Symmetries and Integrability Properties of Generalized Fisher Type Nonlinear Diffusion Equation

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Nonlinear reaction-diffusion systems are known to exhibit very many novel spatiotemporal patterns. Fisher equation is a prototype of diffusive equations. In this contribution we investigate the integrability properties of the generalized Fisher type equation to obtain physically interesting solutions using Lie symmetry analysis. In particular, we report several travelling wave patterns, static patterns and localized structures depending upon the choice of the parameters involved.

1 Introduction

Nonlinear partial differential equations are frequently used to model a wide variety of phenomena in physics, chemistry, biology and other fields [1, 2, 3]. In such models, when large aggregates of microstructures consisting of particles, atoms, molecules, defects, dislocations, etc. are able to move and/or interact, the evolution of the concentration of the species can be shown to obey nonlinear diffusion equations of reactive type. These equations play an important role in dissipative dynamical systems. Many interesting physical phenomena, such as wall propagation in liquid crystals, nerve impulse propagation in nerve fibres, pattern formation in dissipative systems, nucleation kinetics and neutron action in the reactor, are closely connected with the study of nonlinear diffusion equations. The underlying systems give rise to very many simple/complex patterns which are essentially distinct structures on a suitable space-time scale and they arise as collective and cooperative phenomena due to the underlying large number of constituent subsystems. These structures tell us a lot about the dynamics as well as about the microscopic behaviour of the underlying systems to some extent. As the interactions among the constituents are nonlinear, novel structures which can mimic naturally occurring patterns arise. These structures can be stationary or changing with time.

Generally, in the study of dissipative systems, one of the challenging problems is the selection mechanism. That is, one would like to know the kinds of evolving velocity and emerging patterns that would be selected in a kinetic process when the system is suddenly quenched into an unstable state. Aronson and Weinberger's work on the Fisher type nonlinear diffusion equation [4] has shown the existence of distinct selection mechanism, that is the solution $u(x,t)$ of the Fisher equation in $(1+1)$ dimensions,

$$u_t = u_{xx} + u(1-u), \tag{1}$$

converges to a local travelling wave with a definite speed from a wide class of initial data. Further it is known that equation (1) has a travelling wave solution called a cline [3] which is nothing but a wave travelling in the x-direction with $c \geq c_{\min} = 2$. However, the first explicit analytic form for a cline solution was obtained by Ablowitz and Zeppetella [5], who showed that an exact propagating wavefront solution (see Fig. 1) is of the form

$$u(x,t) = 1 - \left[1 + \frac{k}{\sqrt{6}} \exp\left(\frac{x - \frac{5}{\sqrt{6}}t}{\sqrt{6}}\right)\right]^{-2}, \tag{2}$$

where k is an arbitrary constant. Here the authors made use of the Painlevé singularity structure analysis of equation (1) to find the exact solution in the year 1979.

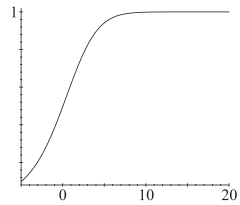

Figure 1. An exact wavefront solution [5] of the Fisher equation $\left(\xi = x - \frac{5}{\sqrt{6}}t\right)$.

There is continuing interest in recent literature [6] to investigate more general forms of the Fisher equation. For instance, there is an interesting generalization of the Fisher equation in the description of bacterial colony growth, chemical kinetics and many other natural phenomena and it is of the general form [10]

$$\frac{\partial u(\vec{r},t)}{\partial t} = D\triangle u(\vec{r},t) + \Lambda(u)[\nabla u(\vec{r},t)]^2 + \lambda u(\vec{r},t)G(u,\vec{r},t), \tag{3}$$

where D is the diffusion coefficient, Λ is the nonlocal growth rate, λ is the local growth rate, $G(u,\vec{r},t)$ is the local growth function, ∇ and $\triangle$ are gradient and Laplacian operators respectively. As a special case of equation (3), we obtain the generalized Fisher type equation

$$u_t - \triangle u - \frac{m}{1-u}(\nabla u)^2 - u(1-u) = 0, \tag{4}$$

where the subscript denotes partial differentiation with respect to time. In the study of population dynamics, $u(\vec{r},t)$ refers to the population density at point $\vec{r}$ at time t. In equation (4), the linear term modelling the birth rate gives rise to an exponential growth in time while the quadratic term that models competition between individuals for food, etc. leads to a stable, homogeneous value $u = 1$ at long times and the diffusion term models the spatial variation of the population. This introduces the possibility of spatial pattern formation between the homogeneous regions with $u = 1$ and $u = 0$ for appropriate initial conditions. Further, the classical Fisher equation ($m = 0$) occurs in models of population growth [3], neurophysiology [7], Brownian motion [8] and nuclear reactors [9]. Besides allowing for exact solutions, the $m = 2$ case finds its application in real systems such as the bacterial colony growth [10] where the square-gradient term corresponds to the nonlocal growth occuring at concentration gradients which is similar to the nonlinear terms in the Kuramoto–Sivashinsky equation for propagating flame and in the theory of growing interfaces. Moreover, models which admit exact solutions are of considerable importance for understanding general behaviour of nonlinear dissipative systems. In one dimensional space, such models have received considerable attention. But many realistic models are two or three dimensional in nature and in this direction, Brazhnik and Tyson [6] considered equation (4) in two spatial dimensions and explored five kinds of travelling wave patterns namely plane, V and Y-waves, a separatrix and space oscillating propagating structures. All these structures were found when the medium is unbounded and spatially homogeneous. Further they show that when the medium is bounded and no flux is allowed through the boundaries, only plane and oscillating waves survive because the frontline of the wave must approach the boundary orthogonally.

In general, obtaining solutions for reaction-diffusion systems is more complex than that for pure dispersive systems. For the latter there are several analytical methods like the inverse scattering transform method [11], the Hirota method [12], Bäcklund transformation method [13], Lie–Bäcklund symmetries method and so on. On the other hand, for nonlinear diffusive systems, no such formal techniques are available to solve them analytically. Very often perturbation analysis or numerical techniques are used to treat them. There is therefore an urgent need to isolate and identify integrable nonlinear reaction-diffusion systems which can act as model systems to deal with more complicated cases. In this connection, symmetry analysis can play a very crucial role.

Consider for example, the well known case of Burgers equation

$$u_t = \nu u_{xx} + u u_x,$$

where ν is the diffusion coefficient. It can be considered to be integrable in the sense that it is linearizable: Under the Cole–Hopf transformation $u = -\nu v_x/v$, it reduces to the linear heat equation. It possesses interesting Lie point symmetry structures and infinite number of Lie–Bäcklund symmetries. So, it will be quite interesting to know about other such integrable reaction-diffusion equations and the role of symmetries that allows the system to exhibit different spatiotemporal patterns and structures which usually possess some kind of symmetry. In this direction, the method of Lie groups is the most powerful method to analyse nonlinear partial differential equations (PDEs) and hence we make use of it and the singularity structure analysis to investigate the integrability properties and hence the dynamics/patterns of (4). We report in this paper that the $m = 2$ case of equation (4) possesses infinite dimensional Lie symmetry structure, which allows one to linearize it both in $(1 + 1)$ and $(2 + 1)$ dimensions and to obtain a large class of exact solutions. We also obtain several exact solutions for the $m \neq 2$ case.

The plan of the paper is as follows. In Section 2, we briefly recall some of the important reaction-diffusion equations exhibiting novel/complex patterns. Then in Section 3, by carrying out the singularity structure analysis, we point out that the PDE (4) is free from movable critical singular manifolds for the specific value $m = 2$. More interestingly, we point out that the Bäcklund transformation deduced from the Laurent expansion gives rise to the linearizing transformation for this case in a natural way. In Sections 4 and 5, we discuss different underlying patterns via symmetry analysis and similarity reductions for the generalized Fisher type equation in 1- and 2-spatial dimensions, respectively. Finally we summarize our results in Section 6.

2 Reaction-diffusion systems and various patterns

The general form of the nonlinear reaction-diffusion equation is given by

$$\frac{\partial \underline{C}}{\partial t} = \vec{\nabla} \cdot (D \vec{\nabla} \underline{C}) + \vec{F}(\underline{C}^T, \vec{r}, t), \qquad \underline{C} = (c_1, c_2, \ldots, c_n)^T,$$
$$D = \operatorname{diag}(D_1, D_2, \ldots, D_n), \qquad \vec{F} = (f_1, f_2, \ldots, f_n)^T.$$

Here $\underline{C}$ represents the population or concentration densities of the species and D and $\vec{F}$ are, in general, nonlinear functions of $\underline{C}$ representing the diffusivity and the reaction kinetics respectively. In such a case, the dynamics is dominated by the onset of patterns. Inspite of the absence of rigorous analytical tools as in the case of soliton systems, combined local analysis and numerical investigations on such systems have been found to exhibit a number of important spatiotemporal patterns.

Some of the dominant patterns exhibited by these systems are homogeneous or uniform steady states, travelling waves, spiral waves, Turing patterns (rolls, stripes, hexagons, rhombs, etc.), localized structures, spatiotemporal chaos and so on. A few of the well known models include the following:

2.1 The Oregonator model

This model explains the various features of the Belousov–Zhabotinsky reaction and was introduced by Fields, Körös and Noyes of University of Oregon, USA in 1972. In its simplest version it reads as [14]

$$
\begin{aligned}
u_{1t} &= D_1\nabla^2 u_1 + \eta^{-1}\left[u_1(1-u_1) - \frac{bu_2(u_1-a)}{(u_1+a)}\right], \\
u_{2t} &= D_2\nabla^2 u_2 + u_1 - u_2.
\end{aligned} \tag{5}
$$

Here u_1 is the concentration of the autocatalytic species $HBrO_2$, u_2 is the concentration of the transition ion catalyst in the oxidised state Ce^{3+} and Fe^{3+} and η, a and b are parameters. This model is the most popular among the pattern forming chemical reactions. In particular, (5) exhibits 'propagating pulse solutions' that can travel through the system without attenuation. Besides, it admits periodic wave trains, target patterns and in two dimensions they generate spiral waves.

2.2 Gierer–Meinhardt model

It describes possible interaction between an activator a and a rapidly diffusing inhibitor h and is of the form [15]

$$
\begin{aligned}
a_t &= D_a\nabla^2 a + \rho_a\frac{a^2}{(1+K_a a^2)} - \mu_a a + \sigma_a, \\
h_t &= D_h\nabla^2 h + \rho_h a^2 - \mu_h h + \sigma_h,
\end{aligned}
$$

where D_a and D_b are the two diffusion coeffcients, ρ_a and ρ_b are the removal rates and σ_a and σ_b are the basic production terms of the activator and inhibitor respectively. Further K_a corresponds to the saturation constant. This model is mainly used in the study of the development of an organism in biological pattern formation. They are also used to model cell differentiation, cell movement, shape changes of cells and tissues and so on.

2.3 Brusselator model

Among the various reaction-diffusion type model systems, this is one of the best studied models for the formation of chemical patterns theoretically [16]. It is based on the chemical reactions

$$
A \longrightarrow X, \qquad B + X \longrightarrow Y, \qquad 2X + Y \longrightarrow 3X, \qquad X \longrightarrow E,
$$

where the concentration of the species A, B and E are maintained constant. Thus they form the real constant parameters of the system. The evolution of the active species X and Y can be described by

$$
\begin{aligned}
X_t &= A - (B+1)X + X^2Y + D_X\nabla^2 X, \\
Y_t &= BX - X^2Y + D_Y\nabla^2 Y,
\end{aligned} \tag{6}
$$

after proper rescaling. Here D_X and D_Y are diffusion coefficients. This model exhibits heterogeneous patterns through Turing instability.

2.4 Lotka–Volterra predator-prey model

Taking into consideration the interaction of two species in which the population of the prey is dependent on the predator and vice-versa, the model equations [17] become

$$S_{1t} = D_1 S_{1xx} + a_1 S_1 - b_1 S_1 S_2,$$
$$S_{2t} = D_2 S_{2xx} - a_2 S_2 + b_2 S_1 S_2.$$

Here S_1 and S_2 are the population densities of prey and predator. D_1 and D_2 are the diffusivities of the two populations, respectively. The parameters a_1, a_2 are the linear ratio of birth and death rates of the individual species while b_1, b_2 are the nonlinear decay and growth factors due to interaction.

2.5 FitzHugh–Nagumo nerve conduction model

The Hodgkin–Huxley model describes the propagation of the electrical impulses along the axonal membrane of a nerve fibre. FitzHugh–Nagumo nerve conduction equation [18] is the simplest version of the above model and is represented by the following set of equations:

$$\begin{aligned} V_t &= V_{xx} + V - \frac{V^3}{3} - R + I(x,t), \\ R_t &= c(V + a - bR). \end{aligned} \tag{7}$$

Here the membrane potential is $V(x,t)$, R corresponds to the lumped refractory variable and $I(x,t)$ is the external injected current. The parameters a and b are positive constants while c stands for the temperature factor. The above model has been widely used to study various phenomena in neurophysiology and cardiophysiology. This system exhibits travelling wave pulses [3]. In particular, the two-dimensional version of (7) admits ring wave patterns as well as spiral wave patterns for a variety of special initial conditions.

As mentioned in the introduction, symmetries can play a very important role in determining the underlying dynamics of nonlinear systems. Particularly they can help to identify integrable cases of the above type of reactive-diffusive systems, if they exist. As an important case study, we now investigate integrability and symmetry properties of the generalized Fisher type equation (4).

3 Singularity structure analysis

This analysis separates out the $m = 2$ case for both the $(1+1)$ and $(2+1)$ dimensions as the only system for which the Fisher equation (4) is free from movable critical singular manifolds satisfying the Painlevé property [13]. By locally expanding the solution in the neighbourhood of the non-characteristic singular manifold $\phi(x,t) = 0$, $\phi_x, \phi_t \neq 0$ in the form of the Laurent series [19]

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+p},$$

the possible values of the power of the leading order term are found to be

$$(i)\ p = -2,$$
$$(ii)\ p = \frac{1}{1-m}, \qquad m \neq 1,$$
$$(iii)\ p = 0.$$

For all these leading orders, only for the value $m = 2$ the solution is free from movable critical singular manifolds since for $p = -1$ the leading order coefficient u_0 becomes arbitrary besides the arbitrary singular manifold ϕ. In all other cases only one arbitrary function exists for $m = 2$ thereby leading to special solutions.

More interestingly, from the Laurent series expansion if we cut off the series at "constant" level term, that is $j = -p$ for the leading order $p = 1/(1-m) = -1$, $m = 2$, one can deduce the Bäcklund transformation that gives rise to the linearizing transformation in a natural way. Thus, defining the relation

$$u = \frac{u_0}{\phi} + u_1, \tag{8}$$

we demand that if u_1 is a solution of equation (4) for the case $m = 2$, then u is also a solution, from which the Bäcklund transformation is deduced. Now starting from the trivial solution, $u_1 = 0$ of (4), we find that the equations for u_0 and ϕ in equation (8) are consistent for the choice $u_0 = \phi$, giving rise to the new solution $u = 1$. This is nothing but an exact solution of equation (4). Then with $u_1 = 1$ as the new seed solution, one can check from equations satisfied by u_0 and ϕ that

$$u_0 = -1, \qquad \phi_t - \phi_{xx} - \phi + 1 = 0. \tag{9}$$

Choosing $\phi = 1 + \chi$, equation (9) can be rewritten as the linear heat equation,

$$\chi_t - \chi_{xx} - \chi = 0. \tag{10}$$

Thus the transformation

$$u = 1 - \frac{1}{1+\chi}, \tag{11}$$

where χ satisfies the linear heat equation (10), is the linearizing transformation for equation (1) in $(1 + 1)$ dimensions for the choice $m = 2$ in an automatic way. We note that this is exactly the transformation given in ref. [20] in an adhoc way. Here we have given an interpretation for the transformation in terms of the Bäcklund transformation. The same transformation (11) linearizes equation (3) in $(2 + 1)$ dimensions (for $m = 2$) as well, where χ satisfies the two dimensional linear heat equation $\chi_t - \chi_{xx} - \chi_{yy} - \chi = 0$. Further equation (11) transforms equation (4) in $(3 + 1)$ dimensions to the 3-dimensional heat equation as well; however, we do not study the case further here.

4 Symmetries and integrability properties of $(1 + 1)$ dimensional generalized Fisher equation

The generalized Fisher equation (4) in its $(1 + 1)$ dimensional form reads as

$$u_t - u_{xx} - \frac{m}{1-u} - u + u^2 = 0. \tag{12}$$

An invariance analysis of equation (12) under the infinitesimal transformations

$$x \longrightarrow X = x + \varepsilon\xi(t,x,u), \qquad t \longrightarrow T = t + \varepsilon\tau(t,x,u),$$
$$u \longrightarrow U = u + \varepsilon\phi(t,x,u), \qquad \varepsilon \ll 1,$$

separates out the $m = 2$ case in that it possesses a nontrivial infinite-dimensional Lie algebra of symmetries

$$\tau = a, \qquad \xi = b, \qquad \phi = c(t,x)(1-u)^2.$$

Here a, b are arbitrary constants and $c(t,x)$ is any solution of the linear heat equation $c_t - c_{xx} - c = 0$. For all other values of m in equation (12) one gets trivial translation symmetries

$$\tau = a, \qquad \xi = b, \qquad \phi = 0.$$

In order to obtain solutions of physical importance and corresponding patterns, we make use of the method of similarity reductions. This leads to the similarity reduced variables for the $m = 2$ case as

$$z = ax - bt, \qquad u = 1 - \frac{a}{a + v(z) + \int c(t,x)dt}. \tag{13}$$

Using (13), equation (12) can be reduced for the $m = 2$ case to the similarity reduced ordinary differential equation (ODE)

$$a^2 v'' + bv' + v = 0$$

whose general solution is

$$v = I_1 e^{m_1 z} + I_2 e^{m_2 z}, \qquad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a^2}}{2a^2},$$

where I_1 and I_2 are integration constants thereby leading to

$$u = \begin{cases} 1 - \dfrac{a}{a + I_1 e^{m_1(ax-bt)} + I_2 e^{m_2(ax-bt)} + \int c(t,x)dt}, & b^2 - 4a^2 > 0; \\ 1 - \dfrac{a}{a + e^{p(ax-bt)}\left(I_1 + I_2(ax-bt)\right) + \int c(t,x)dt}, & b^2 - 4a^2 = 0; \\ 1 - \dfrac{a}{a + e^{p(ax-bt)}\left(I_1 \cos q(ax-bt) + I_2 \sin q(ax-bt)\right) + \int c(t,x)dt}, & b^2 - 4a^2 < 0 \end{cases}$$

with $p = -b/2a^2$, $q = \sqrt{4a^2 - b^2}/2a^2$, as the solution to the original PDE (12). Here the similarity reduced variable (13) is nothing but the linearizing transformation (11).

Proceeding in a similar fashion for all the other (nonintegrable) cases ($m \neq 2$), the similarity variables $z = ax - bt$ and $u = w(z)$ reduce equation (12) to the ODE

$$a^2 vv'' - ma^2 v'^2 + bvv' - (1-v)v^2 = 0, \qquad v = 1 - w, \tag{14}$$

which is in general nonintegrable except for $m = 0$ and $b/a = 5/\sqrt{6}$. This special choice leads to the cline solution (2) obtained by Ablowitz and Zeppetella [5]. In the static case ($b = 0$), one obtains elliptic function solutions. Besides, a particular solitary wave solution

$$u = 1 - \frac{(3-2m)}{(2-2m)}\left[\operatorname{sech}^2\left(I_2 - \frac{x}{2}\sqrt{\frac{1}{1-m}}\right)\right], \qquad m < 1$$

with I_2 as the second integration constant, which is a limiting case of a elliptic function solution, is also obtained (refer Fig. 2). In the general case, as equation (14) is of nonintegrable nature, we make use of numerical techniques to study the underlying dynamics. Here we obtain typical periodic wave trains for $b/a = 0$ which is in accordance with the fact that reaction-diffusion systems exhibiting limit cycle motion in the absence of diffusion exhibits travelling wave patterns (Fig. 3a,b). For $b/a = 1$, we get a propagating pulse (Fig. 3c) and the corresponding phase portrait ($v - v'$) shows a stable spiral equilibrium point (Fig. 3d). On increasing the value of b/a, that is, at $b/a \geq 2$ ($b = 2.041$) [5], the system supports a travelling wave front (Fig. 3e) and the trajectories in the phase plane ($v - v'$) correspond to a stable node (Fig. 3f).

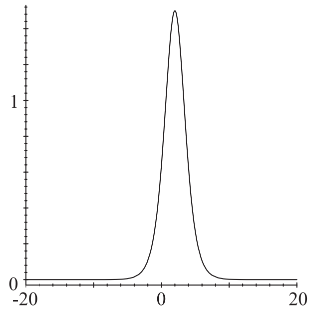

Figure 2. A static solitary wave pulse for $m = 1/2$ of the generalized Fisher equation (12).

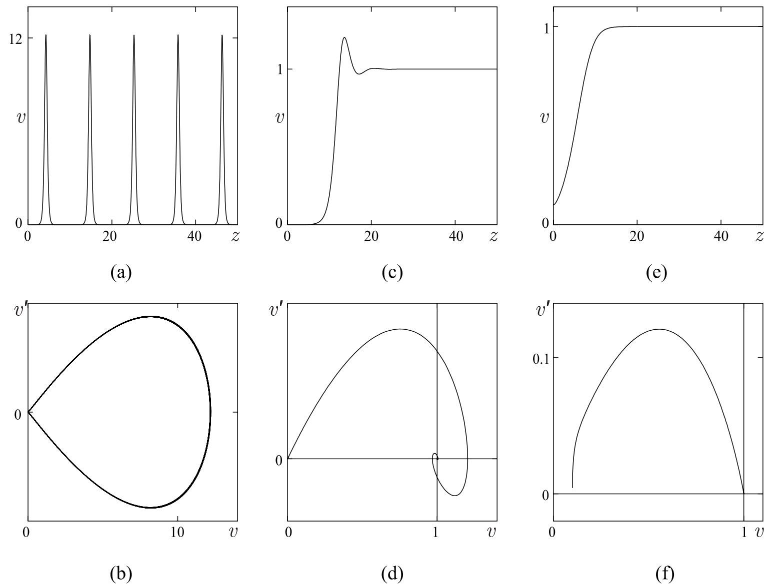

Figure 3. Propagating patterns and corresponding phase portraits in the $v - v'$ plane of equation (14): (a) periodic pulses; (b) limit cycle; (c) travelling pulse; (d) stable spiral; (e) travelling wavefront; (f) stable node.

5 The $(2+1)$ dimensional generalized Fisher equation

Extending a similar analysis to the $(2+1)$ dimensional case of the generalized Fisher equation

$$u_t - u_{xx} - u_{yy} - \frac{m}{1-u}\left(u_x^2 + u_y^2\right) - u + u^2 = 0, \tag{15}$$

one finds that the invariance analysis of equation (15) under the infinitesimal transformation singles out the special value $m = 2$ for which the Lie point symmetries are

$$\tau = a, \qquad \xi = b_3 y + b_4, \qquad \eta = -b_3 x + d_4, \qquad \phi = c(t,x,y)(1-u)^2,$$

where η is the infinitesimal symmetry associated with the variable y, $c(t,x,y)$ is the solution of the two dimensional linear heat equation $c_t - c_{xx} - c_{yy} - c = 0$ and b_3, b_4 and d_4 are arbitrary constants. But for all other choices of m ($\neq 2$) we get

$$\tau = a, \qquad \xi = b_3 y + b_4, \qquad \eta = -b_3 x + d_4, \qquad \phi = 0.$$

In a similar fashion as that for the $(1+1)$ dimensional case, the similatiry variables for the $m = 2$ case

$$z_1 = \frac{b_3}{2}(x^2 + y^2) + b_4 y - d_4 x, \qquad z_2 = -t - \frac{a}{b_3}\sin^{-1}\left(\frac{d_4 - b_3 x}{\sqrt{d_4^2 + 2b_3 z_1 + b_4^2}}\right),$$

$$u = 1 - \frac{a}{w(z_1, z_2) + \int c(t, x, y)dt} \tag{16}$$

reduce the PDE (15) to

$$w_{z_2} + 2b_3 w_{z_1} + \left(2b_3 z_1 + b_4^2 + d_4^2\right) w_{z_1 z_1} + \frac{a^2 w_{z_2 z_2}}{2b_3 z_1 + b_4^2 + d_4^2} + w - a = 0. \tag{17}$$

Here too one can obtain the linear heat equation

$$\chi_t - \chi_{xx} - \chi_{yy} - \chi = 0,$$
$$\chi = \frac{1}{a}\left[w(z_1, z_2) + \int c(t, x, y)dt\right],$$

from the similarity form (16). Such a transformation can be interpreted as the linearizing transformation from a group theoretical point of view.

Carrying out a Lie symmetry analysis for equation (17) also, one can obtain the new similarity variables

$$\zeta = \bar{z}_1, \quad w = a + e^{\left(\frac{c_1 \bar{z}_2}{c_3}\right)}\left[f(\zeta) + \frac{1}{c_3}\int \hat{c}_2(\bar{z}_1, \bar{z}_2) e^{\left(-\frac{c_1}{c_3}\bar{z}_2\right)} d\bar{z}_2\right],$$
$$\bar{z_1} = 2b_3 z_1 + b_4^2 + d_4^2, \qquad \bar{z}_2 = z_2, \qquad b_3, d_4 \neq 0,$$

where f satisfies the linear second order ODE of the form

$$\zeta^2 f'' + \zeta f' + (A + B\zeta)f = 0,$$
$$A = (ac_1/2b_3c_3)^2, \qquad B = (1 + c_1/c_3)/4b_3^2, \tag{18}$$

with prime denoting differentiation w.r.t. ζ. Thus the solution to the original PDE reads as

$$u = 1 - a\left[a + e^{\left(\frac{c_1}{c_3}\bar{z}_2\right)}\left(I_1 Z_1\left(2\sqrt{B\bar{z}_1}\right) + I_2 Z_2\left(2\sqrt{B\bar{z}_1}\right)\right.\right.$$
$$\left.\left. - \int \frac{\hat{c}_2(\bar{z}_1, \bar{z}_2)}{c_3} e^{\left(\frac{c_1}{c_3}\bar{z}_2\right)} d\bar{z}_2\right) + \int c(t, x, y)dt\right]^{-1}. \tag{19}$$

In the limit $b_4 = d_4 = c_1 = 0$ the system is found to exhibit circularly symmetric structures given in Fig. 4.

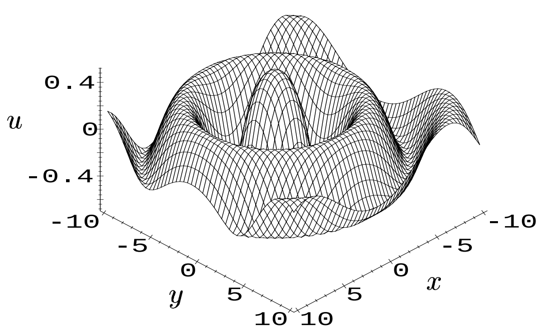

Figure 4. Circularly symmetric patterns of the $(2+1)$ dimensional generalized Fisher equation (15) for $m = 2$.

More interestingly, in the special case $b_3 = 0$, $d_4 = 0$ the system exhibits propagating wave structures and the corresponding forms are

$$u = \begin{cases} 1 - a\Bigg\{a + \exp\left[-k\left(\dfrac{a}{b_4}x - t\right)\right]\Bigg[I_1\cos(\sqrt{k_1}c_5b_4y) + I_2\sin\left(\sqrt{k_1}c_5b_4y\right) \\ \qquad + \displaystyle\int \frac{\hat{c}_3(z_1,z_2)}{c_5}e^{kz_2}dz_2\Bigg] + \int c(t,x,y)dt\Bigg\}^{-1}, \quad k_1 < 0, \\ 1 - a\Bigg\{a + \exp\left[-k\left(\dfrac{a}{b_4}x - t\right)\right]\Bigg[I_1e^{\sqrt{k_1}c_5b_4y} + I_2e^{-\sqrt{k_1}c_5b_4y} \\ \qquad + \displaystyle\int \frac{\hat{c}_3(z_1,z_2)}{c_5}e^{kz_2}dz_2\Bigg] + \int c(t,x,y)dt\Bigg\}^{-1}, \quad k_1 > 0, \\ 1 - a\Bigg\{a + \exp\left[-k\left(\dfrac{a}{b_4}x - t\right)\right]\Bigg[I_1c_5b_4y + I_2 \\ \qquad + \displaystyle\int \frac{\hat{c}_3(z_1,z_2)}{c_5}e^{kz_2}dz_2\Bigg] + \int c(t,x,y)dt\Bigg\}^{-1}, \quad k_1 = 0, \end{cases} \tag{20}$$

where the parameter $k_1 = \frac{1}{b_4^2c_5^2}\left[k - \left(\frac{ak}{b_4}\right)^2 - 1\right]$ with $k = -c_2/c_5$ and $z_1 = b_4y$, $z_2 = \frac{a}{b_4}x - t$. Here c_2, c_4, c_5 are arbitrary constants of integration. Equation (20), in particular exhibits the five classes of bounded travelling wave solutions reported by Brazhnik and Tyson [6] for certain choice of the parameters involved along with the specific assumptions of the functions $\hat{c}_3(z_1,z_2) = 0$ and $c(t,x,y) = 0$. The corresponding solutions are given below.

Among the classes of solutions, the simplest travelling wave solution (Fig. 5a)

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right) \pm \sqrt{k_1}c_5b_4y\right]}, \qquad k_1 > 0$$

can be constructed by assuming either $I_1 = 0$ or $I_2 = 0$. For $I_1 = I_2$ ($\neq 0$), we obtain a V-wave pattern (Fig. 5b)

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]\cosh\left(\sqrt{k_1}c_5b_4y\right)}, \qquad k_1 > 0.$$

Again the case $I_2 = 0$ and $k_1 < 0$ leads to a wave front oscillating in space (Fig. 5c) and is represented by

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]\left|\cos\left(\sqrt{k_1}c_5b_4y\right)\right|}.$$

But when $I_1 \neq 0$ and $I_2 = 0$ we get a separatrix (Fig. 5d)

$$u = 1 - \frac{1}{1 + A|y|\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]}.$$

Finally for positive k_1 and $I_1 = -I_2$ the Y-wave solution (Fig. 5e) becomes

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]\left|\sinh\left(\sqrt{k_1}c_5b_4y\right)\right|}.$$

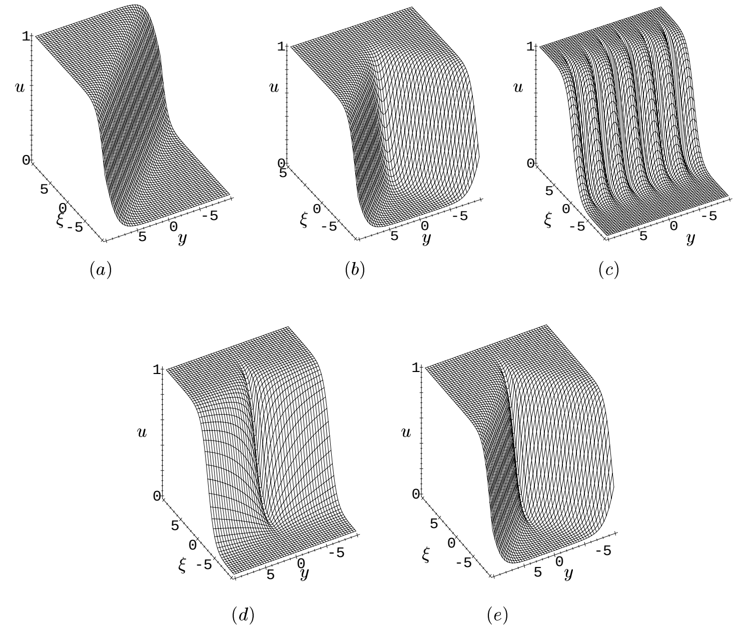

Figure 5. Five interesting classes of propagating wave patterns as obtained in ref. [6], which follow from equations (20): (a) travelling waves; (b) V-waves; (c) oscillating front; (d) separatrix solution; (e) Y-waves with $\xi = -k(\frac{a}{b_4}x - t)$.

In each of the above solutions A is a positive constant. It is a well known fact about Fisher equation is that it forms a basis for many nonlinear models of different nature. As a result, the above solutions are reminiscent of patterns from different fields. In particular, V-waves are characterized in the framework of geometrical crystal growth related models [21] and in excitable media [22] while space oscillating fronts are relevant to cellular flame structures and patterns in chemical reaction diffusion systems [23]. Further it has been shown in [24] with a geometrical model that excitable media can support space-oscillating fronts. Several static structures can also be obtained as limiting cases of the above solutions (19) and (20).

Finally, a similar analysis for the nonintegrable $(m \neq 2)$ case yields static patterns/structures in (x, y) variables. Here one has to look for certain special solutions due to its nonintegrable nature. That is, for $b_3 = 0$ and $d_4 = 0$ with the similarity variables $z_1 = b_4 y$, $z_2 = \frac{a}{b_4}x - t$, $u = w(z_1, z_2)$, the reduced ODE reads as

$$
\begin{aligned}
&Df'' + \frac{Dm}{1-f}f'^2 - c_1 f' + f(1-f) = 0,\\
&D = \left(\frac{a^2}{b_4^2}{c_1}^2 + b_4^2 {c_2}^2\right), \qquad ()' = d/d\zeta,
\end{aligned}
$$

with $\zeta = -c_1\left(\frac{a}{b_4}x - t\right) + c_2 b_4 y$ and $w = f(\zeta)$, giving rise to plane wave structures. For $b_3 = 0$, the similarity variables $z_1 = d_4 x - b_4 y$, $z_2 = ax - b_4 t$ and $u = w(z_1, z_2)$ reduces the PDE to an ODE

$$
Af_1'' + Bf_1' - \frac{Am}{f_1}f_1'^2 - f_1 + f_1^2 = 0, \qquad ()' = d/d\zeta
$$

with $f_1 = 1 - f$, $A = a^2\left(c_1^2 d_4^2 + c_2^2 b_4^2\right)$, $B = -d_4 b_4(c_1 + c_2)$ and $\zeta = ac_2(d_4 x - b_4 y) - d_4(c_1 + c_2)(ax - b_4 t)$, $w = f(\zeta)$. Then the system is found to possess elliptic function solutions including the limiting case of the solitary pulse for certain choices of the constants involved.

6 Conclusion

Our studies on the integrability/symmetry properties of the the generalized Fisher type nonlinear reaction-diffusion equation show that the system under consideration possesses interesting Lie point symmetries that could form infinite dimensional Lie algebra for the particular choice of the system parameter $m = 2$, thereby exhibiting various interesting patterns and dynamics. Besides, the singularity structure analysis singles out the $m = 2$ case as the only system parameter for which the generalized Fisher type equation is free from movable critical singular manifolds. The generalized Fisher equation is found to possess a large number of interesting wave patterns. It will be of interest to consider other physically interesting reaction-diffusion systems from the Lie symmetry point of view and to study the underlying patterns.

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