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On Differential Equations of First- and Second-Order in the Space $M(1,3) \times R(u)$ with Nontrivial Symmetry Groups

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The differential equations of the first order in the space $M(1,3) \times R(u)$ which are invariant under splitting subgroups of the group $P(1,4)$ have been constructed. For majority of these subgroups the differential equations of the second-order in the same space have also been described.

The differential equations with nontrivial symmetry groups play an important role in theoretical and mathematical physics, mechanics, gas dynamics (see, for example, [1–8]).

In many cases these equations can be written in the following form:

$$F(J_1, J_2, \ldots, J_t) = 0, \tag{1}$$

where F is an arbitrary enough smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ are functional bases of differential invariants of the corresponding symmetry groups.

Differential invariants of the local Lie groups of the point transformations have been studied in many works (see, for example, [1, 4, 9–12]).

The present work is devoted to the construction of differential equations of the first- and second-order in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the generalized Poincaré group $P(1,4)$.

In order to give some of the results obtained we must consider the Lie algebra of the group $P(1,4)$.

1 The Lie algebra of the group $P(1,4)$ and its continuous subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ and P'_μ $(\mu, \nu = 0, 1, 2, 3, 4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma} P'_\nu - g_{\nu\sigma} P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho} M'_{\nu\sigma} + g_{\nu\sigma} M'_{\mu\rho} - g_{\nu\rho} M'_{\mu\sigma} - g_{\mu\sigma} M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

Let us consider following representation of the Lie algebra of the group $P(1,4)$

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial u}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right), \qquad x_4 \equiv u.$$

More details about this representation can be found in [8].

Further we will use following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0}, \qquad (a = 1, 2, 3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \qquad (k = 1, 2, 3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [13]. Splitting subgroups of the group $P(1,4)$ have been found in [14, 15].

2 The differential equations of the first-order in the space $M(1,3) \times R(u)$

The differential equations of the first-order in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$ have been constructed. These equations can be written in the form (1), where $\{J_1, J_2, \ldots, J_t\}$ are functional bases of differential invariants of the first-order of the splitting subgroups of the group $P(1,4)$.

Below, for some splitting subgroups of the group $P(1,4)$, we write the basis elements of its Lie algebras and corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F.

1. $\langle L_3 \rangle$,
$$J_1 = x_0, \quad J_2 = x_3, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = u, \quad J_5 = x_1 u_2 - x_2 u_1,$$
$$J_6 = u_0, \quad J_7 = u_3, \quad J_8 = u_1^2 + u_2^2, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3;$$

2. $\langle P_3 + C_3, L_3 \rangle$,
$$J_1 = x_0, \quad J_2 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_3 = \left(x_3^2 + u^2\right)^{1/2}, \quad J_4 = \frac{u_3 u + x_3}{u - x_3 u_3},$$
$$J_5 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, \quad J_6 = \frac{u_1^2 + u_2^2}{u_0^2}, \quad J_7 = \frac{u_3^2 + 1}{u_0^2};$$

3. $\langle P_1, P_2, X_3 \rangle$,
$$J_1 = x_0 + u, \quad J_2 = \left(x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2}, \quad J_3 = \frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1},$$
$$J_4 = \frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1}, \quad J_5 = \frac{u_3}{u_0 + 1}, \quad J_6 = \frac{u_1^2 + u_2^2 + 2(u_0 + 1)}{(u_0 + 1)^2};$$

4. $\langle G, P_1, P_2, P_3 \rangle$,
$$J_1 = \left(x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2\right)^{1/2}, \quad J_2 = x_1 + \frac{x_0 + u}{u_0 + 1} u_1, \quad J_3 = x_2 + \frac{x_0 + u}{u_0 + 1} u_2,$$
$$J_4 = x_3 + \frac{x_0 + u}{u_0 + 1} u_3, \quad J_5 = \left(u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1\right)\left(\frac{x_0 + u}{u_0 + 1}\right)^2;$$

5. $\langle L_3, P_1, P_2, P_3, X_4 \rangle$,
$$J_1 = x_0 + u, \quad J_2 = \frac{x_3}{x_0 + u} + \frac{u_3}{u_0 + 1},$$
$$J_3 = \left(\frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1}\right)^2 + \left(\frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1}\right)^2, \quad J_4 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2};$$

6. $\langle G, P_3, L_3, X_1, X_2, X_4 \rangle$,
$$J_1 = x_3 + \frac{x_0 + u}{u_0 + 1} u_3, \quad J_2 = \left(u_1^2 + u_2^2\right)\left(\frac{x_0 + u}{u_0 + 1}\right)^2, \quad J_3 = \frac{u_0^2 - u_3^2 - 1}{u_1^2 + u_2^2};$$

7. $\langle G, L_3, P_1, P_2, P_3, X_3, X_4\rangle$,
$$J_1 = \left(x_1 + \frac{x_0+u}{u_0+1}u_1\right)^2 + \left(x_2 + \frac{x_0+u}{u_0+1}u_2\right)^2,$$
$$J_2 = \left(u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1\right)\left(\frac{x_0+u}{u_0+1}\right)^2;$$
8. $\langle G, L_3, P_1, P_2, X_1, X_2, X_3, X_4\rangle$,
$$J_1 = \frac{x_0+u}{u_0+1}u_3, \quad J_2 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2};$$
9. $\langle G, L_1, L_2, L_3, X_0, X_1, X_2, X_3, X_4\rangle$,
$$J_1 = \frac{u_1^2 + u_2^2 + u_3^2}{u_0^2 - 1};$$
10. $\langle L_1, L_2, L_3, P_1 - C_1, P_2 - C_2, P_3 - C_3, X_1, X_2, X_3, X_0 + X_4\rangle$,
$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2;$$
11. $\langle L_1, L_2, L_3, P_1 + C_1, P_2 + C_2, P_3 + C_3, X_0, X_1, X_2, X_3, X_4\rangle$,
$$J_1 = \frac{u_1^2 + u_2^2 + u_3^2 + 1}{u_0^2}.$$

3 On differential equations of the second-order in the space $M(1,3) \times R(u)$

Some of the differential equations of the second-order in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$ have been described. The equations obtained have the form (1), where $\{J_1, J_2, \ldots, J_t\}$ are functional bases of differential invariants of the second-order of corresponding splitting subgroups of the group $P(1,4)$.

In the following, for some splitting subgroup of the group $P(1,4)$, we give the basis elements of its Lie algebra and corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F.

$\langle L_3, X_0\rangle$,
$$J_1 = x_3, \quad J_2 = x_0 - u, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0,$$
$$J_6 = u_3, \quad J_7 = u_1^2 + u_2^2, \quad J_8 = (x_1u_1 - x_2u_2)u_{01} + (x_1u_2 + x_2u_1)u_{02},$$
$$J_9 = 2\sqrt{2}\arctan\frac{u_1}{u_2} - \arctan\left(\frac{u_{11} - u_{22}}{\sqrt{2}u_{12}}\right), \quad J_{10} = u_{00}, \quad J_{11} = u_{03}, \quad J_{12} = u_{33},$$
$$J_{13} = u_{11} + u_{22}, \quad J_{14} = u_{01}^2 + u_{02}^2, \quad J_{15} = u_{13}^2 + u_{23}^2, \quad J_{16} = u_{11}^2 + u_{12}^2 + u_{22}^2,$$
$$J_{17} = u_{02}u_{13} - u_{01}u_{23}, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu, \nu = 0, 1, 2, 3.$$

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