to start with an assumed girder position and then calculate cam shaft angles required to obtain it and what the potentiometer readings should be.

To simplify the discussion we will first show how this works first for linear potentiometers and then study case of cam motion, both in 2-D. Finally, we shall see that by using a simple extrapolation, a 2-D analysis is all we really need to completely describe the complete 3-D girder motion.

2.1 Linear Potentiometers

Imagine a plane figure, such as the one in Figure 2, in contact at three points with the ends of the sliding shafts of three linear potentiometers. The locations of the points of contact and the angles of contact with the edge of the figure are arbitrary. The bodies of the linear potentiometers are considered fixed in space so that If the body moves, there will be a change shaft lengths. For a 2-D body there are only three degrees of freedom, translations in x and y coordinates, and roll. So three linear potentiometers are necessary to determine the motion.

Now refer to Figure 3 showing a close-up of a single linear potentiometer in contact with a body in both the moved and initial positions. The unit vector $\mathbf{u}$ is defined as perpendicular to the surface of the body at the initial contact point $P1$ and is directed into the body. We assume that the linear potentiometer is pointed in this same direction for simplicity. Since only differential motion is assumed, there is no significant change in the direction of the unit vector $\mathbf{u}$. The increase in the length of the potentiometer shaft, LP, is then just the dot product of the displacement of point $P1$ and the unit vector $\mathbf{u}$. From equation 2 applied to point $P1$ we can write down the differential change in the position of the initial contact point and relate it to the change in the potentiometer reading,

$$\begin{aligned} \mathbf{R_2} - \mathbf{R_1} &= d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P1}} && (3) \\ LP &= \mathbf{u1} \cdot (\mathbf{R_2} - \mathbf{R_1}) && (4) \\ LP &= \mathbf{u1} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P1}}), && (5) \end{aligned}$$

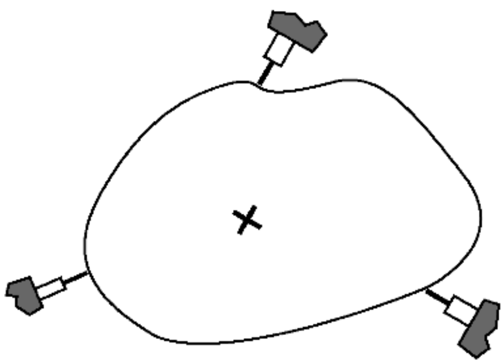

Figure 2: Three linear potentiometers at arbitrary positions can be used for measuring motions of a 2-D body.

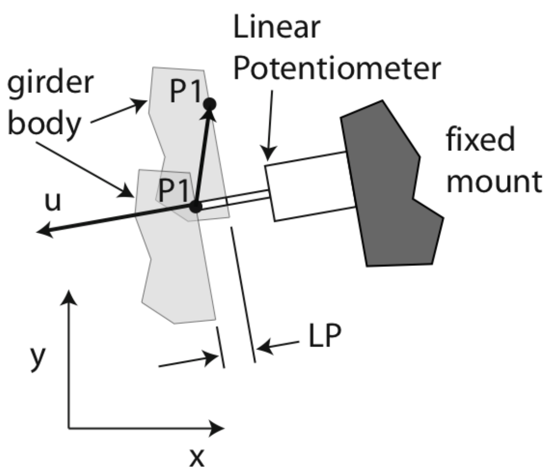

Figure 3: Motion of initial contact point $P1$ and the linear potentiometer displacement, LP.

where $\mathbf{r_{P1}}$ is a vector from some yet-to-be-chosen origin fixed in the girder.

Equation 5 is a generic equation that connects a linear poteniometer reading with the overall translation and rotation of the body. If we write down three such equations for the three linear potentiometers in Figure 2, with readings: $LP1$, $LP2$, and $LP3$; then we have in total,

$$\begin{aligned} LP1 &= \mathbf{u1} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P1}}) && (6) \\ LP2 &= \mathbf{u2} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P2}}) && (7) \\ LP3 &= \mathbf{u3} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P3}}). && (8) \end{aligned}$$

Once the origin is chosen, $\mathbf{r}$'s and $\mathbf{u}$'s can be determined from the geometry of the body. Equations 6-8 show that for a given 2-D vector $d\mathbf{T}$ and rotation angle $d\theta$, the increases to the lengths of the linear potentiometers can be calculated.

By inverting equations 6-8 we can go from linear potentiometer readings to body translations and rotations. Generally the equations, which are linear, will be invertible unless there is a particularly poor choice for the locations and angles of the linear potentiometers, e.g., two are co-linear.

Any arbitrary point fixed in the body can be chosen for the coordinate system origin which determines the r's and u's, but it is usually convenient to pick one in which the positions of the contact points can be easily be evaluated. In the case of the Undulator system, a convenient choice is the point on the theoretical beamline when the girder is at the 'Home' position and at the z midpoint of the undulator segment. The 'Home' position is defined as the midpoint of the range of motion of the girder.

2.2 Cams

Now turn to Figure 4 which depicts three roller bearings, mounted eccentrically on fixed rotatings shafts, making contact with a 2-D body at three points. The bearings have the property that there is free relative motion between the body and the bearing at the contact point in the tangential direction, but no motion allowed in the perpendicular direction. Because of the eccentricity, when a shaft rotation angle changes

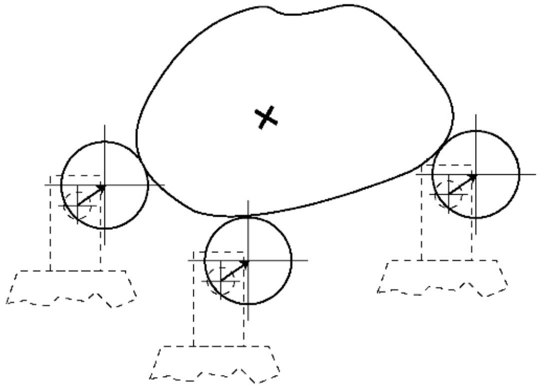

Figure 4: Three roller bearings, depicted as circles, can be used for controlling motions of a 2-D body.

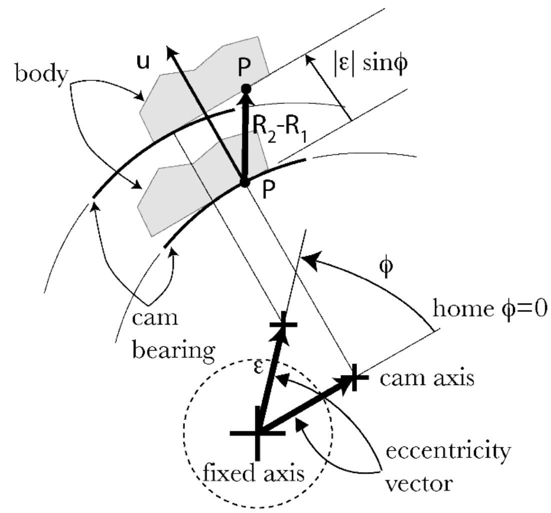

Figure 5: A flat spot on the Body in contact with a cam roller bearing is shown at two angles of the cam bearing axis, 'home' and ϕ.

(shown as small arrows), the center of the corresponding bearing moves and in turn causes the body to move. Since there are three degrees of freedom, there is a unique position of the body for each set of three shaft angles (cam angles), provided the body stays in contact with the roller bearings. If any one of the roller bearings was not making contact with the body, the body would be free to move and its position would be indeterminate.

If we look closely at the contact point of one cam, as shown in Figure 5, we can see that the cam angle ϕ controls the distance from the fixed shaft axis to the flat spot on the edge of the body. The eccentricity vector, ϵ is the vector from the axis of the shaft, which is fixed, to the axis of the cam bearing, which is moving.

When the cam shaft angle ϕ defined such $\phi = 0$ corresponds to ϵ being perpendicular to $\mathbf{u}$, the unit surface vector at the initial contact point, then the relationship between motion of the initial contact point, $\mathbf{R_2} - \mathbf{R_1}$, and the cam shaft angle is:

$$\mathbf{u} \cdot (\mathbf{R_2} - \mathbf{R_1}) = \mathbf{u} \cdot \epsilon = |\epsilon| \sin \phi.$$

Note that $\mathbf{u}$ and the vector of body motion $\mathbf{R_2} - \mathbf{R_1}$ are not generally co-linear.

As before $\mathbf{R_2} - \mathbf{R_1} = d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P1}}$. Applying these relations to all three cams, labeled 1,2,3, yields

$$\begin{aligned}
\mathbf{u1} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P1}}) &= |\epsilon_1| \sin \phi_1 \qquad (9) \\
\mathbf{u2} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P2}}) &= |\epsilon_2| \sin \phi_2 \qquad (10) \\
\mathbf{u3} \cdot (d\mathbf{T} + d\theta \mathbf{e_z} \times \mathbf{r_{P3}}) &= |\epsilon_3| \sin \phi_3 \qquad (11)
\end{aligned}$$

These equations can be used in different ways, depending on what is known and what is to be calculated. For example, if you know the cam angles ϕ_1, ϕ_2, ϕ_3, the locations of the contact points $\mathbf{r_{P1}}, \mathbf{r_{P2}}, \mathbf{r_{P3}}$, and the roll and translation of the body, then you can calculate the magnitude of the eccentricity vectors. Similarly, if you know the magnitude of the eccentricity vectors, cam angles and the locations of the contact points, then you can calculate the translation and roll of the body. In the most common cases, the locations of the contact points are known from the design geometry, and the magnitude of the eccentricity vectors are known from measurements on the cam shafts prior to assembly. Some of the basic geometry of the cams and girders is given in Figure 7 with respect to the girder coordinate system.

If we write out equation 9 explicitly with horizontal and vertical components of $d\mathbf{T}$, (dT_x and dT_y), we get,

$$dT_x u1_x + dT_y u1_y + d\theta \mathbf{u1} \cdot \mathbf{e_z} \times \mathbf{r_{P1}} = |\epsilon_1 \sin\phi_1|$$

Analogous equations can be written for cams 2 and 3. Taking all three equations together, a matrix expression can be constructed. We define the matrix M_3 (which refers to the 3-cam plane) as

$$M_3 = \begin{bmatrix} u1_x & u1_y & \mathbf{u1} \cdot \mathbf{e_z} \times \mathbf{r_{P1}} \\ u2_x & u2_y & \mathbf{u2} \cdot \mathbf{e_z} \times \mathbf{r_{P2}} \\ u3_x & u3_y & \mathbf{u3} \cdot \mathbf{e_z} \times \mathbf{r_{P3}} \end{bmatrix}. \tag{12}$$

Then we can write

$$M_3 \begin{bmatrix} dT_x \\ dT_y \\ d\theta \end{bmatrix} = \begin{bmatrix} |\epsilon_1| \sin\phi_1 \\ |\epsilon_2| \sin\phi_2 \\ |\epsilon_3| \sin\phi_3 \end{bmatrix} \tag{13}$$

and

$$M_3^{-1} \begin{bmatrix} |\epsilon_1| \sin\phi_1 \\ |\epsilon_2| \sin\phi_2 \\ |\epsilon_3| \sin\phi_3 \end{bmatrix} = \begin{bmatrix} dT_x \\ dT_y \\ d\theta \end{bmatrix}. \tag{14}$$

With this information, given the cam angles we can solve for the motion of the girder in the 3-cam plane, or inversely we can find the cam angles that will generate a given motion.

We shall see that the full motion of the girder can be determined by additionally solving the motion in the 2-cam plane. However there are only two degrees of freedom in the 2-cam plane so we can not generally solve for 2D transverse motion *and* roll. Since roll is uniquely determined in the 3-cam plane, we will take it as a known quantity when doing calculations in the 2-cam plane. As in the 3-cam plane we define a matrix connecting the cam angles with the motion:

$$M_2 = \begin{bmatrix} u4_x & u4_y & \mathbf{u4} \cdot \mathbf{e_z} \times \mathbf{r_{P4}} \\ u5_x & u5_y & \mathbf{u5} \cdot \mathbf{e_z} \times \mathbf{r_{P5}} \end{bmatrix}, \tag{15}$$

so that

$$M_2 \begin{bmatrix} dT_x \\ dT_y \\ d\theta \end{bmatrix} = \begin{bmatrix} |\epsilon_4| \sin\phi_4 \\ |\epsilon_5| \sin\phi_5 \end{bmatrix}. \tag{16}$$

To invert this relationship and find the displacements as a function of angles for a given $d\theta$ we note that,

$$\begin{aligned} u4_x dT_x + u4_y dT_y &= d\theta|\epsilon_4| \sin\phi_4 - \mathbf{u4} \cdot \mathbf{e_z} \times \mathbf{r_{P4}} \\ u5_x dT_x + u5_y dT_y &= d\theta|\epsilon_5| \sin\phi_5 - \mathbf{u5} \cdot \mathbf{e_z} \times \mathbf{r_{P5}}. \end{aligned}$$

Identifying the 2×2 matrix,

$$A = \begin{bmatrix} u4_x & u4_y \\ u5_x & u5_y \end{bmatrix}, \tag{17}$$

we have the desired inverse relationship for the 2-cam plane.

$$A^{-1} \begin{bmatrix} |\epsilon_4| \sin\phi_4 - d\theta \mathbf{u4} \cdot \mathbf{e_z} \times \mathbf{r_{P4}} \\ |\epsilon_5| \sin\phi_5 - d\theta \mathbf{u5} \cdot \mathbf{e_z} \times \mathbf{r_{P5}} \end{bmatrix} = \begin{bmatrix} dT \\ dT_y \end{bmatrix}. \tag{18}$$

3 Girder Motion—3D

So far we have only looked at kinematics of 2D objects. The girders are, of course, 3D objects. But because the undulator cams are arranged in two planes whose z separation is much larger than the transverse motion, the 3D motion can easily and accurately be calculated from the 2D motion in each of the two cam-planes.

To see how this is done, consider Figure 6 which depicts a displaced line that should be thought of as moving with the girder and intially co-linear with the ideal beam centerline z axis. We will define the 'girder axis' as this line. The transverse coordinates of the girder axis in the planes $z = Z_A$ and $z = Z_B$ are given by the components of vectors $\mathbf{T_A}$ and $\mathbf{T_B}$. From the figure it is clear that the transverse displacement of any point on the girder axis $\mathbf{T}(z)$ can be obtained from $\mathbf{T_A}$, $\mathbf{T_B}$, and the z positions of the point and the planes, via,

$$\mathbf{T}(z) = (\mathbf{T_B} - \mathbf{T_A}) \frac{z - z_A}{z_B - z_A}. \tag{19}$$

So knowledge of the displacement of the girder axis in any two planes is sufficient to determine the displacement of the girder axis at any point. For example, the motion of points of interest, such as the quadrupole center or the BPM center, can calculated using equation 19, where z_A is the downstream cam-plane coordinate, z_B is the upstream cam-plane coordinate, and z is the coordinate of the quadrupole center or bpm center.

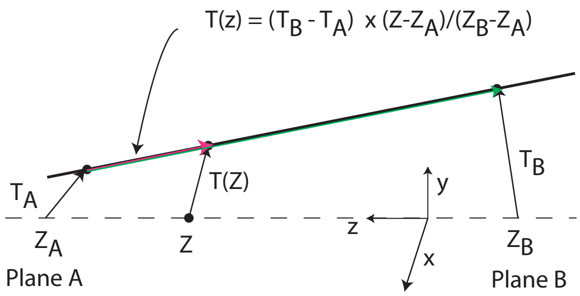

Figure 6: Any point on the girder axis can be calculated from the displacments of the axis at two planes A and B as shown in this figure.

4 Conclusion

Equations 6-8 relate the linear potentiometer readings to the motion of the girder. Equations 9-11 relate the cam shaft angles to the motion of the girder. Both sets are easily inverted to either obtain the girder motion from the angles or readings, or, to find the angles and readings that would give a desired motion. The motion of any point on the girder can be calculated by applying either sets of equations to the two cam-planes and extrapolating in the z coordinate using equation 19. The formulation of the equations is quite general and easily coded via matrix and vector methods. They form the basis of the girder matlab software on the LCLS control system.

Acknowledgements

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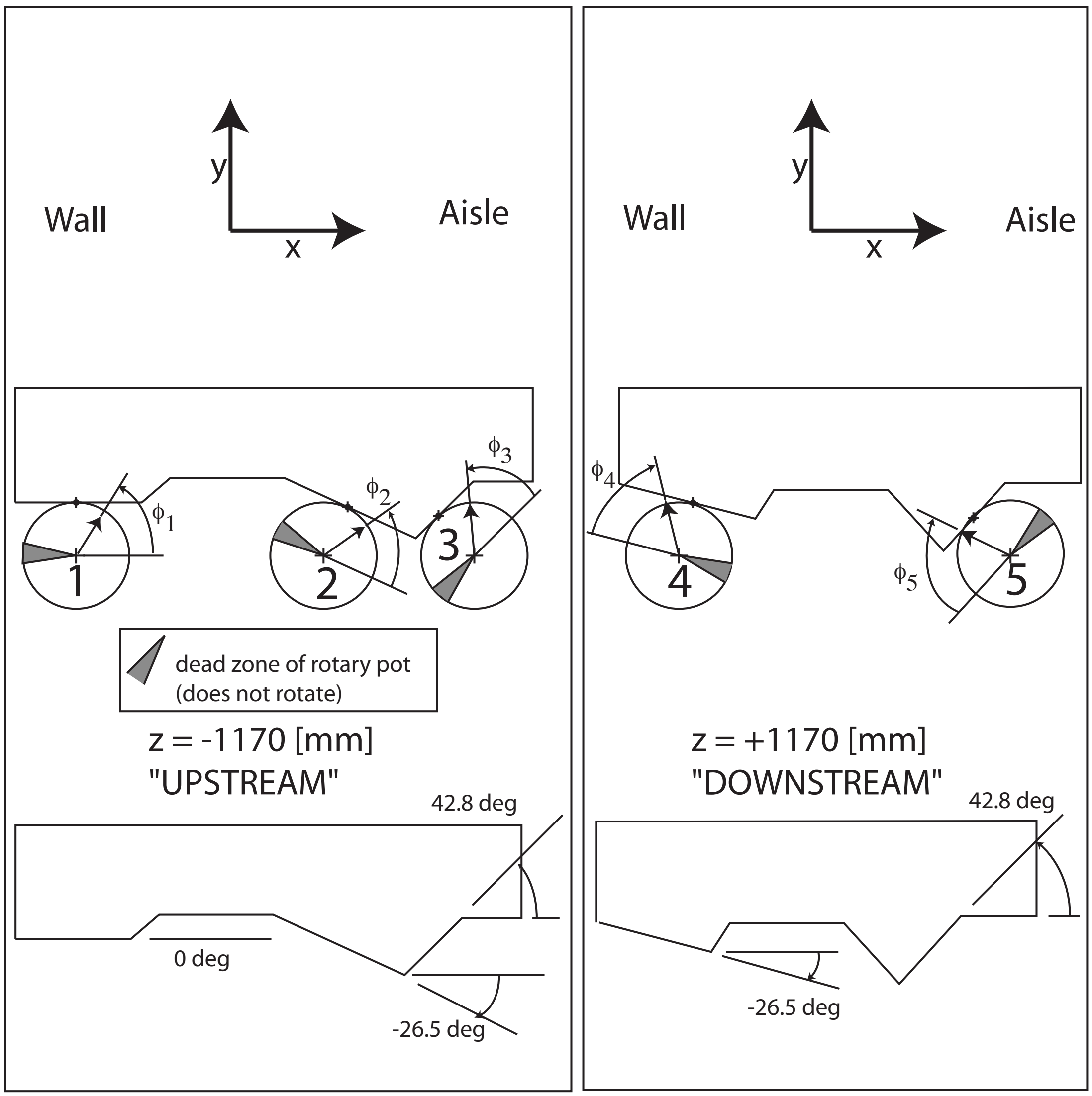
References

[1] H. Goldstein, *Classical Mechanics*, p163



Coordinate system x=y=0 at ideal beam center. z = 0 at segment midplane. (from D. Schafer special markups 10/7/08)

Cam contact points	x	y	z
Cam 1	-151.4	-384.2	-1170
Cam 2	301.5	-388.7	-1170
Cam 3	357.7	-397.1	-1170
Cam 4	-207.0	-388.7	1170
Cam 5	358.3	-397.1	1170
Quad center	0.0	0.0	1808
BPM center	0.0	0.0	1939
BFW center	0.0	0.0	-1795

Control System
Definitions of cam angles, cam numbers, and flat angles.

J.Welch 11/12/08

Figure 7: Configuration of cams and linear potentiometers.