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Classification of Systems of Nonlinear Evolution Equations Admitting Higher-Order Conditional Symmetries

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Algorithm for construction of conditionally invariant systems of evolution equations and their subsequent reduction to the systems of ordinary differential equations is suggested. Classification and reduction theorems are formulated for n-order evolution equations and for systems of two evolution equations. Two classes of conditionally invariant second order systems of evolution equations are given, and their reduction to the systems of four ordinary differential equations is carried out.

1 Introduction

Modelling of dynamic processes in physics, chemistry and other fields of science requires solving evolution equations. Provided equations under study are linear, the methodology of constructing exact solutions is developed quite well. In the case of nonlinear equations, there are no general methods for finding their solutions. Among the most efficient methods for constructing exact solutions of nonlinear evolution equations are those based on their conditional symmetries [1, 2].

A number of Galaktionov's papers are devoted to constructing exact solutions of equations

$$u_t = F\left(u, u_x, u_{xx}\right), \qquad u_t = \frac{\partial u}{\partial t}, \qquad u_x = \frac{\partial u}{\partial x}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

with quadratic nonlinearities. To this end the technique based on the concept of the invariant subspace [3] is employed. New approach to reduction of nonlinear evolution equations (1) using their higher symmetries was suggested in [4]. With the help of this approach, classification of evolution equations [5] and in accordance with results presented in [6] reduction of initial-value problem for them to Cauchy problem for system of ordinary differential equations (ODEs) [7] was carried out. A number of exact solutions of equation (1) with quadratic nonlinearities were obtained in [8] with the aid of ansatzes, being solutions of third-order linear ODEs.

In all the above mentioned papers the right-hand sides of equation (1) are quadratic polynomials or can be transformed to them by a certain change of variables. Classes of systems

$$u_t = u_{xx} + F\left(u, v, u_x, v_x\right), \qquad v_t = -v_{xx} + G\left(u, v, u_x, v_x\right),$$

admitting fourth-order symmetries, were described in [9]. F, G are fifth order polynomials.

In this paper we propose algorithm for construction of classes of systems of evolution equations, admitting conditional symmetries, and formulate classification and reduction theorems for systems of evolution equations, which are analogous to theorems, proved in [4]. With help of this algorithm we classify nonlinear equations

$$u_t = F\left(t, x, u, u_x, u_{xx}\right), \tag{2}$$

which admit reduction to systems of ODEs. To this end we consider these equations together with the condition

$$u_{xxx} = f\left(t, x, u, u_x, u_{xx}\right). \tag{3}$$

Equation (3) can be considered as an ODE with parameter t.

We also give examples for constructing of classes of conditionally invariant systems

$$u_t = u_{xx} + F\left(x, u, v, u_x, v_x\right), \qquad v_t = -v_{xx} + G\left(x, u, v, u_x, v_x\right) \tag{4}$$

and carry out their reduction to systems of four first-order ODEs.

2 Classification algorithm

Let us consider a system of partial differential equations (PDEs)

$$u_{it} = F_i\left(t, x, u_1, \ldots, u_n\right) \tag{5}$$

under additional conditions for functions u_i

$$u_{ix} = f_i\left(t, x, u, \ldots, u_n\right). \tag{6}$$

Here and henceforth we assume, unless otherwise specified, that $i = \overline{1, n}$.

Let f_i, F_i be continuously-differentiable functions of their arguments in some open domain Ω and $\bar{\boldsymbol{f}} \neq 0$ in any point of this domain, $u_i = u_i\left(t, x\right)$ are twice continuously-differentiable functions. Differentiating (5) with respect to x, (6) with respect to t and equating right-hand sides of obtained equalities we arrive at following compatibility condition for the system (5), (6)

$$F_{ix} + u_{1x} F_{i\,u_1} + \cdots + u_{nx} F_{i\,u_n} = f_{i\,t} + u_{1t} f_{i\,u_1} + \cdots + u_{n\,t} f_{i\,u_n}.$$

Taking into account (5), (6), we rewrite it in form

$$F_{ix} + f_1 F_{i\,u_1} + \cdots + f_n F_{i\,u_n} = f_{i\,t} + f_{i\,u_1} F_1 + \cdots + f_{i\,u_n} F_n. \tag{7}$$

By change of variables $\eta = x$, $\omega_i = \omega_i\left(t, x, u_1, \ldots, u_n\right)$, where ω_i are first integrals of (6):

$$L\omega_i = \omega_{ix} + f_1 \omega_{iu_1} + \cdots + f_n \omega_{iu_n} = 0,$$

(7) is transformed to system

$$F_{i\eta} = g_{i0} + g_{i1} F_1 + \cdots + g_{in} F_n, \tag{8}$$

where $g_{ij}\left(t, \omega_1, \ldots, \omega_n, \eta\right) = f_{iu_j}$, $g_{i0}\left(t, \omega_1, \ldots, \omega_n, \eta\right) = f_{it}$.

By assumption that functions f_i are known and system (5), (6) is compatible, F_i must satisfy of linear system (8), that can be considered as an ODE with parameters $t, \omega_1, \ldots, \omega_n$. Thus

$$F_i = \sum_{j=1}^{n} G_j\left(t, \omega_1, \ldots, \omega_n\right) p_{ij}\left(\eta, t, \omega_1, \ldots, \omega_n\right). \tag{9}$$

Here $\left(\bar{\boldsymbol{p}}_1, \ldots, \bar{\boldsymbol{p}}_n\right)$ is a fundamental system of solutions of (7) and $G_1, \ldots, G_n$ are arbitrary smooth functions.

Substituting general solutions of (6) into (5), (9) we obtain system of ODEs that is equivalent

$$\dot{C}_i\left(t\right) = g_i\left(t, , C_1\left(t\right), \ldots, C_n\left(t\right)\right).$$

Now we consider the case that right-hand sides of equations (6) do not depend on t explicitly:

$$u_{ix} = f_i\left(x, u_1, \ldots, u_n\right). \tag{10}$$

Then system (8) is homogenous ($f_{it} = 0$ and consequently $g_{i0} = 0$).

Theorem 1. *Let* $Q = \xi(x,u_1,\dots,u_n)\partial_x + \sum_{l=1}^{n}\varphi_l(x,u_1,\dots,u_n)\partial_{u_l}$ *be symmetry operator of system (10) and in system (5)* $F_i = \varphi_i - \xi f_i$. *Then system (5), (10) is compatible. Here and in the sequel* $\partial_x = \frac{\partial}{\partial x}$, $\partial_{u_l} = \frac{\partial}{\partial u_l}$.

Proof. Since Q is a symmetry operator of system (10), then $\mathrm{Pr}^{(1)}Q(u_{ix} - f_i) = 0$, for $u_{ix} = f_i$.

$$\mathrm{Pr}^{(1)}Q = Q + \sum_{l=1}^{n}\varphi^l\partial_{u_{lx}}, \qquad \varphi^l = D_x(\varphi_l - \xi u_{lx}) + \xi u_{lxx}$$

is first prolongation of Q. D_x signifies total derivative with respect to x [10].

For system (10) we have

$$\mathrm{Pr}^{(1)}Q(u_{ix} - f_i) = \left[\xi\partial_x + \sum_{l=1}^{n}\varphi_l\partial_{u_l} + \sum_{l=1}^{n}\left(D_x(\varphi_l - \xi u_{lx}) + \xi u_{lxx}\right)\partial_{u_{lx}}\right](u_{ix} - f_i)$$
$$= D_x(\varphi_i - \xi u_{ix}) + \xi u_{ixx} - \xi f_{ix} - \sum_{l=1}^{n}\varphi_l f_{iu_l} = D_x(\varphi_i - \xi u_{ix}) + \xi D_x f_i - \xi f_{ix} - \sum_{l=1}^{n}\varphi_l f_{iu_l}$$
$$= D_x(\varphi_i - \xi u_{ix}) + \xi\sum_{l=1}^{n} f_{iu_l}u_{lx} - \sum_{l=1}^{n}\varphi_l f_{iu_l} = D_x(\varphi_i - \xi f_i) - \sum_{l=1}^{n}(\varphi_l - \xi f_l)f_{iu_l} = 0.$$

Hence $D_x(\varphi_i - \xi f_i) = \sum_{l=1}^{n}(\varphi_l - \xi f_l)f_{iu_l}$, that is equivalent to (7), that is compatibility condition for the system (5), (6) ($f_{it} = 0$). ■

Theorem 2. *Let (10) admit n independent symmetry operators* $Q_1,\dots,Q_n$, *where* $Q_j = \xi_j\partial_x + \sum_{l=1}^{n}\varphi_{lj}\partial_{u_l}$. *Then functions* $\bar{\boldsymbol{P}}_j = \bar{\varphi}_{\mathbf{j}} - \xi_{\mathbf{j}}\bar{\boldsymbol{f}}$ *form fundamental system of solutions of (8)* ($g_{i0} = 0$) *and its general solution (compatibility condition of (5), (10)) has a form*

$$F_i = \sum_{j=1}^{n} G_j(t,\omega_1,\dots,\omega_n)(\varphi_{ij} - \xi_j f_i), \tag{11}$$

where $\omega_i = \omega_i(x,u_1,\dots,u_n)$, $G_1,\dots,G_n$ *are arbitrary smooth functions and substitution of solution of (10)* $u_i = U_i(x, C_1(t),\dots,C_n(t))$ *in (5), (11) gives following system of ODEs*

$$\dot{C}_i = \sum_{j=1}^{n} G_j(t,C_1,\dots,C_n)\,g_j(C_1,\dots,C_n) = \sum_{j=1}^{n} G_j Q_j(\omega_i)\,|_{u_i=U_i}\ . \tag{12}$$

Proof. The first assertion of theorem (condition (11)) follows from Theorem 1 and fact, that

$$D_x\omega_i = \omega_{ix} + \sum_{l=1}^{n}\omega_{iu_l}u_{lx} = \omega_{ix} + \sum_{l=1}^{n}\omega_{iu_l}f_l = L\omega_i = 0.$$

Let us prove (12). By assumption that right-hand side (10) does not vanish anywhere in Ω, this system has n independent first integrals, hence $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\omega}}$ are mutually inverse functions. Thus, if we substitute $u_i = u_i(x,\omega_1,\dots,\omega_n)$ in (5) and differentiate obtained equalities with respect to t, then we have the system

$$\sum_{j=1}^{n}\frac{\partial u_i}{\partial\omega_j}D_t\omega_j = \sum_{j=1}^{n} G_j(\varphi_{ij} - \xi_j f_i). \tag{13}$$

$\det \left\| \frac{\partial u_i}{\partial \omega_j} \right\| \neq 0$, because in the opposite case functions $u_1, \ldots, u_n$ are linearly dependent and number of independent first integrals are smaller than n. Thus, system (13) as a system of linear algebraic equations has the unique solution

$$\overline{D_t \omega} = \left\| \frac{\partial u_i}{\partial \omega_j} \right\|^{-1} \sum_{j=1}^{n} G_j \overline{P_j}.$$

According to inverse function theorem, $\left\| \frac{\partial u_i}{\partial \omega_j} \right\|^{-1} = \left\| \frac{\partial \omega_i}{\partial u_j} \right\|$ and consequently this solution can be rewritten component-wise as follows:

$$\begin{aligned} D_t \omega_i &= \sum_{j=1}^{n} G_j \sum_{l=1}^{n} (\varphi_{lj} - \xi_j f_l)\, \omega_{iu_l} = \sum_{j=1}^{n} G_j \sum_{l=1}^{n} (\varphi_{lj} \omega_{iu_l} - \xi_j f_l \omega_{iu_l}) \\ &= \sum_{j=1}^{n} G_j \left(\sum_{l=1}^{n} \varphi_{lj} \omega_{iu_l} + \xi_j \omega_{ix} \right) = \sum_{j=1}^{n} G_j Q_j (\omega_i). \end{aligned} \tag{14}$$

The same result we obtain by immediate differentiating $\omega_i (x, u_1, \ldots, u_n)$ with respect to t in consideration of (11). Taking into account, that $D_x D_t \omega_i = D_t D_x \omega_i = 0$, we conclude that right-hand side of (14) does not depend on x explicitly. After that, to complete proof, we change $u_1, \ldots, u_n$ for $U_1, \ldots, U_n$ taking into account, that

$$C_i (t) = \omega_i (x, U_1, \ldots, U_n), \qquad \dot{C}_i = D_t \omega_i (x, U_1, \ldots, U_n). \qquad \blacksquare$$

Thus, we formulate the following algorithm for constructing of classes of conditionally invariant systems of evolution equations and their reduction to systems of ODEs:

- calculate symmetry algebra of equation (10);
- find its first integrals;
- integrate (10) (if (10) admit n-parametric solvable symmetry algebra then it can be integrated in quadratures [10]);
- determine F_i by formula (11);
- write system of ODEs (12) for functions $C_1 (t), \ldots, C_n (t)$.

3 Classification and reduction of equations (2)

Now we go to the problem classification of equations (2), which are conditionally invariant under condition (3). First we consider auxiliary systems

$$u_t = F (t, x, u, v, w), \qquad v_t = G (t, x, u, v, w), \qquad w_t = H (t, x, u, v, w); \tag{15}$$

$$u_x = v, \qquad v_x = w, \qquad w_x = f (t, x, u, v, w). \tag{16}$$

Note, that system (16) is equivalent (3). Compatibility condition of the given system is

$$\begin{aligned} &F_x + vF_u + wF_v + fF_w = G, \qquad G_x + vG_u + wG_v + fG_w = H, \\ &H_x + vH_u + wH_v + fH_w = f_t + f_u F + f_v G + f_w H. \end{aligned}$$

First integrals of systems (16) are functionally independent solutions of the equation

$$\omega_{ix} + v\omega_{iu} + w\omega_{iv} + f\omega_{iw} = 0.$$

If (15), (16) is compatible, then F, G, H satisfy the system

$$F_\eta = G, \qquad G_\eta = H, \qquad H_\eta = f_t + f_u F + f_v G + f_w H,$$

that is equivalent equation

$$F_{\eta\eta\eta} - f_{u_{xx}} F_{\eta\eta} - f_{u_x} F_\eta - f_u F = f_t. \tag{17}$$

A solution of linear equation (17) has the form $F = F^g + F^p$, where F^p is a partial solution of equation (17) and

$$F^g = G_1 p_1 + G_2 p_2 + G_3 p_3, \qquad G_j = G_j\left(t, \omega_1, \omega_2, \omega_3\right), \qquad p_j = p_j\left(\eta, \omega_1, \omega_2, \omega_3\right)$$

is the general solution of corresponding homogeneous equation.

Thus, having solved (17) we obtain in the explicit form the function F, for which system (2), (3) is compatible. According to theorem, proved in [4], substitution of ansatz which is a solution of equation (3), into (2), reduces (2) to a system of three ODEs.

Assertion analogous to Theorem 2 can be formulated for equations

$$u_t = F\left(t, x, u, u^{(1)}, \ldots, u^{(n-1)}\right), \tag{18}$$

$$u^{(n)} = f\left(x, u, u^{(1)}, \ldots, u^{(n-1)}\right), \tag{19}$$

where $u^{(i)} = \frac{\partial^i u}{\partial x^i}$, $F \in C^{n+1}(\Omega)$, $f \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^{n+1}$, $u = u(x,t) \in C^{n+1}(\Omega')$, $\Omega' \subset \mathbb{R}^2$.

Theorem 3. *Let $\omega_1, \ldots, \omega_n$ be first integrals and $Q_1, \ldots, Q_n$ be independent symmetry operators of equations (19): $Q_j = \xi_j(x,u)\,\partial_x + \varphi_j(x,u)\,\partial_u$. If*

$$F = \sum_{j=1}^{n} G_j\left(t, \omega_1, \ldots, \omega_n\right)\left(\varphi_j - \xi_j u_x\right), \tag{20}$$

G_j are arbitrary sufficiently smooth functions, then system (18), (19) is compatible and substitution of general solution (19) $u{=}U\left(x, C_1(t), \ldots, C_n(t)\right)$ in (18) reduces it to system of ODEs

$$\dot{C}_i = \sum_{j=1}^{n} G_j\left(t, C_1, \ldots, C_n\right) g_j\left(C_1, \ldots, C_n\right) = \sum_{j=1}^{n} G_j \mathrm{Pr}^{(n-1)} Q_j\left(\omega_i\right)|_{u=U}\ .$$

Proof follows from the fact that if Q is symmetry operator of (19), then $\mathrm{Pr}^{(n-1)}Q\,\big|_{u^{(i)}=u_i}$ is symmetry operator of system

$$u_{1x} = u_2, \quad u_{2x} = u_3, \quad \ldots, \quad u_{nx} = f\left(x, u_1, \ldots, u_n\right), \quad \text{where} \quad u_1 = u.$$

We apply obtained result for classification and reduction equations (2) under additional condition

$$u_{xxx} = f\left(x, u, u_x, u_{xx}\right). \tag{21}$$

It is well-known that solution of third order ODE admitting three-parametrical solvable symmetry group can be constructed in quadratures. Using normal forms of ODEs (21), which admit three-parametrical solvable symmetry algebrae, we constructed (by formula (20)) nine classes of evolution equations (2) that are conditionally invariant under these types of (21). We also reduced obtained classes of evolution equations to system of three ODEs. Here we do not adduce these results, as they is cumbersome. This problem will be considered in further papers.

4 Examples of reduction of systems of evolution equations

Consider system

$$u_t = F\left(t, x, u, v, u_x, v_x\right), \qquad v_t = G\left(t, x, u, v, u_x, v_x\right) \tag{22}$$

under additional conditions

$$u_{xx} = f\left(x, u, v, u_x, v_x\right), \qquad v_{xx} = g\left(x, u, v, u_x, v_x\right). \tag{23}$$

We apply described procedure and obtain following system for determining functions F, G

$$F_{\eta\eta} - f_{u_x} F_\eta - f_{v_x} G_\eta - f_u F - f_v G = 0, \qquad G_{\eta\eta} - g_{u_x} F_\eta - g_{v_x} G_\eta - g_u F - g_v G = 0.$$

Theorem 4. *Let $\omega_j = \omega_j\left(x, u, v, u_x, v_x\right)$ are first integrals and*

$$Q_j = \xi_j\left(x, u, v\right)\partial_x + \varphi_j\left(x, u, v\right)\partial_u + \psi_j\left(x, u, v\right)\partial_v$$

are independent symmetry operators of system (23), $j = \overline{1,4}$. If

$$F = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\varphi_j - \xi_j u_x\right), \qquad G = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\psi_j - \xi_j v_x\right),$$

R_j are arbitrary twice continuously-differentiable function, then system (22), (23) is compatible and substituting of solutions of (23) $u = U\left(x, C_1(t), \ldots, C_4(t)\right)$, $v = V\left(x, C_1(t), \ldots, C_4(t)\right)$ into (22) reduces it to system of four ODE

$$\dot{C}_i = \sum_{j=1}^{4} R_j\left(t, C_1, \ldots, C_4\right) g_j\left(C_1, \ldots, C_4\right) = \sum_{j=1}^{4} R_j \mathrm{Pr}^{(1)} Q_j\left(\omega_i\right)|_{u=U, v=V}, \quad i = \overline{1,4}.$$

Proof of Theorem 4 is analogous to proof of Theorem 3 ((23) can be changed into equivalent first-order system).

Remark 1. For a system (4), (23) compatibility condition is

$$F = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\varphi_j - \xi_j u_x\right) - f\left(x, u, v, u_x, v_x\right),$$
$$G = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\psi_j - \xi_j v_x\right) + g\left(x, u, v, u_x, v_x\right).$$

There are more than sixty nonequivalent classes of systems (23) with four dimensional solvable symmetry algebras. Here we give some examples of application of exposed algorithm to construction and reduction of classes of systems (4). We write systems (23), their symmetries, general solutions and first integrals, functions F, G and reduced systems.

1. $u_{xx} = \xi''(x)\ln v_x + f(x), \qquad v_{xx} = g'(x)\, v_x,$

$Q_1 = \partial_u, \quad Q_2 = \partial_v, \qquad Q_3 = x\partial_u, \qquad Q_4 = \xi(x)\,\partial_u + v\partial_v,$

$u = \ln C_1(t)\,\xi(x) + \int^x\!\!\int^z \left(\xi''(y)\, g(y) + f(y)\right) dy dz + C_3(t)\, x + C_4(t),$

$$v = C_1(t)\int^x e^{g(y)}dy + C_2(t), \qquad \omega_1 = v_x e^{-g(x)}, \qquad \omega_2 = v - \omega_1\int^x e^{g(y)}dy,$$
$$\omega_3 = u_x - \xi'(x)(\ln v_x - g(x)) - \int^x \left(\xi''(y)g(y) + f(y)\right)dy,$$
$$\omega_4 = u - x\omega_3 - \xi(x)(\ln v_x - g(x)) - \int^x\!\!\int^z \left(\xi''(y)g(y) + f(y)\right)dydz,$$
$$F = \xi(x)R_4 + xR_3 + R_1 - \xi''(x)\ln v_x - f(x), \qquad G = vR_4 + R_2 + g'(x)v_x,$$
$$\dot C_1 = C_1R_4, \qquad \dot C_2 = C_2R_4 + R_2, \qquad \dot C_3 = R_3, \quad \dot C_4 = R_1.$$

2. $u_{xx} = x^{-1}f_x(v_x) + x^{-1}\ln x g(v_x), \qquad v_{xx} = x^{-1}g(v_x),$

$$Q_1 = \partial_u, \qquad Q_2 = \partial_v, \qquad Q_3 = x\partial_u, \quad Q_4 = x\partial_x + (u+v)\partial_u + v\partial_v,$$
$$u = \frac{1}{C_1(t)}\int^{C_1(t)x}\!\!\int^z \frac{f\left(H^{-1}(y)\right) + \ln y g\left(H^{-1}(y)\right)}{y}dydz - \frac{\ln C_1(t)}{C_1(t)}\int^{C_1(t)x} H^{-1}(y)\,dy$$
$$+ C_3(t)x + C_4(t), \qquad v = \frac{1}{C_1(t)}\int^{C_1(t)x} H^{-1}(y)\,dy + C_2(t), \qquad H(y) = e^{\int^y \frac{dz}{g(z)}},$$
$$\omega_1 = \frac{H(v_x)}{x}, \qquad \omega_2 = v - \frac{1}{\omega_1}\int^{H(v_x)} H^{-1}(y)\,dy,$$
$$\omega_3 = u_x - \int^{H(v_x)} \frac{f\left(H^{-1}(y)\right) + \ln y g\left(H^{-1}(y)\right)}{y}dy + \frac{\ln(H(v_x))}{x}v_x,$$
$$\omega_4 = u - x\omega_3 - \frac{x}{H(v_x)}\int^{H(v_x)}\!\!\int^z \frac{f\left(H^{-1}(y)\right) + \ln y g\left(H^{-1}(y)\right)}{y}dydz$$
$$+ \frac{x}{F(v_x)}\ln\left(\frac{H(v_x)}{x}\right)\int^{H(v_x)} F^{-1}(y)\,dy,$$
$$F = (u + v - xu_x)R_4 + xR_3 + R_1 - x^{-1}f_x(v_x) + x^{-1}\ln x g(v_x),$$
$$G = (v - xv_x)R_4 + R_2 + x^{-1}g(v_x),$$
$$\dot C_1 = -C_1R_3, \qquad \dot C_2 = C_2R_4 + R_2, \qquad \dot C_3 = R_3, \qquad \dot C_4 = (C_2 + C_4)R_4 + R_1.$$

In conclusion we note, that Theorem 4 can be easy generalized for classification and reduction of system of evolution equations

$$u_{it} = F_i\left(t, x, u_1, \ldots, u_n, u_1^{(1)}, \ldots, u_n^1, \ldots, u_1^{(k-1)}, \ldots, u_n^{(k-1)}\right)$$

under additional conditions

$$u_i^{(k)} = f_i\left(x, u_1, \ldots, u_n, u_1^{(1)}, \ldots, u_n^1, \ldots, u_1^{(k-1)}, \ldots, u_n^{(k-1)}\right),$$

admitting kn independent symmetries. Here $u_i^{(j)} = \frac{\partial^j u_i}{\partial x^j}$.

[1] Fushchych W.I. and Zhdanov R.Z., Anti–reduction of the nonlinear wave equation, *Proc. Acad. of Sci. Ukraine*, 1993, N 11, 37–40.

[2] Fushchych W.I. and Zhdanov R.Z., Conditional symmetry and anti-reduction of nonlinear heat equation, *Proc. Acad. of Sci. Ukraine*, 1994, N 5, 40–44.

[3] Galaktionov V.A., Invariant subspaces and new explicit solution to evolution equations with quadratic nonlinearities, *Proceedings of Royal Society of Edinburgh A*, 1995, V.125, 225–246.

[4] Zhdanov R.Z., Conditional Lie–Bäcklund symmetry and reduction of evolution equations, *J. Phys. A: Math. Gen.*, 1995, V.28, N 13, 3841–3850.

[5] Andreytsev A., On classification and reduction of conditionally invariant nonlinear evolution equations, in Mathematical Modeling in Education and Science, Sankt-Petersburg, 2000, 16–22.

[6] Basarab-Horwath P. and Zhdanov R.Z., Initial-value problems for evolutionary partial differential equations and higher-order conditional symmetries, *J. Math. Phys.*, 2001, V.42, 376–389.

[7] Zhdanov R.Z. and Andreytsev A.Yu., Non-classical reductions of initial-value problems for a class of nonlinear evolution equations, *J. Phys. A: Math. Gen.*, 2000, V.33, 5763–5781.

[8] Cherniha R.M., A constructive method for obtaining new exact solutions of nonlinear evolution equations, *Rep. on Math. Phys.*, 1996, V.38, N 3, 301–312.

[9] Sokolov V.V. and Wolf Th., A symmetry test for quasilinear coupled systems, *Inverse Problems*, 1999, V.15, 5–11.

[10] Olver P.J., Application of Lie groups to differential equations, New York, Springer, 1986.