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GRADIENT PERTURBATIONS OF THE SLC ARC*

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1. THE DISTURBED BETATRON FUNCTION

As the beam passes through the arcs, the gradient it encounters at each magnet differs from the design value. This deviation may be in part random and in part systematic. In this note we make estimates of the effects to be expected from both kinds of errors.

Since large changes in β will not be tolerated in any case, we can use an approximation that is valid for deviations $\delta\beta$ which are small relative to β itself. We define g to be the <u>relative</u> deviation, namely g $= \delta\beta/\beta$. It is shown in Courant and Snyder[1] that so long as g is small, it satisfies a differential equation which is particularly simple if we use the betatron phase $\phi = \int ds/\beta$ as our independent longitudinal coordinate. Writing $\dot{g}$ for dg/dϕ, the equation for g is

$$\ddot{g} + 4g = -2\,\beta_\circ^2\,\delta K \tag{1}$$

where $\beta_\circ(\phi)$ is the unperturbed function and $\delta K(\phi)$ is the perturbation of the focussing function K at the longitudinal position s(ϕ).

2. EFFECTS OF RANDOM GRADIENT ERRORS

We make now an estimate of the perturbed betatron function produced by randomly occuring gradient errors. To do this we will assume that the effect of a gradient error in each magnet can be approximated by an impulse error occuring in the center of the magnet.

Suppose then that we have an impulse perturbation of strength $\delta K \Delta\phi$ located at some betatron phase ϕ_i. It will produce a small disturbance Δg with an oscillation at twice the betatron frequency and with some amplitude, say ΔA_i, namely

$$\Delta g = \Delta A_i \sin 2\,(\phi - \phi_i); \quad (\phi > \phi_i). \tag{2}$$

The initial slope $\Delta\dot{g}$ of this oscillation is just $2A_i$ which must, by Eq. (1), be equal to $-2\beta_\circ^2\,\delta K \Delta\phi$.

$$\Delta\, \dot{g}_i(\phi_i) = 2\,\Delta A_i = -2\beta_\circ^2\,\delta K \Delta\phi. \tag{3}$$

We can use the impulse approximation so long as the extent $\Delta\phi$ of the perturbation is less than about 1. If $\Delta\phi$ is larger, we can replace $\beta_\circ \Delta\phi$ by an integral that gives the Fourier component of the perturbation at twice the betatron frequency. And because of the symmetry of the variation of β with respect to the magnet centers, we can place the effective impulse at the center of each magnet. For each magnet, then, we get an induced oscillation with amplitude ΔA_i

$$\Delta A_i = q_i \delta K_i \tag{4}$$

where q_i is the effective value of $(\beta_\circ^2 \Delta\phi)$ for the i–th magnet and δK_i is the focussing error of that magnet.

Consider now the value of g at the end of the arcs. Under the assumption of small errors, it will be the sum of the contribution from each magnet in the arc. Namely

$$g_L = \sum_{i=1}^{N} \Delta A_i \sin(\phi_L - \phi_i) \tag{5}$$

where ϕ_L is the (undisturbed) betatron phase at the exit of the arc. Let's now assume that all focussing magnets are equivalent and give an r.m.s. contribution ΔA_F, and similarly for the defocussing magnets, for which we have an r.m.s. contribution ΔA_D

The contributions of separate magnets combine as the squares. The contribution of each focussing magnet is

$$\langle \Delta A^2 \sin^2 \Delta\phi \rangle_F = \Delta A_F^2 \langle \sin^2 \Delta\phi \rangle_F \tag{6}$$

where $\Delta\phi$ is the phase change from each magnet to the exit. And similarly for the defocussing magnets. It turns out that the average of $\sin^2\Delta\phi$ for both F and

D magnets is just 1/2. (This average actually applies to one achromat.) Let N be the total number of magnets (N/2 of each type). Then the mean square g at the exit is

$$\langle g_L^2 \rangle = \frac{N}{4}(\Delta A_F^2 + \Delta A_D^2). \tag{7}$$

To get ΔA_F and ΔA_D we need to know the expected δK and the value of $q = \beta_o^2 \Delta\phi$ for each class of magnet. From the SLC Design Handbook,[2] the amplitude function of two cells of the arc lattice is reproduced in Fig. 1. The phase advance as function of distance is shown in Fig. 2. Notice that the phase advance in the focus magnet is different from that of a defocus magnet. To evaluate q it is more convenient to work with amplitude function as a function of phase advance ϕ which is shown in Fig. 3. Using the beta–function shown in Fig. 3, we find that

$$q_F = 18.6m^2; \quad q_D = 2.42m^2$$

So it is clear that gradient errors in the focussing magnet are by far the most important for horizontal β–function.

One of us has estimated the construction errors to be expected in the arc magnet.[3] (See CN–313). It turns out that the <u>random</u> gradient errors expected from magnet <u>construction</u> errors, after shuffling and alignment offset, are likely to be smaller than the expected gradient errors introduced by the distortion of the central orbit of the beam. If we assume the usual alignment tolerance of $100\mu m$, and the usual orbit correction scheme, we know that the orbit will have a random offset in the magnets of about $150\mu m$.[4] [5] Such displacements translate to r.m.s. focussing errors of

$$\delta K_F = 1.5 \times 10^{-3} m^{-2}; \qquad \delta K_D = 2.4 \times 10^{-3} m^{-2}.$$

Using these values we find that

$$\sigma_g = \sqrt{\langle g_L^2 \rangle} = \frac{\delta\beta_{\rm rms}}{\beta} = 0.35 \tag{8}$$

The expected error in final β due to random gradient errors is about 35%.

Focussing errors will also produce an error in the derivative of β. Since the disturbed g is a free oscillation at twice the betatron frequency but with unknown phase, we expect that

$$\dot{g}_{rms} = 2g_{rms}, \tag{9}$$

which translates to

$$\delta\beta'_{\rm rms} = 2\frac{\delta\beta_{\rm rms}}{\beta}\sqrt{1+\beta'^2/2}, \tag{10}$$

If we choose the exit point to have the design $\beta' = 0$, the $\delta\beta'$ from the errors is just twice the relative error in β, or about 0.7. This is to be compared with a typical β' in the arcs of about 5. The tilt of the ellipse is, however, not negligible. For $\beta' = 0.7$ the correlation coefficient r = 0.33.

We have considered so far only the perturbations to the horizontal beta function. The analysis of the effects of perturbation in the vertical focussing strength is essentially identical. And indeed the expected random gradient errors are also the same, so the final results above apply equally to the vertical beta.

We should remark that the changes in both vertical and horizontal focussing are caused by the same magnet displacement and so are highly correlated. It turns out, however, that $\delta\beta_x$ is not strongly correlated to $\delta\beta_y$, because $\delta\beta_x$ has a strong sensitivity to displacements of the focussing magnets and a weak dependence of the position of the defocussing magnets, while the reverse is true for $\delta\beta_y$.

3. THE EFFECTS OF SYSTEMATIC GRADIENT ERRORS IN THE ARCS

If the sources of gradient error are known and static in time, their effect on the amplitude function β can be studied by solving equation (1). In case the source of perturbation is periodic, and small, the superposition principle holds, then the perturbing term can be Fourier decomposed and the problem can be solved for each harmonic component of the perturbation.

In the arc design there are two possible sources of such systematic errors. One is the systematic error in field strength of focus magnet to that of defocus magnet. The second one is the strength variation due to the discontinuity of the synchrotron radiation correction tapers. The effects of those systematic errors will be investigated after brief description of the solution of Eq (1) to a sinusoidal perturbation.

3*A*. Response of a Periodic Transport System to a Sinusoidal Perturbation

Recall that the equation of motion of the relative variation g is given by Eq. (1),

$$\frac{d^2g}{d\phi^2} + 4g = -2\beta_o^2\delta K \tag{1}$$

where the period in angle ϕ is chosen in such a way that one period in $\phi(0 \rightarrow 2\pi)$ corresponds to one betatron oscillation. For Fourier decomposition of a systematic error, we will introduce a new independent variable ψ one period of which corresponds to the fundamental period of the perturbation. The period of perturbation, for the two examples mentioned above, is one achromat; therefore we choose the period of the new independent variable ψ to be $0 \rightarrow 2\pi$ in one achromat. Because there are three betatron oscillation in one achromat, the tune in Eq. (1) should be three, *i.e.* $\nu = 3$. The equation of motion now becomes

$$\frac{d^2g}{d\psi^2} + 4\nu^2 g = -2\nu^2\beta_o^2\delta K = F(\psi) \tag{11}$$

and the relationship between ψ and ϕ is

$$\phi = \nu\psi \tag{12}$$

Since the perturbing term is a periodic function of ψ, it can be Fourier decomposed to give

$$\begin{aligned} F(\psi) &= -\,2\nu^2\beta_0^2\,\delta K(\psi) \\ &= \frac{A_0}{2} + \sum_n A_n \cos n\psi + \sum_n B_n \sin n\psi \end{aligned} \tag{13}$$

where

$$A_n = \frac{1}{\pi}\int_0^{2\pi} F(\phi)\cos n\psi d\psi, \quad n = 0, 1, 2, \ldots \tag{14}$$

and

$$B_n = \frac{1}{\pi}\int_0^{2\pi} F(\psi)\sin n\psi d\psi \quad n = 1, 2, \ldots \tag{15}$$

Because of the linearization approximation introduced in section 2, the problem becomes linear and the response will be the superposition of responses to each harmonic component.

We are interested in the variation of the amplitude function β in a periodic transport system when the beam are perfectly matched at the entrance. In other words; the equation satisfies

$$g(0) = 0, \quad g'(0) = 0 \tag{16}$$

The solutions of Eq. (1) under single sinusoidal forcing term satisfying the initial condition (16) are summarized in Table 1.

Table 1: Solution of Eq (11) and (16)

Case	$F(\psi)$	Solution $g(\psi)$	
1	A_0	$\frac{A_0}{4\nu^2}(1-\cos 2\nu\psi)$	(17)
2	$A_n \cos n\psi$ $(n \neq 2\nu)$	$\frac{A_n}{4\nu^2 - n^2}(\cos n\psi - \cos 2\nu\psi)$	(18)
3	$A_n \cos n\psi$ $(n = 2\nu)$	$\frac{A_n}{4\nu}\psi \sin 2\nu\psi$	(19)
4	$B_n \sin n\psi$ $(n \neq 2\nu)$	$\frac{B_n}{4\nu^2 - n^2}\left(\sin n\psi - \frac{n}{2\nu}\sin 2\nu\psi\right)$	(20)
5	$B_n \sin n\psi$ $(n = 2\nu)$	$\frac{B_n}{4\nu}\left(\frac{1}{2\nu}\sin 2\nu\psi - \psi\cos 2\nu\psi\right)$	(21)

If the perturbing frequency n is different from the natural frequency 2ν, the solution is simply an oscillatory function. However if $n = 2\nu$, it satisfies the resonant condition, there the amplitude will grow linearly with time and will be modulated by a sinusoidal function of phase. Therefore the resonant situation is the most dangerous one and any systematic error rich in second harmonic of betatron frequency should be avoided!

3*B*. Difference of Gradient Strength Between Focus and Defocus Magnets

The first example we consider for the arc is the possible gradient strength difference between focus and defocus magnets caused by possible calibration error of the measurement coil. The uncertainty in the calibration is about 2×10^{-3}.

Although this is a small number, the purpose of this calculation is to see whether there is any danger of exciting the second harmonic resonance.

Assuming that the gradient strength of the focus magnet is stronger than that of the defocusing magnet by a constant amount ΔK and that the gradient strength of the defocus magnets is set at the correct value, then the perturbing function is

$$\begin{aligned} F(\psi) &= -2\nu^2 \beta_0^2(\psi)\,\Delta K \qquad &\text{at } F \\ &= 0 &\text{at } D \end{aligned} \tag{22}$$

In order to do Fourier decomposition of Eq. (22) in one achromat we have to know $\beta(\psi)$ as function of ψ. This can be obtained from Fig. 3. Given the relationship between ψ and ν from Eq. (14), the ψ-integration for the Fourier decomposition can be expressed in ϕ-integration. For example,

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_0^{2\pi} F(\psi) \cos n\psi d\psi \\ &= \frac{1}{\pi} \int_0^{2\pi\nu} F(\phi) \cos n\frac{\phi}{\nu} \left(\frac{d\phi}{\nu}\right) \end{aligned} \tag{23}$$

The result of the calculation is summarized in Table 2.

Table 2. Fourier Coefficients of $\mathbf{F}(\psi)$

n	0	1	2	3	4	5
A_n^*	3.1×10^6	-2.63	-2.50	-2.10	-2.25	-1.72
B_n^*	0	0.68	1.38	1.99	3.59	5.44

n	6	7	8	9	10
A_n^*	-0.26	1.54	5.88	14.3	2.42×10^6
B_n^*	4.86	7.60	12.1	16.8	1.70×10^6

$*$ In Unit of 10^{-7}

Except for n = 0 and n = 10, the coefficients are all very small and consequently the effect on amplitude function β are negligible. When n = 10, the coefficients are much larger because the amplitude function has an intrinsic period of 10 within one achromat. From Eq. (18) the maximum effect on g in the arc (for n = 10) is

$$g(\psi \approx 46\pi) = \frac{0.242 \times 2}{36 - 100} \approx 8 \times 10^{-3} \tag{24}$$

which is small enough not to cause worry. It is also easy to check that the variation of β is neglible when n = 0.

In case n = 6, the resonant build–up can take place. At the end of arc, the phase ψ goes through 23 periods and is very close to 46π. Therefore, the dominating term is the last term in Eq (21) which gives,

$$\begin{aligned} g(\phi = 46\pi) &= \frac{\Delta\beta}{\beta} = -\frac{4.86}{4 \times 3} \times 10^{-7}(46\pi) \\ &\approx 0.59 \times 10^{-5} \end{aligned} \tag{25}$$

It is also a very small number. The conclusion is that a systematic error in gradient strength of 2×10^{-3} would not hurt the amplitude function.

3C. EFFECT OF DISCONTINUITY OF SYNCHROTRON RADIATION TAPER

When the electrons (or positrons) pass through the arc, they lose energy through synchrotron radiation. If we want the final energy of the beam to be 50 GeV at collision point, their energy at the end of Linac should be 51.42 GeV. Such energy loss corresponds to 2.84% change of particle energy through one of the arcs which can not be accommodated by the optical bandpass of the arcs. The plan is to set the main power supply to the arc magnet system at 50.71 GeV and further correct the excitation by adjusting the trim windings on an achromat by achromat basis. Because there are 23 achromats in the arc, the resolution of the excitation of each achromat is then

$$\frac{\Delta E}{E} = \frac{1.42}{50.71} \times \frac{1}{23} = 0.0012 \tag{26}$$

Which means that the magnitude of gradient perturbation will be

$$\Delta K = 0.0012 \cdot K = 5 \times 10^{-4} m^{-2} \tag{27}$$

If the trim supply is set to the correct energy at the entrance of the achromat, the excitation will be 0.12% higher at the end of the achromat. From inspection it is easy to see that this arrangement results in a large DC component of the perturbation. To minimize this DC component, the magnet excitation should be set to correct value at the middle of one achromat, then the exciting term to the gradient perturbation can be expressed as

$$F(\psi) = -2\nu^2\beta_\circ^2(\psi)\left[5\times 10^{-4}\frac{(s-L/2)}{L}\right] \tag{28}$$

where L is the length of one achromat and s is the distance from the beginning of the achromat. The result of Fourier decomposition of Eq. (29) is summarized in Table 3.

Table 3. Fourier Coefficients of F(ψ)

n	0	1	2	3	4	5
A_n	−0.002	−0.006	−0.006	−0.0065	−0.0069	−1.72
B_n	0.0	−0.043	−0.020	−0.008	−0.0047	5.44

n	6	7	8	9	10
A_n	0.0075	−0.0085	−0.011	−0.017	0.089
B_n	−0.0016	0.002	0.008	0.023	0.033

Although the maximum amplitude of the gradient error is 1.2×10^{-3}, not any larger than the assumed systematic error, these errors are much richer in Fourier components. All the components, except $n=6$, contribute little to the growth of β. In case $n=6$, the growth at the end of arc can be calculated from Eq. (19), giving

$$\begin{aligned} g(\psi = 46\pi) = \frac{\Delta\beta}{\beta} &\approx \frac{0.0075}{4\times 3}(46\pi) \\ &\approx 0.092 \end{aligned} \tag{29}$$

This is about a 10% blow–up of the amplitude function at the end of the arc.

Compared to random gradient errors discussed in section 1, the effect of systematic errors is not as serious. If the systematic and random gradient errors are both present in a system, the random effect should be calculated with respect to the perturbed solution with systematic errors. In other words, the total perturbation should add algebraically. Therefore, for the arc the total effects due to both random and systematic gradient errors could be 45%. It is advisable to introduce gradient corrections at suitable locations in the arc to minimize the growth of the amplitude function.

Results of simulations which model gradient errors due to random misalignment of magnets as well as DC offsets of BPM yield similiar results on β–distortions.[6]

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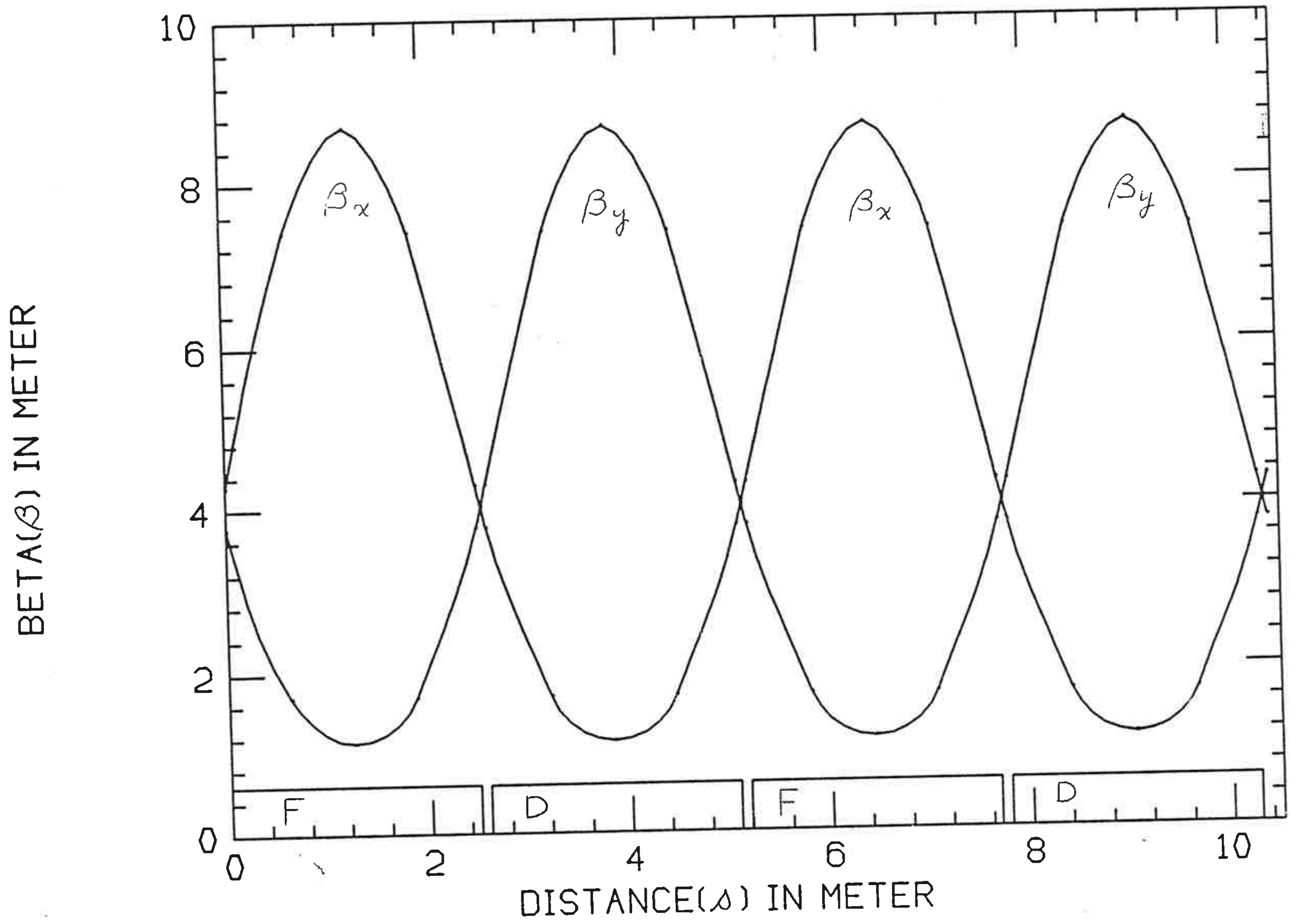

FIG. 1 **β** IN TWO CELLS OF THE ARC AS FUNCTION OF DISTANCE

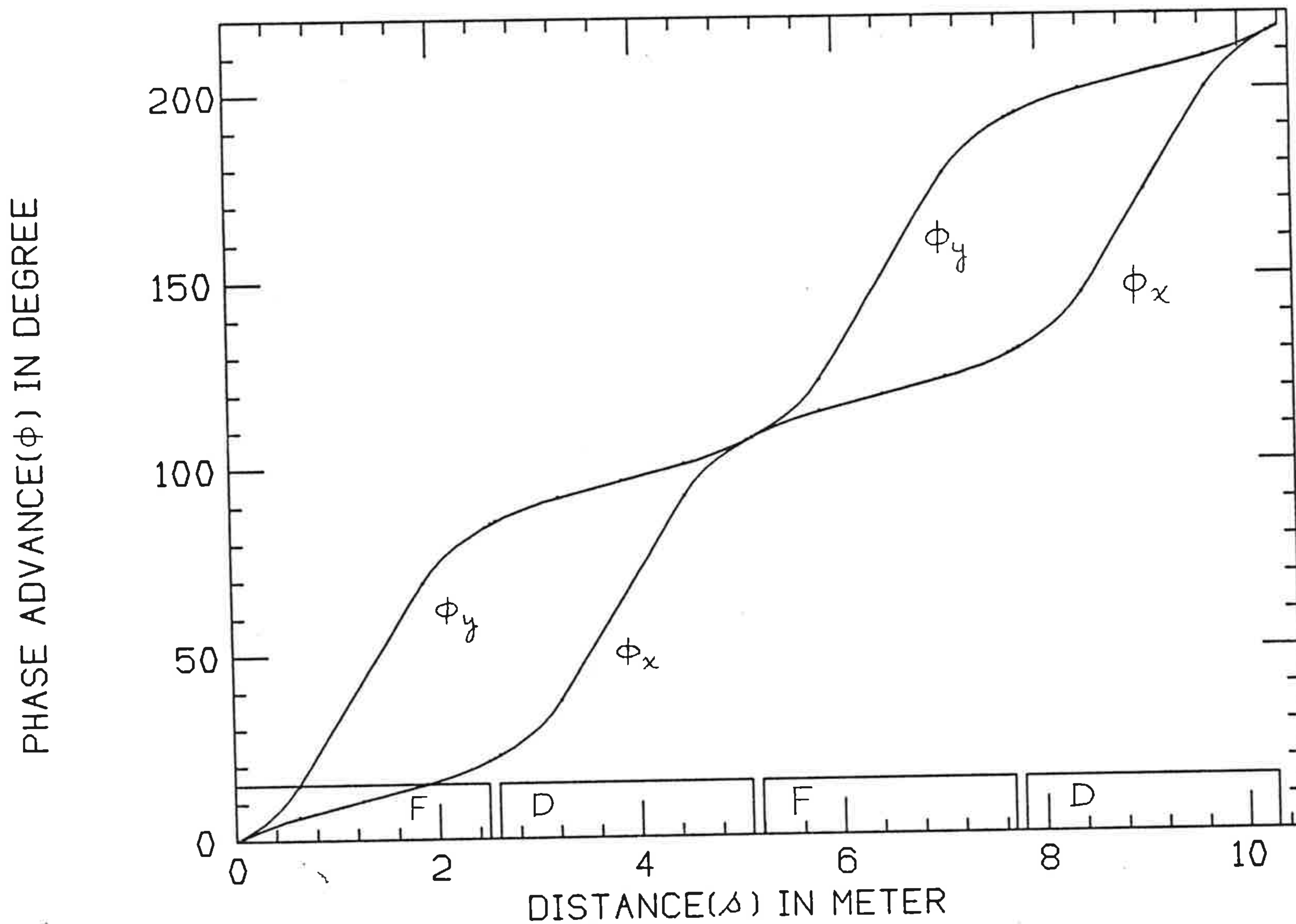

FIG. 2 PHASE ADVANCE AS FUNCTION OF DISTANCE

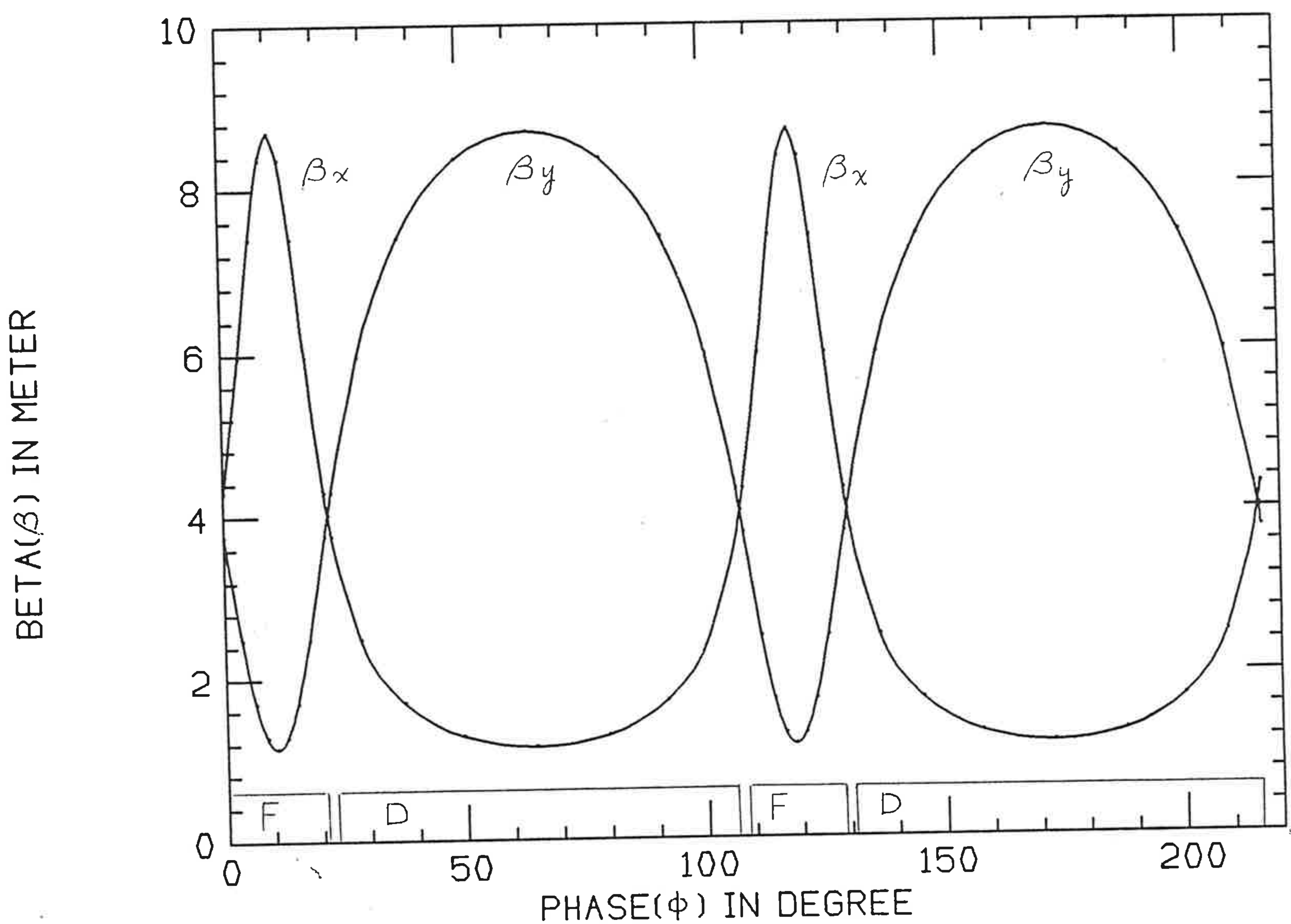

FIG. 3 **β** IN TWO CELLS OF THE ARC AS FUNCTION OF PHASE

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