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A Vibrating Wire System For Quadrupole Fiducialization

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Abstract

A vibrating wire system is being developed to fiducialize the quadrupoles between undulator segments in the LCLS. This note provides a detailed analysis of the system.

1 Introduction[1]

The LCLS will have quadrupoles between the undulator segments to keep the electron beam focused. If the quadrupoles are not centered on the beam axis, the beam will receive transverse kicks, causing it to deviate from the undulator axis. Beam based alignment will be used to move the quadrupoles onto a straight line, but an initial, conventional alignment must place the quadrupole centers on a straight line to 100 μm[2]. In the fiducialization step of the initial alignment, the position of the center of the quadrupole is measured relative to tooling balls on the outside of the quadrupole. The alignment crews then use the tooling balls to place the magnet in the tunnel. The required error on the location of the quadrupole center relative to the tooling balls must be less than 25 μm[3].

In this note, we analyze a system under construction for the quadrupole fiducialization. The system uses the vibrating wire technique to position a wire onto the quadrupole magnetic axis. The wire position is then related to tooling balls using wire position detectors. The tooling balls on the wire position detectors are finally related to tooling balls on the quadrupole to perform the fiducialization. The total 25 μm fiducialization error must be divided between these three steps. The wire must be positioned onto the quadrupole magnetic axis to within 10 μm, the wire position must be measured relative to tooling balls on the wire position detectors to within 15 μm, and tooling balls on the wire position detectors must be related to tooling balls on the quadrupole to within 10 μm[4]. The techniques used in these three steps will be discussed.

The note begins by discussing various quadrupole fiducialization techniques used in the past and discusses why the vibrating wire technique is our method of choice. We then give an overview of the measurement system showing how the vibrating wire is positioned onto

[1] Work supported in part by the DOE Contract DE-AC02-76SF00515. This work was performed in support of the LCLS project at SLAC.

[2] LCLS parameter database http://www-ssrl.slac.stanford.edu/htbin/rdbweb/LCLS_params_DB_public.

[3] H. D. Nuhn et al., "General Undulator System Requirements", LCLS Physics Requirements Document 1.4-001.

[4] ibid.

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the quadrupole axis, how the wire position detectors locate the wire relative to tooling balls without touching the wire, and how the tooling ball positions are all measured. The novel feature of this system is the vibrating wire which we discuss in depth. We analyze the wire dynamics and calculate the expected sensitivity of the system. The note should be an aid in debugging the system by providing calculations to compare measurements to.

2 Comparison Of Quadrupole Fiducialization Techniques

A number of techniques have been used in the past to fiducialize quadrupoles. The rotating coil technique and several different stretched wire techniques are discussed here.

A very common quadrupole fiducialization technique uses a rotating coil. If the quadrupole center is not on the axis of rotation of the coil, a dipole component in the quadrupole field is measured[5]. If either the quadrupole is moved onto the coil axis until the measured dipole component is zero, or the measured dipole component is used in a calculation to determine the center position relative to the coil, then the quadrupole tooling balls can be related to the coil rotation axis to fiducialize the quadrupole. This technique has very high sensitivity and quadrupole motions of a fraction of a micron can be resolved[6]. It is fairly difficult, however, to determine the axis of rotation of the coil to a few microns in a global coordinate system. This axis information is required for LCLS fiducialization, however, making the technique difficult at best for our use.

Because of the difficulty in locating the axis of a rotating coil at the micron level, many groups have used a single stretched wire for fiducialization which can be located at the micron level. The HERA quadrupoles were fiducialized with a moving wire technique[7] in which a stretched wire was translated in the quadrupole and the voltage induced in the wire was integrated to give the flux change in the circuit. By performing precision motions of the wire, the position of the magnetic center of the quadrupole was determined. The magnetic center position was then related to tooling balls on the magnet for fiducialization. This technique worked very well for the measurement of the HERA quadrupoles. The magnets involved, however, were superconducting quadrupoles several meters long. In spite of the size and field strength of the magnets, the measured signals in the single, slowly moving wire were very small and special care had to be taken so that thermal emfs did not cause many microns of error. Because the LCLS quadrupoles are much weaker than the HERA quadrupoles, this technique appeared extremely challenging and we did not pursue it.

Another elegant way to fiducialize quadrupoles involves the pulsed wire technique[8,9]. A small diameter Cu-Be wire is stretched through the quadrupole and a short pulse of current is sent through the wire. If the wire is in a magnetic field, it will experience a force which causes the wire to move. The magnet can be moved until the wire is stationary after

[5] A. K. Jain, "Basic Theory Of Magnets", Proc. CERN Accelerator School on Measurement and Alignment of Accelerator and Detector Magnets, Anacapri, April, 1997, CERN 98-05 (1998) 1-26.

[6] C. Rago et al., "High Reliability Prototype Quadrupole For The Next Linear Collider", SLAC-PUB-8990 (2002).

[7] H. Brueck et al., "Methods For Magnetic Measurement Of The Superconducting HERA Magnets", Kerntech **56** (1991) 248-256.

[8] R. Warren, C. Elliot, "New System for Wiggler Fabrication and Testing", LA-UR 87-2981 (1987).

[9] C. Fortgang, "Taut Wire Alignment Of Multiple Permanent Magnet Quadrupoles", LA-UR-89-2696 (1989).

the current pulse, and then the quadrupole is centered on the wire. The tooling balls on the magnet are located relative to the wire to fiducialize the magnet. We set up such a system in the SLAC magnetic measurements lab and we could very easily detect by eye on an oscilloscope motion of as little as 5 μm of a prototype LCLS quadrupole. The method was fairly insensitive, however, to pitch and yaw of the quadrupole. Without pitch and yaw information, the measurement leads to a line through the quadrupole on which the integrated transverse field is zero. This line is not unique, as an infinite number of lines have zero integrated transverse field. The next time the magnet is fiducialized, a different line might be found. This is of no consequence to the electron beam, but it would be beneficial for alignment to get a unique, reproducible result.

A technique used at SLAC for the SLC final focus quadrupoles involved a vibrating wire[10]. The wire was mechanically vibrated with audio speakers. The magnet was moved until the voltage induced in the wire at the frequency of wire vibration went to zero. The quadrupole was then centered on the wire. This technique was very sensitive to magnet motion. It was difficult, however, to determine a line representing the axis of the moving wire. The technique led to a line for which the transverse field integral was zero, but the line was not unique and could be pitched and yawed relative to the quadrupole as discussed above.

Our method of choice for the LCLS quadrupoles is another vibrating wire technique[11,12]. In concept, it is similar to the pulsed wire technique. Instead of a large current pulse in the wire, however, an AC current is used. The alternating current frequency is set to the natural frequency of vibration of the wire. When the magnet is centered on the wire, the transverse magnetic field along the wire is zero and the wire does not experience a force and does not move. When the quadrupole is moved, however, the current in the magnetic field produces an alternating force at the natural frequency of vibration of the wire. Since the wire vibrates at its resonant frequency, the technique is extremely sensitive. In addition, the pitch and yaw of the magnet can be determined, as will be shown, resulting in a unique fiducialization. In this note, the vibrating wire technique is described in detail.

The note continues with an overview of the measurement system. Practical aspects such as how the wire is located are discussed. Then the equations of motion for the stretched wire with an alternating applied force are derived and solved. Effects such as gravity and vibration damping are included. A model vibrating wire system is discussed and values of parameters such as wire diameter and length are inserted into the general equations so that estimates of the performance of the system can be made.

3 Overview of the Quadrupole Fiducialization System

The components of the vibrating wire quadrupole fiducialization system are shown in figure 1. A wire is tensioned between fixed end points and through the quadrupole being fiducialized. The quadrupole is on a mover which positions the magnetic axis of the quadrupole

[10] G. Fischer et al., "Precision Fiducialization of Transport Components", SLAC-PUB-5764 (1992).

[11] A. Temnykh, "Vibrating Wire Field-Measuring Technique", NIM A399 (1997) 185-194.

[12] A. Temnykh, "The Use of Vibrating Wire Technique for Precise Positioning of CESR Phase III Super-Conducting Quadrupoles at Room Temperature", Proceedings of the 2001 Particle Accelerator Conference, Chicago, pp. 3469-3471.

onto the wire using a number of instruments, as described below. Wire position detectors on both sides of the quadrupole locate the wire relative to tooling balls on the detector housings without touching the wire. The entire system is in a coordinate measuring machine which relates the positions of the tooling balls on the wire position detectors to tooling balls on the quadrupole. We now discuss each step of the fiducialization along with the associated system components.

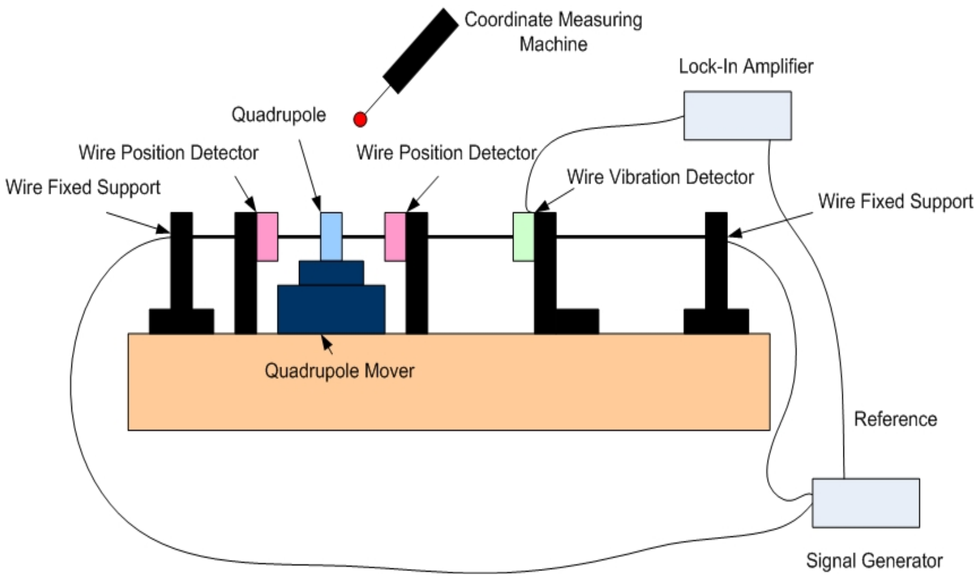

Figure 1: Overview of the vibrating wire system.

3.1 Step 1: Move the Quadrupole Magnetic Axis onto the Wire

The first step in the quadrupole fiducialization is to move the magnetic axis of the quadrupole onto the wire. As noted above, this alignment is determined by when the wire has no force on it and ceases vibrating. The vibrations are driven by alternating current in a magnetic field. In order to maximize the sensitivity of the system, the alternating current frequency is set to the resonant frequency of vibration of the wire. Since the wire is excited at resonance, even small offsets of the quadrupole from center cause large vibrations of the wire, which are easily detected.

As shown in figure 1, a signal generator is connected to the wire. The wire current is given by the output voltage divided by the wire resistance. No current amplifier is needed, due to the high sensitivity of the system. Wire vibration detectors, which are described below, output a signal proportional to the wire position. This signal is sent to a lock-in amplifier which measures the magnitude and phase of the signal. The lock-in reference comes from the signal generator. The frequency of the signal generator is adjusted until the wire vibration signal is maximum and 90° out of phase with the wire current, which determines the resonance condition. The quadrupole is on a mover and the quadrupole position is adjusted until the signal measured by the lock-in amplifier goes to zero. This process is repeated for magnet motions in x, y, pitch, and yaw. At his point the quadrupole is aligned to the wire and the first step of the fiducialization is complete.

The wire vibration detector assembly consists of two orthogonal detectors, one for vertical wire motion and one for horizontal wire motion. Each detector assembly consists of a laser shining onto a slit, with a photodiode behind the slit as illustrated in figure 2. The

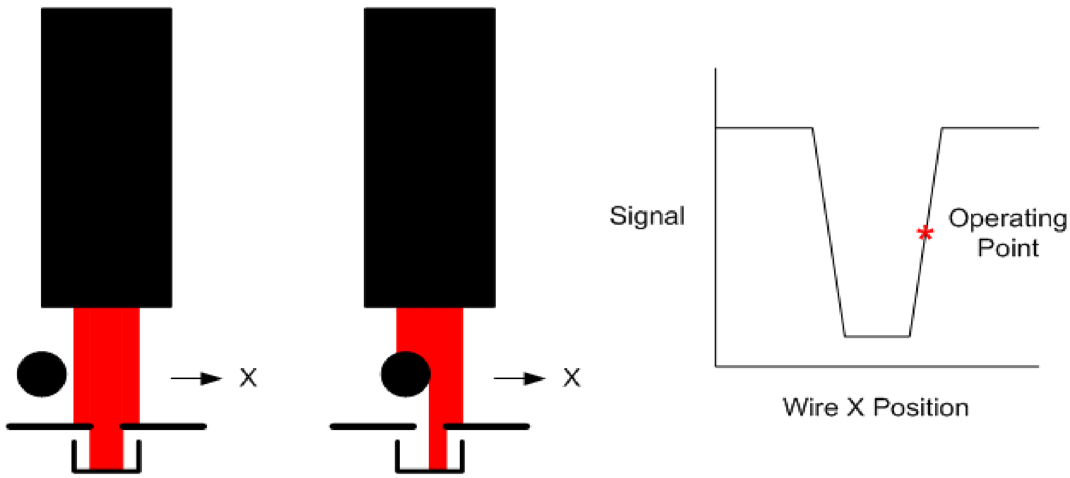

Figure 2: Wire vibration detector. As the wire moves across the part of the laser beam hitting the detector, the signal decreases. The signal is near zero when the wire covers the slit. It rises again as the slit is uncovered. The DC operating point of the detector is located where the signal is halfway through its range.

slit is much narrower than the wire. When the wire enters the part of the laser beam going through the slit, it casts a shadow and the signal from the photodiode decreases. This is illustrated in the figure. The signal goes from its full value to nearly zero as the wire moves a distance corresponding to the width of the slit. In practice, there are diffraction effects and a sensitivity reduction effect when the slit is not exactly parallel to the wire, but we ignore these effects when describing the basic operation of the detector.

Even with the imperfections noted above, this arrangement leads to a very sensitive wire vibration detector. For example, in the ideal case of a 10 μm wide slit and an output of 1 V from the detector with no wire shadow, the signal changes by approximately 0.1 V/μm as the wire edge moves past the slit. In practice, the slit can not be perfectly aligned to the wire and sensitivities of 0.01 V/μm are more common. This is still a large signal, allowing wire motions below 1 μm to be easily detected. The static operating point is where the wire covers half the slit. Then, when the wire vibrates, the output is a sinusoidal signal corresponding to the wire vibration.

3.2 Step 2: Locate the Wire Relative to Tooling Balls Without Touching the Wire

The second step in the quadrupole fiducialization is to locate the wire position relative to tooling balls, which effectively locates the magnetic axis relative to the tooling balls. This is done with wire position detectors, which are very similar to the wire vibration detectors. Note that the coordinate measuring machine can not locate the wire directly because it locates objects by touching them, and touching the wire will move it. The wire position detectors optically locate the wire relative to tooling balls, which the coordinate measuring machine can touch.

The wire position detector consists of a laser, slit, and photodiode in a housing with tooling balls on it. The assembly, illustrated in figure 3, is mounted on a stage with a

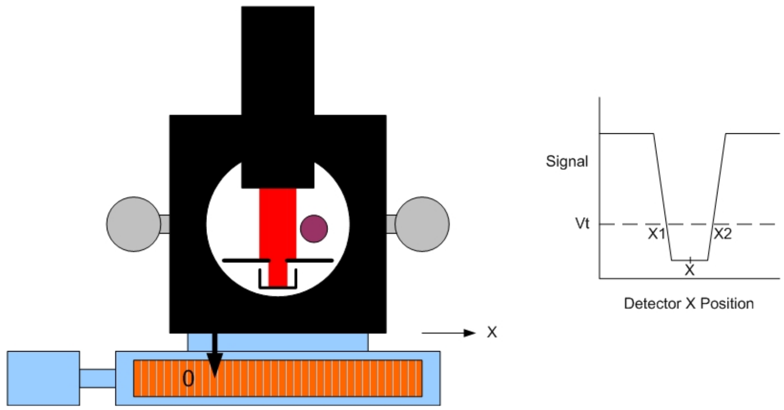

Figure 3: A wire position detector measures the distance from a tooling ball to the wire without touching the wire.

linear scale on it. The position of the wire as measured on the scale is obtained from both edges of the wire. To measure the wire position, the detector assembly is moved until the shadow of the wire causes the signal to decrease to a threshold voltage V_t, then the scale reading is recorded, x_1. The detector assembly is then moved across the diameter of the wire and when the signal again reaches V_t, the scale is recorded again, x_2. The position of the center of the wire is calculated as $x = \frac{1}{2}(x_1 + x_2)$.

The exact location of the slit and photodiode relative to the scale can not be directly measured. The scale reading, as described so far, is not an absolute measure of the wire position. Differences in the scale readings, however, would be an accurate measure of the distance between two wires. Suppose we set the zero position of the scale at the center of the first wire. Then the scale reading is the position of the wire being measured relative to the first wire. It is helpful to keep this picture in mind when discussing the $x = 0$ position on the scale and the distance from a tooling ball to the $x = 0$ position. To use the detector, we must know the distance from a tooling ball to the $x = 0$ position, and then add the scale reading to get the distance from a tooling ball to the wire. The distance from a tooling ball to the $x = 0$ position is found with a calibration.

The calibration of the wire position detector is performed as follows. The detector is put into a special fixture which positions one ball in a vee and the other on a flat as shown in figure 4. A wire is built into the fixture at a fixed position x_w relative to the vee. The ball labeled R, for reference, is placed into the vee. The detector is used to measure the wire position x_a on the scale. The position of the wire relative to ball R in the vee is $x_w = x_0 + x_a$. The detector is now flipped so that ball R is on the flat and the other ball is

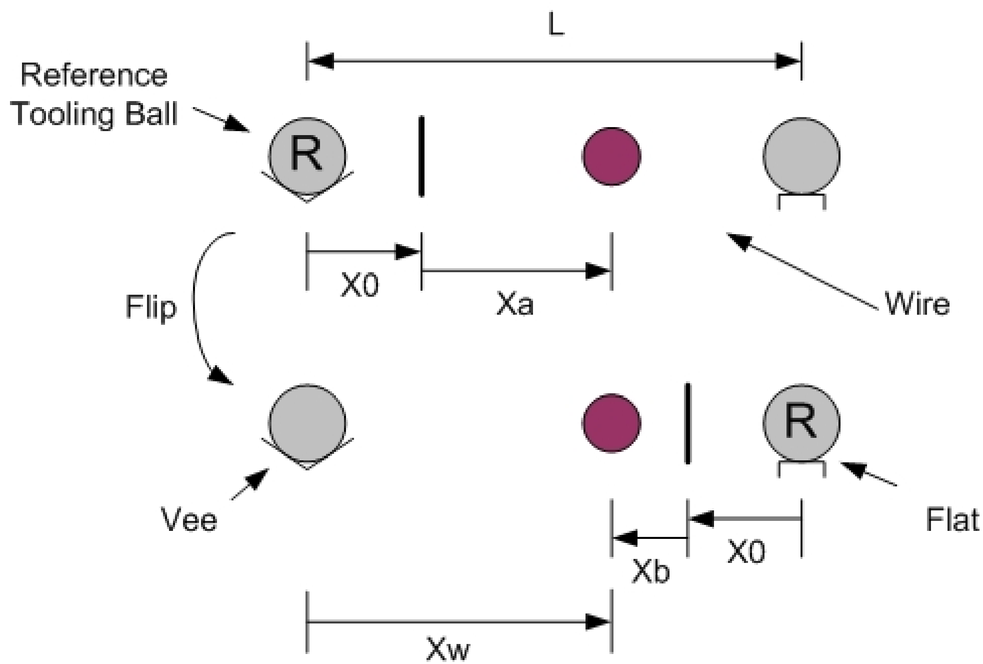

Figure 4: Schematic of the calibration procedure for the wire position detector.

in the vee. The two balls are a distance L apart, as measured using a coordinate measuring machine. The wire position is again determined on the scale, this time it is x_b. Relative to the ball in the vee, the wire is at $x_w = L - x_0 - x_b$. Since the wire position relative to the vee is fixed, x_w is the same in both cases, so

$$x_0 + x_a = L - x_0 - x_b \tag{1}$$

Solving for x_0, we get

$$x_0 = \frac{L}{2} - \frac{x_a + x_b}{2} \tag{2}$$

Once x_0 is known from the calibration fixture, the wire position detector can be used to locate an arbitrary wire relative to the reference ball using $x_w = x_0 + x$, where x is the scale reading of the center of the wire. Note that the wire position must be given in both the x and y directions on both sides of the quadrupole, resulting in four detectors. All detectors operate the same way.

Similar wire position detectors have previously been built and used for measurements. The position detectors were repeatable to 1.5 μm[13]. Based on this experience, we believe we can locate the wire position relative to the tooling balls on the detector housing to within a few microns. In order to verify this, a special fixture will be built which contains a short wire whose position is reliably given by the ends of the wire. A coordinate measuring

[13]Z. Wolf et al., "Alignment Tools Used To Locate A Wire And A Laser Beam In The Visa Undulator Project", Proceedings of the 6'th International Workshop on Accelerator Alignment-IWAA99 (1999).

machine will locate the dowel pins positioning the ends of the wire and we will compare to the value obtained from the wire position detector.

3.3 Step 3: Locate All Tooling Balls

The positions of the tooling balls on the wire position detectors and on the quadrupole are measured with a coordinate measuring machine. These machines are highly specialized and are designed for such measurements. We leave these measurements to the experts, but machine specifications show that the tooling ball locations will be measured to better than 10 μm[14].

One might wonder whether other devices, such as laser trackers could be used for this measurement. A cursory look at laser tracker specifications shows that they are not as accurate as coordinate measuring machines. Their typical position measurement accuracy is greater than 25 μm[15], which is outside our accuracy specification. Other choices for high accuracy measurements do not come to mind.

4 Equations of Motion for the Wire

4.1 Differential Equation, Vertical Wire Motion

The equations of motion describing simple vibrating wires are derived in a number of textbooks[16], however, the analysis of our specific problem including damping and gravity is not common. We derive the equations of motion for the vibrating wire and solve them in this note for completeness and consistency of notation so that the equations describing quadrupole fiducialization will be clear. We first consider the vertical motion of the wire and then discuss the horizontal motion.

Consider a section of wire of length dz as shown in figure 5. The slope of the wire is $(\partial y/\partial z)|_z$ on the left side of the segment and $(\partial y/\partial z)|_{z+dz}$ on the right side of the segment. The tension force on the wire is T acting in the z direction and is the same at both ends of the segment because the segment is assumed to not accelerate in the z direction. In the vertical y direction, the force on the segment is $-T(\partial y/\partial z)|_z$ from the wire to the left, and $T(\partial y/\partial z)|_{z+dz}$ from the wire to the right. The mass per unit length of the wire is m_l, the acceleration due to gravity is g, and the force per unit length from the current in the wire interacting with a magnetic field is F_l. As the wire moves, it experiences resistive forces such as air resistance. We model all such velocity dependent forces per unit length as $-\alpha_l \partial y/\partial t$ acting in the direction opposite to the velocity. The acceleration of the wire segment is $\partial^2 y/\partial t^2$. Writing Newton's law for the wire segment in the y-z plane gives

$$T \left(\frac{\partial y}{\partial z}\right)\bigg|_{z+dz} - T \left(\frac{\partial y}{\partial z}\right)\bigg|_{z} - m_l\, dz\, g + F_l\, dz - \alpha_l \frac{\partial y}{\partial t}\, dz = m_l\, dz\, \frac{\partial^2 y}{\partial t^2} \tag{3}$$

Dividing through by dz and rearranging terms we find

$$m_l \frac{\partial^2 y}{\partial t^2} + \alpha_l \frac{\partial y}{\partial t} - T \frac{\partial^2 y}{\partial z^2} = -m_l\, g + F_l \tag{4}$$

[14] See, for instance, Prismo 10 specifications, Carl Zeiss Industrial Metrology, Oberkochen, Germany.
[15] See, for instance, the LT800 from Leica Geosystems, Unterentfelden, Switzerland.
[16] See, for instance, P. Wallace, **Mathematical Analysis of Physical Problems**, Dover, 1984.

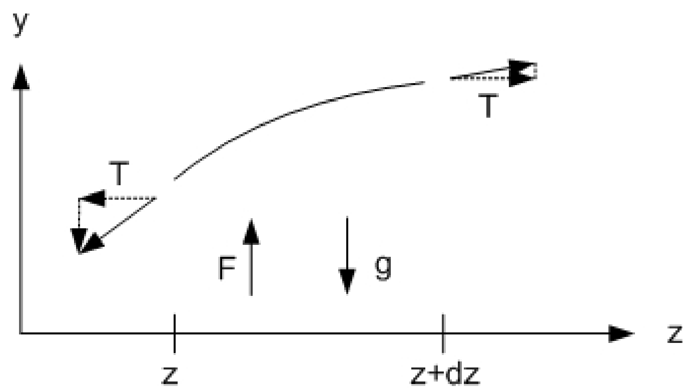

Figure 5: Segment of the wire showing the tension T acting at the ends, and the force per unit length F and gravity acting in the y direction.

The force per unit length F_l in the y direction from a current $I(t)$ in the z direction and a magnetic field $B_x(z)$ in the x direction is $F_l = I(t)B_x(z)$. Thus, the equation we wish to solve is

$$m_l \frac{\partial^2 y}{\partial t^2} + \alpha_l \frac{\partial y}{\partial t} - T\frac{\partial^2 y}{\partial z^2} = -m_l\, g + I(t)B_x(z) \tag{5}$$

To solve this equation, we must also specify the boundary conditions. (The initial conditions are not important for our present solution since we are interested in the steady state.) We fix one end of the wire at $z = 0$ and the other end at $z = L$ for a wire of length L. The boundary conditions are then $y(z = 0, t) = 0$ and $y(z = L, t) = 0$. Given these boundary conditions, we solve the differential equation for $y(z, t)$ by first finding the homogeneous solution and then finding the particular solutions for the two driving terms on the right hand side.

4.2 Homogeneous Solution

The homogeneous solution to the differential equation is found by setting the driving terms to zero. The differential equation for the homogeneous solution $y_h(z, t)$ is

$$m_l \frac{\partial^2 y_h}{\partial t^2} + \alpha_l \frac{\partial y_h}{\partial t} - T\frac{\partial^2 y_h}{\partial z^2} = 0 \tag{6}$$

with boundary conditions $y_h(z = 0, t) = 0$ and $y_h(z = L, t) = 0$. We use separation of variables to write $y_h(z, t) = Y_z(z)Y_t(t)$. Dividing by the tension and rearranging terms, we find

$$\frac{1}{Y_t}\left(\frac{m_l}{T}\frac{d^2Y_t}{dt^2} + \frac{\alpha_l}{T}\frac{dY_t}{dt}\right) = \frac{1}{Y_z}\frac{d^2Y_z}{dz^2} \tag{7}$$

Since both sides are functions of different variables, they must be constant. We set both sides equal to $-k^2$.

The equation for Y_z becomes

$$\frac{d^2Y_z}{dz^2} + k^2 Y_z = 0 \tag{8}$$

with $Y_z(0) = 0$ and $Y_z(L) = 0$. The solution is

$$Y_z(z) = A_n^{'} \sin\left(\frac{n\pi z}{L}\right), \ (n = 1, 2, 3, ...) \tag{9}$$

for given n, with $A_n^{'}$ an arbitrary constant.

The equation for Y_t for given n is

$$\frac{m_l}{T}\frac{d^2 Y_t}{dt^2} + \frac{\alpha_l}{T}\frac{dY_t}{dt} + \left(\frac{n\pi}{L}\right)^2 Y_t = 0 \tag{10}$$

Let $\alpha = \alpha_l/m_l$ and $\omega_n^2 = (T/m_l)\,(n\pi/L)^2$. With these substitutions, the equation becomes

$$\frac{d^2 Y_t}{dt^2} + \alpha\frac{dY_t}{dt} + \omega_n^2 Y_t = 0 \tag{11}$$

Using trial solution $Y_t = e^{st}$ and solving for s, we find $s = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \omega_n^2}$. We take $\alpha \ll \omega_n$, so $s = -\frac{\alpha}{2} \pm i\sqrt{\omega_n^2 - \left(\frac{\alpha}{2}\right)^2}$. The solution for Y_t for given n is then

$$Y_t(t) = A_n^{''} e^{-\frac{\alpha}{2}t} e^{i\sqrt{\omega_n^2 - \left(\frac{\alpha}{2}\right)^2}\,t} + B_n^{''} e^{-\frac{\alpha}{2}t} e^{-i\sqrt{\omega_n^2 - \left(\frac{\alpha}{2}\right)^2}\,t} \tag{12}$$

with $A_n^{''}$ and $B_n^{''}$ arbitrary complex constants.

Combining terms, the homogeneous solution is given by

$$y_h(z,t) = \mathrm{Re}\sum_{n=1}^{\infty}\left[A_n e^{-\frac{\alpha}{2}t} e^{i\sqrt{\omega_n^2 - \left(\frac{\alpha}{2}\right)^2}\,t} + B_n e^{-\frac{\alpha}{2}t} e^{-i\sqrt{\omega_n^2 - \left(\frac{\alpha}{2}\right)^2}\,t}\right]\sin\left(\frac{n\pi z}{L}\right) \tag{13}$$

with $\alpha = \alpha_l/m_l$, $\omega_n^2 = (T/m_l)\,(n\pi/L)^2$, A_n and B_n arbitrary complex constants, and Re representing the real part of the expression.

The homogeneous solution can be used to measure the decay constant α. With no current in the wire, the wire can be plucked and the time for the vibration to decay away can be measured. If τ is the decay time, $\alpha = 2/\tau$. In addition, the vibration frequency of the plucked wire can be measured. The lowest frequency of vibration is $\sqrt{\omega_1^2 - \left(\frac{\alpha}{2}\right)^2}$. Knowing α, this allows us to determine ω_1. Since the wire tension, mass per unit length, and length can be determined independently, measuring ω_1 provides a check on the behavior of the system.

4.3 Particular Solution Due To Gravity

We now seek a solution to the differential equation for the wire motion with only the first driving term due to gravity

$$m_l\frac{\partial^2 y_g}{\partial t^2} + \alpha_l\frac{\partial y_g}{\partial t} - T\frac{\partial^2 y_g}{\partial z^2} = -m_l\, g \tag{14}$$

Since the driving term is time independent, we seek a time independent solution $y_g(z)$ of the equation

$$\frac{d^2 y_g}{dz^2} = \frac{m_l\, g}{T} \tag{15}$$

Integrating once, we get

$$\frac{dy_g}{dz} = \frac{m_l\, g}{T} z + C_1 \tag{16}$$

where C_1 is a constant. Integrating again, we get

$$y_g = \frac{m_l\, g}{2T} z^2 + C_1 z + C_2 \tag{17}$$

where C_2 is another constant.

Using the boundary condition that $y_g(0) = 0$ and $y_g(L) = 0$, we determine C_1 and C_2 and arrive at the solution

$$y_g(z) = \frac{m_l\, g}{2T} z(z - L) \tag{18}$$

Note that y_g is negative since $0 < z < L$.

The sag of the wire is given by the minimum value of y_g. This occurs when $dy_g/dz = 0$, which is at $z = L/2$. The negative of the value of y_g at this point is defined to be the sag s.

$$s = \frac{m_l g L^2}{8T} \tag{19}$$

The sag is related to the fundamental frequency of vibration ω_1

$$\omega_1^2 = \frac{\pi^2 T}{m_l L^2} \tag{20}$$

Writing s in terms of ω_1, we find the relation

$$s = \frac{g\pi^2}{8\omega_1^2} \tag{21}$$

Using the relation $\omega_1 = 2\pi f_1$, we can rewrite this relation as

$$s = \frac{g}{32 f_1^2} \tag{22}$$

Since the vibration frequency f_1 can be accurately determined and the gravity constant g is well known, equation 22 leads to an accurate determination of the sag, which in our case of a short, light wire under high tension would otherwise be difficult to measure directly.

4.4 Particular Solution Due To Wire Current And External Magnetic Field

We next seek a solution to the differential equation for the wire motion with the second driving term due to the current in the wire interacting with the magnetic field of the magnet

$$m_l \frac{\partial^2 y_B}{\partial t^2} + \alpha_l \frac{\partial y_B}{\partial t} - T \frac{\partial^2 y_B}{\partial z^2} = I(t) B_x(z) \tag{23}$$

The current in the wire varies sinusoidally with time and we may represent it as $I(t) = I_0 e^{i\omega t}$, and then use the real part of y_B to represent the solution to the equation. We look for a particular solution with the same time dependence, $y_B(z,t) = Y_B(z)e^{i\omega t}$. Inserting this form for the solution into the differential equation, we get

$$-\omega^2 m_l Y_B + i\omega\alpha_l Y_B - T\frac{d^2 Y_B}{dz^2} = I_0 B_x(z) \tag{24}$$

The boundary conditions are $Y_B(0) = 0$ and $Y_B(L) = 0$.

To solve for $Y_B(z)$, we expand both $Y_B(z)$ and $B_x(z)$ in Fourier sine series.

$$Y_B(z) = \sum_{n=1}^{\infty} Y_{Bn} \sin\left(\frac{n\pi z}{L}\right) \tag{25}$$

$$B_x(z) = \sum_{n=1}^{\infty} B_{xn} \sin\left(\frac{n\pi z}{L}\right) \tag{26}$$

where

$$Y_{Bn} = \frac{2}{L}\int Y_B(z)\,\sin\left(\frac{n\pi z}{L}\right)\,dz \tag{27}$$

$$B_{xn} = \frac{2}{L}\int B_x(z)\,\sin\left(\frac{n\pi z}{L}\right)\,dz \tag{28}$$

Equation 24 then becomes

$$\left[-\omega^2 m_l + i\omega\alpha_l + T\left(\frac{n\pi}{L}\right)^2\right] Y_{Bn} = I_0 B_{xn} \tag{29}$$

Dividing through by m_l and using the definitions $\alpha = \alpha_l/m_l$ and $\omega_n^2 = (T/m_l)\,(n\pi/L)^2$, we get

$$\left[-\omega^2 + i\omega\alpha + \omega_n^2\right] Y_{Bn} = \frac{1}{m_l} I_0 B_{xn} \tag{30}$$

This can be solved for Y_{Bn}

$$Y_{Bn} = \frac{-I_0 B_{xn}}{m_l\left(\omega^2 - \omega_n^2 - i\omega\alpha\right)} \tag{31}$$

With Y_{Bn}, we can write down the expression for $Y_B(z)$

$$Y_B(z) = \sum_{n=1}^{\infty} \frac{-I_0 B_{xn}}{m_l\left(\omega^2 - \omega_n^2 - i\omega\alpha\right)} \sin\left(\frac{n\pi z}{L}\right) \tag{32}$$

Knowing $Y_B(z)$, we can write the particular solution due to the driving term from the magnetic field

$$y_B(z,t) = \mathrm{Re}\sum_{n=1}^{\infty} \frac{-I_0 B_{xn}}{m_l\left(\omega^2 - \omega_n^2 - i\omega\alpha\right)} \sin\left(\frac{n\pi z}{L}\right)\, e^{i\omega t} \tag{33}$$

Note that if the wire current frequency is chosen to be near a natural vibration frequency of the wire, one term in the series will dominate and

$$y_B^{res}(z,t) \simeq \frac{I_0 B_{xn}}{m_l \omega_n \alpha} \sin\left(\frac{n\pi z}{L}\right)\, \cos\left(\omega_n t - \frac{\pi}{2}\right) \tag{34}$$

This motion of the wire is fundamental to the vibrating wire technique and will be discussed further below.

4.5 General Solution

The general solution to the differential equation governing vertical wire motion is given by the sum of the homogeneous solution and the two particular solutions

$$y(z,t) = y_h(z,t) + y_g(z) + y_B(z,t) \tag{35}$$

Writing this out, we get

$$\begin{aligned} y(z,t) \quad = \quad & \mathrm{Re}\sum_{n=1}^{\infty}\left[A_n e^{-\frac{\alpha}{2}t} e^{i\sqrt{\omega_n^2-\left(\frac{\alpha}{2}\right)^2}t} + B_n e^{-\frac{\alpha}{2}t} e^{-i\sqrt{\omega_n^2-\left(\frac{\alpha}{2}\right)^2}t}\right]\sin\left(\frac{n\pi z}{L}\right) \\ & + \frac{m_l\, g}{2T} z(z-L) + \mathrm{Re}\sum_{n=1}^{\infty}\frac{-I_0 B_{xn}}{m_l\left(\omega^2-\omega_n^2-i\omega\alpha\right)}\sin\left(\frac{n\pi z}{L}\right)\, e^{i\omega t} \end{aligned} \tag{36}$$

In this expression $\alpha = \alpha_l/m_l$ and $\omega_n^2 = (T/m_l)\,(n\pi/L)^2$. The coefficients in the homogeneous solution are determined by the initial conditions.

4.6 Horizontal Wire Motion

Up to now, we have been considering the vertical vibrations of the wire. The horizontal vibrations of the wire are treated in an almost identical manner. Only two changes need to be made. First, we no longer need to worry about gravity, so the term involving g is not present. Second, the horizontal force on the wire is caused by the vertical magnetic field. The force per unit length F_l in the x direction from a current $I(t)$ in the z direction and a magnetic field $B_y(z)$ in the y direction is $F_l = -I(t)B_y(z)$. There is a sign change relative to the vertical force. With these changes, we can immediately write down the equation describing the horizontal motion of the wire

$$\begin{aligned} x(z,t) \quad = \quad & \mathrm{Re}\sum_{n=1}^{\infty}\left[\bar{A}_n e^{-\frac{\alpha}{2}t} e^{i\sqrt{\omega_n^2-\left(\frac{\alpha}{2}\right)^2}t} + \bar{B}_n e^{-\frac{\alpha}{2}t} e^{-i\sqrt{\omega_n^2-\left(\frac{\alpha}{2}\right)^2}t}\right]\sin\left(\frac{n\pi z}{L}\right) \\ & + \mathrm{Re}\sum_{n=1}^{\infty}\frac{I_0 B_{yn}}{m_l\left(\omega^2-\omega_n^2-i\omega\alpha\right)}\sin\left(\frac{n\pi z}{L}\right)\, e^{i\omega t} \end{aligned} \tag{37}$$

4.7 Discussion

The general solution for the vertical wire motion consists of three terms, the homogeneous solution and two particular solutions. In general, the homogeneous solution will decay away exponentially and will not be used in the measurements. It can be used, however, to measure α and ω_1, as noted above.

Once ω_1 is known, the wire sag can be calculated as in equation 22. The wire sag is important because it determines whether or not we can use the ends of the wire to determine the wire position for fiducialization. If the sag is too large, we must measure the wire position near the quadrupole we are fiducializing.

The third term in the expression for $y(z,t)$ is the primary one we will use for fiducialization. We will set the wire current frequency to a natural frequency ω_n, so $I(t) = I_0\cos(\omega_n t)$.

We use the wire current as a phase reference and adjust the frequency until the wire motion is 90° out of phase using the signal from the wire position sensor and a lock-in amplifier. The steady state wire motion about the sagged wire position is then given by

$$y_B^{res}(z,t) \simeq \frac{I_0 B_{xn}}{m_l \omega_n \alpha} \sin\left(\frac{n\pi z}{L}\right) \cos\left(\omega_n t - \frac{\pi}{2}\right) \tag{38}$$

$$x_B^{res}(z,t) \simeq \frac{I_0 B_{yn}}{m_l \omega_n \alpha} \sin\left(\frac{n\pi z}{L}\right) \cos\left(\omega_n t + \frac{\pi}{2}\right) \tag{39}$$

As the quadrupole is moved in y, for example, B_{xn} changes. When the quadrupole is centered on the wire $B_{xn} = 0$ and the wire vibration in the vertical direction goes to zero. This outlines the general technique. A full discussion will follow.

5 Quadrupole Fiducialization

5.1 Quadrupole Vertical Position

To determine the vertical position of the axis of the quadrupole, the wire is excited at the second harmonic of the natural frequency of vibration of the wire. The wire will vibrate as shown in figure 6. We use $n = 2$ so that the system is insensitive to constant fields, like the Earth's field, which have $B_{x2} = 0$, and so do not affect the measurement. This helps eliminate offsets in the measurements. For $n = 2$, the quadrupole is placed at one peak of

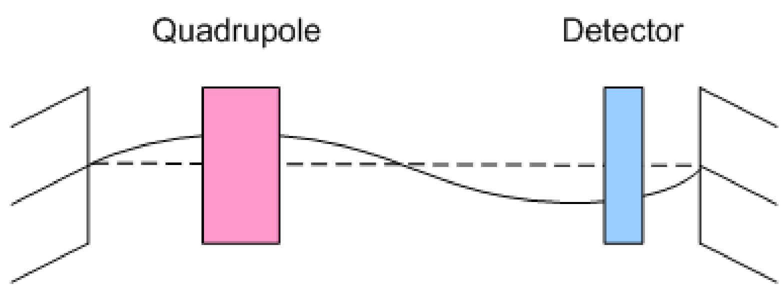

Figure 6: Wire vibrating at the second harmonic.

the vibration at $z = L/4$ and the detector is placed at $z = 7L/8$. The detector location is chosen to be at a peak of vibration of the fourth harmonic. This improves the pitch measurement as shown below, but does not significantly degrade the position measurement.

The wire vibration is described by equation 38 with $n = 2$ since we are driving the wire at the second harmonic.

$$y_B^{res}(z,t) \simeq \frac{I_0 B_{x2}}{m_l \omega_2 \alpha} \sin\left(\frac{2\pi z}{L}\right) \cos\left(\omega_2 t - \frac{\pi}{2}\right)$$

The coefficient B_{x2} in the sine expansion of the magnetic field along the wire is given by

$$B_{x2} = \frac{2}{L} \int_0^L B_x(z) \sin\left(\frac{2\pi z}{L}\right) dz \tag{40}$$

We assume that the quadrupole has constant offset from the wire and so has a constant field B_x along its length. The quadrupole of length L_Q is centered at $z = L/4$. B_{x2} is then given by

$$B_{x2} = \frac{2}{L} \int_{L/4-L_Q/2}^{L/4+L_Q/2} B_x \sin\left(\frac{2\pi z}{L}\right) dz \tag{41}$$

where B_x depends on the y position of the quadrupole, but not z. Performing the integral, we get

$$B_{x2} = -\frac{2}{L} B_x \frac{L}{2\pi} \cos\left(\frac{2\pi z}{L}\right)\Bigg|_{L/4-L_Q/2}^{L/4+L_Q/2} \tag{42}$$

Expanding, we get

$$B_{x2} = \frac{2B_x}{\pi} \sin\left(\frac{\pi L_Q}{L}\right) \tag{43}$$

For $L_Q \ll L$, we approximate $\sin(\frac{\pi L_Q}{L})$ by $\frac{\pi L_Q}{L}$ to get

$$B_{x2} = \frac{2B_x L_Q}{L} \tag{44}$$

If we define the gradient of the quadrupole to be G and the vertical position of the magnet relative to the wire to be y, then the wire is at position $-y$ relative to the quadrupole center and B_x is given by

$$B_x = -Gy \tag{45}$$

B_{x2} can then be expressed as

$$B_{x2} = -\frac{2GL_Q\, y}{L} \tag{46}$$

From equation 38, the wire vibration at z as a function of t is then

$$y_B^{res}(z,t) \simeq -\frac{I_0 2GL_Q}{m_l \omega_2 \alpha L} y \sin\left(\frac{2\pi z}{L}\right) \cos\left(\omega_2 t - \frac{\pi}{2}\right) \tag{47}$$

The detector is at $z = 7L/8$. The vibration of the wire at the detector is given by

$$y_B^{res}\left(z = \frac{7L}{8}, t\right) \simeq \frac{I_0 \sqrt{2} GL_Q}{m_l \omega_2 \alpha L} y \cos\left(\omega_2 t - \frac{\pi}{2}\right) \tag{48}$$

The amplitude of the wire vibration at the detector varies linearly with the y position of the quadrupole and is given by

$$A_{\text{det}} = \left(\frac{\sqrt{2} I_0 GL_Q}{m_l \omega_2 \alpha L}\right) |y| \tag{49}$$

The phase of the vibration of the wire at the detector is

$$\Phi_{\text{det}} = -sign(y)\left(\frac{\pi}{2}\right) \tag{50}$$

For $y > 0$, $\Phi_{\text{det}} = -\pi/2$. For $y < 0$, $\Phi_{\text{det}} = +\pi/2$. Of course, this assumes a convention that $B_x > 0$ for $y < 0$ and $B_x < 0$ for $y > 0$, where y is the quadrupole position. The phase change is very sharp at the quadrupole center.

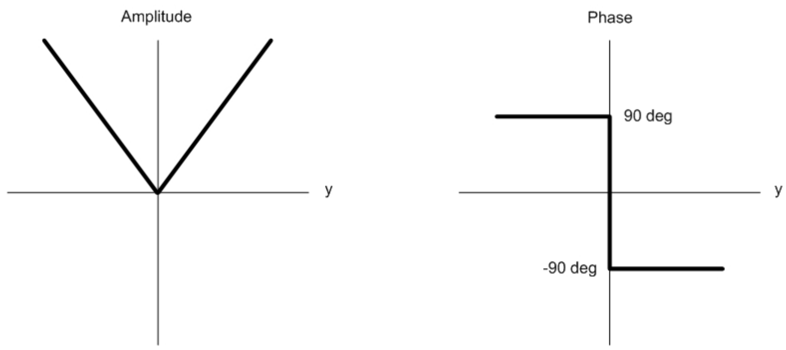

Figure 7: Amplitude and phase of the wire vibration at the detector vs the y position of the quadrupole.

Plotting the amplitude and phase of the wire vibration at the detector gives the result shown in figure 7. The center of the quadrupole is aligned with the wire when the amplitude of the vibration goes to zero and the phase changes by 180°. The slope of the lines in the amplitude vs y graph are given by equation 49.

5.2 Quadrupole Pitch

Quadrupole pitch is illustrated in figure 8. Pitch is a vertical angle between the quadrupole axis and the wire axis in which the y coordinate of the magnetic axis changes with z. We use the fourth harmonic to measure pitch. The quadrupole center is still at at $z = L/4$, as for the vertical position measurement, but it is now centered on a node of the vibration. The detector at $z = 7L/8$ is now at a peak of the vibration.

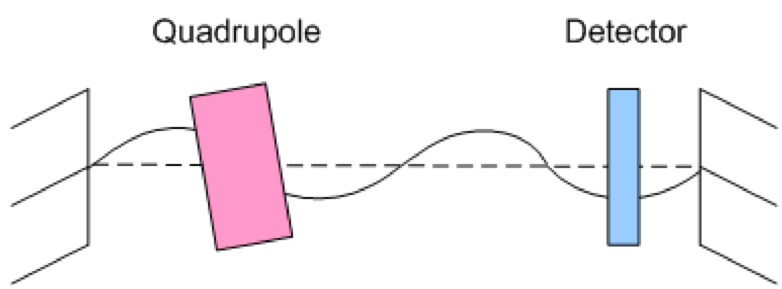

Figure 8: Quadrupole pitch is measured with the fourth harmonic frequency.

The wire vibration is described by equation 38 with $n = 4$ since we are using the fourth harmonic

$$y_B^{res}(z,t) \simeq \frac{I_0 B_{x4}}{m_l \omega_4 \alpha} \sin\left(\frac{4\pi z}{L}\right) \cos\left(\omega_4 t - \frac{\pi}{2}\right) \tag{51}$$

The coefficient B_{x4} in the sine expansion of the magnetic field along the wire is given by

$$B_{x4} = \frac{2}{L}\int_0^L B_x(z) \sin\left(\frac{4\pi z}{L}\right) dz \tag{52}$$

The axis of the quadrupole is given by the line $y_{axis} = \theta(z - L/4)$, where θ is the small pitch angle. The field B_x at the wire is given by the gradient times the y position of the wire relative to the axis, which is $-y_{axis}$. So $B_x(z)$ is given by

$$B_x(z) = -G\theta\left(z - \frac{L}{4}\right) \tag{53}$$

Inserting $B_x(z)$ into equation 52 including the extent of the quadrupole along z, we get

$$B_{x4} = \frac{2}{L}\int_{L/4-L_Q/2}^{L/4+L_Q/2} -G\theta\left(z - \frac{L}{4}\right)\sin\left(\frac{4\pi z}{L}\right)dz \tag{54}$$

Performing the integral, we get

$$B_{x4} = -\frac{2G\theta}{L}\left[-\left(z - \frac{L}{4}\right)\left(\frac{L}{4\pi}\right)\cos\left(\frac{4\pi z}{L}\right) + \left(\frac{L}{4\pi}\right)^2\sin\left(\frac{4\pi z}{L}\right)\right]_{L/4-L_Q/2}^{L/4+L_Q/2} \tag{55}$$

After the limits are put in, the expression for B_{x4} becomes

$$\begin{aligned} B_{x4} \quad = \quad & -\frac{2G\theta}{L}\left[-\left(\frac{L_Q}{2}\right)\left(\frac{L}{4\pi}\right)\cos\left(\pi + \frac{2\pi L_Q}{L}\right) + \left(\frac{L}{4\pi}\right)^2\sin\left(\pi + \frac{2\pi L_Q}{L}\right)\right. \\ & \left. + \left(-\frac{L_Q}{2}\right)\left(\frac{L}{4\pi}\right)\cos\left(\pi - \frac{2\pi L_Q}{L}\right) - \left(\frac{L}{4\pi}\right)^2\sin\left(\pi - \frac{2\pi L_Q}{L}\right)\right] \end{aligned} \tag{56}$$

This equation simplifies to

$$B_{x4} = -\frac{2G\theta}{L}\left[2\frac{L_Q L}{8\pi}\cos\left(2\pi\frac{L_Q}{L}\right) - 2\left(\frac{L}{4\pi}\right)^2\sin\left(2\pi\frac{L_Q}{L}\right)\right] \tag{57}$$

For $L_Q \ll L$, we approximate $\sin(\frac{2\pi L_Q}{L})$ by $\frac{2\pi L_Q}{L} - \frac{1}{6}\left(\frac{2\pi L_Q}{L}\right)^3$ and $\cos(\frac{2\pi L_Q}{L})$ by $1 - \frac{1}{2}\left(\frac{2\pi L_Q}{L}\right)^2$ to get

$$B_{x4} = -\frac{2G\theta}{L}\left[\frac{L_Q L}{4\pi}\left(1 - \frac{1}{2}\left(\frac{2\pi L_Q}{L}\right)^2\right) - 2\left(\frac{L}{4\pi}\right)^2\left(\frac{2\pi L_Q}{L} - \frac{1}{6}\left(\frac{2\pi L_Q}{L}\right)^3\right)\right] \tag{58}$$

This simplifies to

$$B_{x4} = -\frac{2G\theta}{L}\left(-\frac{\pi L_Q^3}{2L} + \frac{\pi L_Q^3}{6L}\right) \tag{59}$$

which further simplifies to the result:

$$B_{x4} = \frac{2\pi}{3}GL_Q\left(\frac{L_Q}{L}\right)^2\theta \tag{60}$$

Inserting this expression for B_{x4} into equation 51, the wire vibration at z as a function of t is given by

$$y_B^{res}(z,t) \simeq \frac{I_0}{m_l \omega_4 \alpha} \frac{2\pi}{3} G L_Q \left(\frac{L_Q}{L}\right)^2 \theta \sin\left(\frac{4\pi z}{L}\right) \cos\left(\omega_4 t - \frac{\pi}{2}\right) \tag{61}$$

The detector is at $z = 7L/8$. The vibration of the wire at the detector is given by

$$y_B^{res}\left(z = \frac{7L}{8}, t\right) \simeq -\frac{I_0}{m_l \omega_4 \alpha} \frac{2\pi}{3} G L_Q \left(\frac{L_Q}{L}\right)^2 \theta \cos\left(\omega_4 t - \frac{\pi}{2}\right) \tag{62}$$

The amplitude of the wire vibration at the detector varies linearly with the pitch angle θ of the quadrupole and is given by

$$A_{\mathrm{det}} = \left(\frac{I_0}{m_l \omega_4 \alpha} \frac{2\pi}{3} G L_Q \left(\frac{L_Q}{L}\right)^2\right) |\theta| \tag{63}$$

The phase of the vibration of the wire at the detector is

$$\Phi_{\mathrm{det}} = sign(\theta) \left(\frac{\pi}{2}\right) \tag{64}$$

For $\theta > 0$, $\Phi_{\mathrm{det}} = \pi/2$, and for $\theta < 0$, $\Phi_{\mathrm{det}} = -\pi/2$.

5.3 Quadrupole Horizontal Position

The horizontal position of the quadrupole affects the horizontal vibration of the wire in an analogous manner to the vertical position, but with the phase change which was mentioned previously. Using the second harmonic for the measurement, we get the following relation

$$x_B^{res}\left(z = \frac{7L}{8}, t\right) \simeq \frac{I_0 \sqrt{2} G L_Q}{m_l \omega_2 \alpha L} x \cos\left(\omega_2 t + \frac{\pi}{2}\right) \tag{65}$$

The amplitude of the wire vibration at the detector varies linearly with the x position of the quadrupole and is given by

$$A_{\mathrm{det}} = \left(\frac{\sqrt{2} I_0 G L_Q}{m_l \omega_2 \alpha L}\right) |x| \tag{66}$$

The phase of the vibration of the wire at the detector is

$$\Phi_{\mathrm{det}} = sign(x) \left(\frac{\pi}{2}\right) \tag{67}$$

For $x > 0$, $\Phi_{\mathrm{det}} = +\pi/2$. For $x < 0$, $\Phi_{\mathrm{det}} = -\pi/2$. This assumed a convention that $B_y > 0$ for $x < 0$ and $B_y < 0$ for $x > 0$, where x is the quadrupole position relative to the wire.

5.4 Quadrupole Yaw

Yaw is the angle of the quadrupole center line relative to the wire in the x direction. It is similar to the pitch angle in the y direction. Adapting the pitch angle equations and again using the fourth harmonic for the measurement, we get

$$y_B^{res}\left(z=\frac{7L}{8},t\right)\simeq-\frac{I_0}{m_l\omega_4\alpha}\frac{2\pi}{3}GL_Q\left(\frac{L_Q}{L}\right)^2\phi\cos\left(\omega_4 t+\frac{\pi}{2}\right) \tag{68}$$

where ϕ is the yaw angle. The amplitude of the wire vibration at the detector varies linearly with the yaw angle ϕ of the quadrupole and is given by

$$A_{\text{det}}=\left(\frac{I_0}{m_l\omega_4\alpha}\frac{2\pi}{3}GL_Q\left(\frac{L_Q}{L}\right)^2\right)|\phi| \tag{69}$$

The phase of the vibration of the wire at the detector is

$$\Phi_{\text{det}}=-sign\left(\phi\right)\left(\frac{\pi}{2}\right) \tag{70}$$

For $\phi>0$, $\Phi_{\text{det}}=-\pi/2$. For $\phi<0$, $\Phi_{\text{det}}=+\pi/2$.

5.5 Quadrupole Roll And z

The vibrating wire technique is not suitable for roll or axial center position measurements. The axial or z position is typically not specified with a tight tolerance. Conventional mechanical measurements are adequate for this coordinate.

The LCLS quadrupoles have a roll angle tolerance of 10 mrad[17]. This is a very loose tolerance and easy to achieve with mechanical measurements of the magnet poles on the coordinate measuring machine during the fiducialization process.

5.6 Effect Of Positioning Errors

In the preceding analysis we assumed that the magnet position and vibration detector position had the specified values without error. We now consider whether the results change appreciably if there is a positioning error.

It is plausible from the above discussion that small errors in the detector position will have a small effect on the pitch and yaw measurements since the detector is placed at a peak in the vibration amplitude. The x and y position sensitivities will change slightly with a small position error, but the location of the magnet center on the wire will not change.

Similarly, small errors in the magnet position along the wire will not affect the horizontal or vertical position sensitivity because if we move the magnet along the wire by Δz, then B_{x2} in equation 41 becomes

$$B_{x2}=\frac{2}{L}\int_{L/4+\Delta z-L_Q/2}^{L/4+\Delta z+L_Q/2}B_x\sin\left(\frac{2\pi z}{L}\right)dz \tag{71}$$

[17] LCLS parameter database http://www-ssrl.slac.stanford.edu/htbin/rdbweb/LCLS_params_DB_public.

It turns out that B_{x2} is independent of Δz to first order. The results given above remain unchanged to first order.

The pitch and yaw measurement assumed the quadrupole to be longitudinally centered on a node of the wire vibration. In this case, one wonders about the effect of an error Δz_M in the magnet position or an error Δz_R in the point the magnet is rotated about. The effect is determined by calculating B_{x4} including the errors

$$B_{x4} = \frac{2}{L} \int_{L/4+\Delta z_M - L_Q/2}^{L/4+\Delta z_M + L_Q/2} -G\theta \left(z - \frac{L}{4} - \Delta z_R \right) \sin \left(\frac{4\pi z}{L} \right) dz \tag{72}$$

After a lengthy calculation, one finds

$$B_{x4} = \frac{2\pi}{3} G L_Q \left(\frac{L_Q}{L} \right)^2 \theta \left[1 + 12 \left(\frac{\Delta z_M}{L_Q} \right)^2 - 12 \left(\frac{\Delta z_R}{L_Q} \right) \left(\frac{\Delta z_M}{L_Q} \right) + \ldots \right] \tag{73}$$

This agrees with our previous result, but with corrections to the sensitivity which are small as long as the position errors Δz_M and Δz_R are small compared to the magnet length. We take this to be the case and conclude that our previous result remains adequate in spite of small positioning errors. Note that even though the sensitivity has a small change, the final position of the magnet at $\theta = 0$, $\phi = 0$ does not change.

6 Sample Calculations

In order to get a feel for the performance of a vibrating wire system, we now insert typical values in the formulas derived so far to determine the sensitivities to quadrupole position. We must first choose the wire type and the system parameters. Sensitivities can then be calculated.

The wire must be non-magnetic so that we don't have steady state forces in the magnetic field gradient of the quadrupole. The wire must also have high tensile strength so that large tension can be applied to minimize sag. The diameter should be small so that the stiffness of the wire is not a factor. On the other hand, the diameter must be large enough so that it can be handled in a production environment. In our experience, 4 mil (100 μm) diameter copper beryllium wire is the best choice. Properties of the wire are listed in table 1[18].

For the LCLS quadrupoles, we assume a typical length of the wire of 1.5 m. We take the wire tension to be 80% of the maximum tension, or 9 N. Given the length of the wire, its electrical resistance will be 14.2 Ω. If we use a signal generator with 10 V amplitude and 50 Ω internal impedance, the current in the wire will have an amplitude of approximately 0.15 A. As a rough estimate, if we pluck the wire, we assume the vibrations will damp out in 1 sec. This means $\alpha = 2$ 1/s. The LCLS quadrupoles have an integrated gradient of 3 T and a length of 0.05 m[19]. We summarize these parameters of the measurement system in table 2.

[18] Little Falls Alloys, Paterson, New Jersey.

[19] LCLS parameter database http://www-ssrl.slac.stanford.edu/htbin/rdbweb/LCLS_params_DB_public.

Property	**Value**	**Units**
Mass density	8.35×10^3	$\mathrm{kg/m^3}$
Resistivity	7.68×10^{-8}	$\Omega\mathrm{m}$
Tensile strength	1.4×10^9	$\mathrm{N/m^2}$
Diameter	1.016×10^{-4}	m
Area	8.11×10^{-9}	$\mathrm{m^2}$
Mass per unit length	6.77×10^{-5}	kg/m
Resistance per unit length	9.47	Ω/m
Tension (max)	11.4 [2.57]	N [lbs]

Table 1: Properties of copper beryllium wire chosen for the measurement system.

Parameter	**Value**	**Units**
Wire length L	1.5	m
Wire tension T	9	N
Wire mass per unit length m_l	6.77×10^{-5}	kg/m
Damping constant α	2	1/s
Gravitational constant g	9.81	$\mathrm{m/s^2}$
Wire current amplitude I_0	0.15	A
Quadrupole integrated gradient GL_Q	3	T
Quadrupole length L_Q	0.05	m

Table 2: Parameters of a typical vibrating wire measurement system for LCLS quadrupoles.

6.1 Fundamental Frequency

The fundamental frequency of vibration of the undamped wire is given by

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{m_l}} \tag{74}$$

Inserting values, we find

$$\begin{aligned} \omega_1 &= 764 \text{ 1/s} \\ f_1 &= 122 \text{ Hz} \end{aligned} \tag{75}$$

6.2 Wire Sag

The wire sag is given by

$$s = \frac{g}{32 f_1^2} \tag{76}$$

Using the value for f_1 and the gravitational constant, we find

$$s = 20.6\ \mu\text{m} \tag{77}$$

Note that the wire sag is larger than the fiducialization accuracy tolerance. Wire position detectors near the quadrupole must be used.

6.3 Quadrupole Position Sensitivity

The sensitivity of the system to both vertical and horizontal quadrupole motion is the same. We define the sensitivity S_{xy} as the amplitude of wire vibration at the detector per unit of vertical or horizontal motion of the quadrupole.

$$S_{xy} = \frac{A_{\mathrm{det},y}}{|y|} = \frac{A_{\mathrm{det},x}}{|x|} = \frac{\sqrt{2} I_0 G L_Q}{m_l \omega_2 \alpha L} \tag{78}$$

The frequency used is twice the fundamental frequency or $\omega_2 = 1528$ 1/s. Inserting the appropriate values, we find

$$S_{xy} = 2.05 \text{ m/m} \tag{79}$$

This means that for every micron that we move the quadrupole, the amplitude of the wire vibration increases by 2.05 microns. We estimated that the wire vibration detector would give an output of 0.01 V per micron of wire motion. Thus, we can expect a signal of 0.021 V per micron of quadrupole motion. This is a large signal which the lock-in amplifier can easily measure.

6.4 Quadrupole Angle Sensitivity

The sensitivity of the system to quadrupole pitch and yaw is given by

$$S_{\theta\phi} = \frac{A_{\mathrm{det},\theta}}{|\theta|} = \frac{A_{\mathrm{det},\phi}}{|\phi|} = \frac{I_0}{m_l \omega_4 \alpha} \frac{2\pi}{3} G L_Q \left(\frac{L_Q}{L}\right)^2 \tag{80}$$

The frequency used for this measurement is four times the fundamental frequency, or $\omega_4 = 3056$ 1/s. Inserting the appropriate values, we find

$$S_{\theta\phi} = 2.53 \times 10^{-3} \text{ m/rad} \tag{81}$$

This means that for every milliradian of pitch, the wire vibration amplitude increases by 2.53 microns. Using the vibration detector sensitivity of 0.01 V per micron of wire motion, the signal we expect is 0.025 V per milliradian of pitch or yaw. Again, this is a large signal which is easy to measure.

7 Conclusion

The vibrating wire technique provides extreme sensitivity for fiducializing the LCLS quadru poles. It is expected that a quadrupole can be centered on the wire at the micron level. Pitch and yaw angles can be reduced below the milliradian level, allowing repeatable fiducialization. Wire position detectors will be required on both sides of the quadrupole. The wire position detectors are expected to locate the wire relative to tooling balls at the 2 μm level. A coordinate measuring machine will locate all tooling balls to better than 10 μm. We expect this system to more than achieve the 25 μm fiducialization accuracy requirement.

The system is being commissioned. Preliminary measurements are in agreement with the equations presented in this note for both the sensitivities and the phases. Repeatability of the fiducialization process must be determined. Tests should also be performed in which the quadrupole is flipped, checking for unforeseen biases in the measurements. Studies of wire position detector accuracy should be continued.

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On a visit to our lab, Alexander Temnykh saw a prototype pulsed wire system we built and suggested that the vibrating wire approach would provide higher sensitivity. He graciously returned for several days and converted the pulsed wire system into a vibrating wire system as a proof of principle test. The results looked very encouraging and led to our present effort to analyze the system and construct it. We are very grateful to Alexander for his assistance. Many thanks also to Kirsten Hacker for help in preparing this note.