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Geometry of Nonlinear Supersymmetry in Curved Spacetime and Unity of Nature

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A new Einstein–Hilbert type action of superon-graviton model (SGM) for space-time and matter is obtained based upon the geometrical arguments of the higher symmetric (SGM) space-time. SGM action is invariant under [global NL SUSY] $\otimes$ [local $GL(4,\mathbb{R})$] $\otimes$ [local Lorentz] $\otimes$ [global $SO(N)$]. The explicit form of SGM action is given in terms of the fields of the graviton and superons by using the affine connection formalism. Some characteristic structures of the gravitational coupling of superons are manifested (in two dimensional space-time) with some details of the calculations. SGM cosmology is discussed briefly.

1 Introduction

To explore the new physics and the new framework for the unification of space-time and matter beyond the standard mode (SM), new (gauge) symmetries and new particles yet to be observed are introduced in the model building. Supersymmetry [1, 2] may be the most promising notion beyond SM, especially for the unification of space-time and matter.

In the previous paper [3] we have introduced a new fundamental constituent with spin 1/2 superon and proposed superon-graviton model (SGM) as a model for unity of space-time and matter. In SGM, the fundamental entities of nature are the graviton with spin-2 and a quintet of superons with spin-1/2. They are the elementary gauge fields corresponding to the local $GL(4,\mathbb{R})$ and the global nonlinear supersymmetry (NL SUSY) with a global $SO(10)$, respectively. All observed elementary particles including gravity are assigned to a single massless irreducible representation of $SO(10)$ super-Poincaré (SP) symmetry and reveal a remarkable potential for the phenomenology, e.g. the three-generations structure of quarks and leptons, stability of proton, mixings, etc. [3]. And except graviton they are supposed to be the (massless) composite-eigenstates of superons of $SO(10)$ SP symmetry [4] of space-time and matter. The uniqueness of $N = 10$ among all $SO(N)$ SP is pointed out. The arguments are group theoretical so far.

In order to obtain the fundamental action of SGM which is invariant at least under local $GL(4,\mathbb{R})$, local Lorentz, global NL SUSY transformations and global $SO(10)$, we have performed the similar arguments to Einstein general relativity theory (EGRT) in the SGM space-time, where the tangent (Riemann-flat) Minkowski space-time is specified by the coset space $SL(2,\mathbb{C})$ coordinates (corresponding to Nambu–Goldstone (N–G) fermion) of NL SUSY of Volkov–Akulov (V–A) [2] in addition to the ordinary Lorentz $SO(3,1)$ coordinates [3], which are locally homomorphic groups. As shown in Ref. [5] the SGM action obtained by the geometrical arguments of SGM space-time is naturally the analogue of Einstein–Hilbert (E–H) action of GR and has the similar concise expression. (The similar systematic arguments are applicable to spin 3/2 N–G case [6].)

In this article, after a brief review of SGM for the self contained arguments we expand SGM action in terms of the fields of graviton and superons in order to see some characteristic structures of our model and to show some details of the calculations. For the sake of simplicity the expansion is performed by the affine connection formalism.

Finally some hidden symmetries and a potential cosmology, especially the birth of the universe are mentioned briefly.

2 Fundamental action of superon-graviton model (SGM)

SGM space-time is defined as the space-time whose tangent(flat) space-time is specified by $SO(1,3)$ Lorentz coordinates x^a and the coset space $SL(2,\mathbb{C})$ coordinates ψ of NL SUSY of Volkov–Akulov (V–A) [2]. The unified vierbein $w_a{}^\mu$ and the unified metric $s^{\mu\nu}(x) \equiv w_a{}^\mu(x) w^{a\nu}(x)$ of SGM space-time are defined by generalizing the NL SUSY invariant differential forms of V–A to the curved space-time [5]. SGM action is given as follows [5]

$$L_{\rm SGM} = -\frac{c^3}{16\pi G}|w|(\Omega + \Lambda), \tag{1}$$

$$|w| = \det w_a{}^\mu = \det(e_a{}^\mu + t_a{}^\mu), \qquad t_a{}^\mu = \frac{\kappa}{2i}\sum_{j=1}^{10}(\bar\psi^j\gamma_a\partial^\mu\psi^j - \partial^\mu\bar\psi^j\gamma_a\psi^j), \tag{2}$$

where κ is an arbitrary constant of V–A up now with the dimension of the fourth power of length, $e_a{}^\mu$ and ψ^j $(j = 1, 2, \ldots, 10)$ are the fundamental elementary fields of SGM, i.e. the vierbein of (EGRT) and the superons of N–G fermion of NL SUSY of Volkov–Akulov [2], respectively. Λ is a cosmological constant which is necessary for SGM action to reduce to V–A model with the first order derivative terms of the superon in the Riemann-flat space-time. Ω is a unified scalar curvature of SGM space-time analogous to the Ricci scalar curvature R of EGRT. SGM action (1) is invariant under the following new SUSY transformations

$$\delta\psi^i(x) = \zeta^i + i\kappa(\bar\zeta^j\gamma^\rho\psi^j(x))\partial_\rho\psi^i(x), \tag{3}$$

$$\delta e^a{}_\mu(x) = i\kappa(\bar\zeta^j\gamma^\rho\psi^j(x))D_{[\rho}e^a{}_{\mu]}(x), \tag{4}$$

where ζ^i, $(i = 1, \ldots, 10)$ is a constant spinor, $D_{[\rho}e^a{}_{\mu]}(x) = D_\rho e^a{}_\mu - D_\mu e^a{}_\rho$ and D_μ is a covariant derivative containing a symmetric affine connection. The explicit expression of Ω is obtained by just replacing $e_a{}^\mu(x)$ in Ricci scalar R of EGRT by the vierbein $w_a{}^\mu(x) = e_a{}^\mu + t_a{}^\mu$ of the SGM curved space-time, which gives the gravitational interaction of $\psi(x)$ invariant under (3) and (4). The overall factor of our model is fixed to $\frac{-c^3}{16\pi G}$, which reproduces E–H action of GR in the absence of superons(matter). Also in the Riemann-flat space-time, i.e. $e_a{}^\mu(x) \to \delta_a{}^\mu$, it reproduces V–A action of NL SUSY[2] with $\kappa_{\rm V\text{–}A}^{-1} = \frac{c^3}{16\pi G}\Lambda$ in the first order derivative terms of the superon. Therefore our model (SGM) predicts a (small) non-zero cosmological constant, provided $\kappa_{\rm V\text{–}A} \sim O(1)$, and posesses two mass scales. Furthermore it fixes the coupling constant of superon (N–G fermion) with the vacuum to $\left(\frac{c^3}{16\pi G}\Lambda\right)^{\frac{1}{2}}$ (from the low energy theorem viewpoint), which may be relevant to the birth of the universe.

It is interesting that our action is the vacuum (matter free) action in SGM space-time as read off from (1) but gives in ordinary Riemann space-time the E–H action with matter (superons) accompanying the spontaneous supersymmetry breaking.

The commutators of new SUSY transformations induce the generalized general coordinate transformations

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\psi = \Xi^\mu\partial_\mu\psi, \tag{5}$$

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]e^a{}_\mu = \Xi^\rho\partial_\rho e^a{}_\mu + e^a{}_\rho\partial_\mu\Xi^\rho, \tag{6}$$

where Ξ^μ is defined by

$$\Xi^\mu = 2ia(\bar\zeta_2\gamma^\mu\zeta_1) - \xi_1^\rho\xi_2^\sigma e_a{}^\mu(D_{[\rho}e^a{}_{\sigma]}). \tag{7}$$

We have shown that our action is invariant at least under [7]

$$[\text{global NL SUSY}] \otimes [\text{local } GL(4,\mathbb{R})] \otimes [\text{local Lorentz}] \otimes [\text{global } SO(N)], \tag{8}$$

which is isomorphic to $N = 10$ extended (global $SO(10)$) SP symmetry through which SGM reveals the spectrum of all observed particles in the low energy [4]. In contrast with the ordinary SP SUSY, SGM SUSY may be regarded as a square root of a generalized $GL(4,\mathbb{R})$. The usual local $GL(4,\mathbb{R})$ invariance is obvious by the construction.

The simple expression (1) invariant under the above symmetry may be universal for the gravitational coupling of Nambu–Goldstone (N–G) fermion, for by performing the parallel arguments we obtain the same expression for the gravitational interaction of the spin-3/2 N–G fermion [6].

Now to clarify the characteristic features of SGM we focus on $N = 1$ SGM for simplicity without loss of generality and write down the action explicitly in terms of ${t^a}_\mu$ (or ψ) and $g^{\mu\nu}$ (or ${e^a}_\mu$). We will see that the graviton and superons (matter) are complementary in SGM and contribute equally to the curvature of SGM space-time. Contrary to its simple expression (1), it has rather complicated and rich structures.

We use the Minkowski tangent space metric $\frac{1}{2}\{\gamma^a,\gamma^b\} = \eta^{ab} = (+,-,-,-)$ and $\sigma^{ab} = \frac{i}{4}[\gamma^a,\gamma^b]$. (Latin $(a,b,\ldots)$ and Greek $(\mu,\nu,\ldots)$ are the indices for local Lorentz and general coordinates, respectively.) By requiring that the unified action of SGM space-time should reduce to V–A in the flat space-time which is specified by x^a and $\psi(x)$ and that the graviton and superons contribute equally to the unified curvature of SGM space-time, it is natural to consider that the unified vierbein ${w^a}_\mu(x)$ and the unified metric $s^{\mu\nu}(x)$ of unified SGM space-time are defined through the NL SUSY invariant differential forms ω^a of V–A [2] as follows:

$$\omega^a = {w^a}_\mu dx^\mu, \tag{9}$$

$${w^a}_\mu(x) = {e^a}_\mu(x) + {t^a}_\mu(x), \tag{10}$$

where ${e^a}_\mu(x)$ is the vierbein of EGRT and ${t^a}_\mu(x)$ is defined by

$${t^a}_\mu(x) = i\kappa\bar{\psi}\gamma^a\partial_\mu\psi, \tag{11}$$

where the first and the second indices of ${t^a}_\mu$ represent those of the γ matrices and the general covariant derivatives, respectively. We can easily obtain the inverse ${w_a}^\mu$ of the vierbein ${w^a}_\mu$ in the power series of ${t^a}_\mu$ as follows, which terminates with t^4 (for 4 dimensional space-time):

$${w_a}^\mu = {e_a}^\mu - {t^\mu}_a + {t^\rho}_a {t^\mu}_\rho - {t^\rho}_a {t^\sigma}_\rho {t^\mu}_\sigma + {t^\rho}_a {t^\sigma}_\rho {t^\kappa}_\sigma {t^\mu}_\kappa. \tag{12}$$

Similarly a new metric tensor $s_{\mu\nu}(x)$ and its inverse $s^{\mu\nu}(x)$ are introduced in SGM curved space-time as follows:

$$s_{\mu\nu}(x) \equiv {w^a}_\mu(x) w_{a\nu}(x) = {w^a}_\mu(x)\eta_{ab}{w^b}_\nu(x) = g_{\mu\nu} + t_{\mu\nu} + t_{\nu\mu} + {t^\rho}_\mu t_{\rho\nu}, \tag{13}$$

$$\begin{aligned} s^{\mu\nu}(x) \equiv {w_a}^\mu(x) w^{a\nu}(x) = {} & g^{\mu\nu} - t^{\mu\nu} - t^{\nu\mu} + t^{\rho\mu}{t^\nu}_\rho + t^{\rho\nu}{t^\mu}_\rho + t^{\mu\rho}{t^\nu}_\rho - t^{\rho\mu}{t^\sigma}_\rho {t^\nu}_\sigma \\ & - t^{\rho\nu}{t^\sigma}_\rho {t^\mu}_\rho - t^{\mu\sigma}{t^\rho}_\sigma {t^\nu}_\rho - t^{\nu\rho}{t^\sigma}_\rho {t^\mu}_\sigma + t^{\rho\mu}{t^\sigma}_\rho {t^\kappa}_\sigma {t^\nu}_\kappa + t^{\rho\nu}{t^\sigma}_\rho {t^\kappa}_\sigma {t^\mu}_\kappa \\ & + t^{\mu\sigma}{t^\rho}_\sigma {t^\sigma}_\rho {t^\nu}_\sigma + t^{\nu\sigma}{t^\rho}_\sigma {t^\sigma}_\rho {t^\mu}_\sigma + t^{\rho\kappa}{t^\sigma}_\kappa {t^\mu}_\rho {t^\nu}_\sigma. \end{aligned} \tag{14}$$

We can easily show

$${w_a}^\mu w_{b\mu} = \eta_{ab}, \qquad s_{\mu\nu}{w_a}^\mu {w_b}^\nu = \eta_{ab}. \tag{15}$$

Furthermore they have generalized $GL(4,\mathbb{R})$ transformations under (3) and (4) [5, 7]. It is obvious from the above general covariant arguments that (1) is invariant under the ordinaly $GL(4,\mathbb{R})$ and under (3) and (4).

By using (10), (12), (13) and (14) we can express SGM action (1) in terms of $e^a{}_\mu(x)$ and $\psi^j(x)$, which describes explicitly the fundamental interaction of graviton with superons. The expansion of the action in terms of the power series of κ (or $t^a{}_\mu$) can be carried out straightforwardly. After the lengthy calculations concerning the complicated structures of the indices we obtain

$$\begin{aligned} L_{\rm SGM} = &-\frac{c^3\Lambda}{16\pi G}e|w_{\text{V-A}}| - \frac{c^3}{16\pi G}eR + \frac{c^3}{16\pi G}e\Bigg[2t^{(\mu\nu)}R_{\mu\nu} + \frac{1}{2}\Big\{g^{\mu\nu}\partial^\rho\partial_\rho t_{(\mu\nu)} - t_{(\mu\nu)}\partial^\rho\partial_\rho g^{\mu\nu} \\ &+ g^{\mu\nu}\partial^\rho t_{(\mu\sigma)}\partial^\sigma g_{\rho\nu} - 2g^{\mu\nu}\partial^\rho t_{(\mu\nu)}\partial^\sigma g_{\rho\sigma} - g^{\mu\nu}g^{\rho\sigma}\partial^\kappa t_{(\rho\sigma)}\partial^\kappa g_{\mu\nu}\Big\} \\ &+ \left(t^\mu{}_\rho t^{\rho\nu} + t^\nu{}_\rho t^{\rho\mu} + t^{\mu\rho}t^\nu{}_\rho\right)R_{\beta\mu} - \Big\{2t^{(\mu\rho)}t^{(\nu}{}_{\rho)}R_{\mu\nu} + t^{(\mu\rho)}t^{(\nu\sigma)}R_{\mu\nu\rho\sigma} \\ &+ \frac{1}{2}t^{(\mu\nu)}\left(g^{\rho\sigma}\partial^\mu\partial_\nu t_{(\rho\sigma)} - g^{\rho\sigma}\partial^\rho\partial_\mu t_{(\sigma\nu)} + \cdots\right)\Big\} \\ &+ \left\{O\left(t^3\right)\right\} + \left\{O\left(t^4\right)\right\} + \cdots + \left\{O\left(t^{10}\right)\right\}\Bigg], \end{aligned} \tag{16}$$

where $e = \det e^a{}_\mu$, $t^{(\mu\nu)} = t^{\mu\nu} + t^{\nu\mu}$, $t_{(\mu\nu)} = t_{\mu\nu} + t_{\nu\mu}$, and $|w_{\text{V–A}}| = \det w^a{}_b$ is the flat space V–A action [2] containing up to $O\left(t^4\right)$ and R and $R_{\mu\nu}$ are the Ricci curvature tensors of GR.

Remarkably the first term can be regarded as a space-time dependent cosmological term and reduces to V–A action [2] with $\kappa_{\text{V–A}}^{-1} = \frac{c^3}{16\pi G}\Lambda$ in the Riemann-flat $e_a{}^\mu(x) \to \delta_a{}^\mu$ space-time. The second term is the familiar E–H action of GR. These expansions show the complementary relation of graviton and (the stress-energy tensor of) superons. The existence of (in the Riemann-flat space-time) NL SUSY invariant terms with the (second order) derivatives of the superons beyond V–A model is manifested. For example, such terms of the lowest order appear in $O\left(t^2\right)$ and have the following expressions (up to the total derivative terms)

$$+\epsilon^{abcd}\epsilon_a{}^{efg}\partial_c t_{(be)}\partial_f t_{(dg)}. \tag{17}$$

Existence of such derivative terms in addition to the original V–A model are already pointed out and exemplified in part in [8]. Note that (17) vanishes in 2 dimensional space-time.

Here we just mention that we can consider two types of the flat space in SGM, which are not equivalent. One is SGM-flat, i.e. $w_a{}^\mu(x) \to \delta_a{}^\mu$, space-time and the other is Riemann-flat, i.e. $e_a{}^\mu(x) \to \delta_a{}^\mu$, space-time, where SGM action reduces to $-\frac{c^3\Lambda}{16\pi G}$ and $-\frac{c^3\Lambda}{16\pi G}|w_{\text{V–A}}| - \frac{c^3}{16\pi G}$ (*derivative terms*), respectively. Note that SGM-flat space-time may allow Riemann space-time, e.g. $t_a{}^\mu(x) \to -e_a{}^\mu + \delta_a{}^\mu$ realizes Riemann space-time *and* SGM-flat space-time. The cosmological implications are mentioned in the discussions.

3 SGM in two dimensional space-time

Now we go to two dimensional SGM space-time to simplify the arguments without loss of generality and demonstrate some details of the computations. It is well known that two dimensional GR has no physical degrees of freedom (due to the local $GL(2,\mathbb{R})$). SGM in SGM space-time is also the case. However the general covariant arguments shed light on the universal characteristic features of the theory in any space-time dimensions. Especialy for SGM, it is also useful to see explicitly the superon-graviton coupling in (two dimensional) Riemann space-time which is realized spontaneously from SGM space-time. We adopt the affine connection formalism. Knowledge of the complete structure of SGM action including the surface terms is useful to linearize SGM into the equivalent linear theory and to find the symmetry breaking of the model.

Following EGRT the scalar curvature tensor Ω of SGM space-time is given as follows

$$\Omega = s^{\beta\mu}\Omega^\alpha{}_{\beta\mu\alpha} = s^{\beta\mu}\left[\left\{\partial_\mu\Gamma^\lambda{}_{\beta\alpha} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\alpha}\right\} - \{\text{ lower indices } (\mu \leftrightarrow \alpha)\}\right], \tag{18}$$

where the Christoffel symbol of the second kind of SGM space-time is

$$\Gamma^{\alpha}{}_{\beta\mu} = \frac{1}{2} s^{\alpha\rho} \left\{ \partial_\beta s_{\rho\mu} + \partial_\mu s_{\beta\rho} - \partial_\rho s_{\mu\beta} \right\}. \tag{19}$$

The straightforward expression of SGM action (1) in two dimensional space-time (which is 3^6 times more complicated than the 2 dimensional GR) is given as follows

$$\begin{aligned} L_{2d\mathrm{SGM}} = &-\frac{c^3}{16\pi G} e \left\{ 1 + t^a{}_a + \frac{1}{2}\left(t^a{}_a t^b{}_b - t^a{}_b t^b{}_a \right) \right\} \left(g^{\beta\mu} - \tilde{t}^{(\beta\mu)} + \tilde{t}^{2(\beta\mu)} \right) \\ &\times \Bigg[\Bigg\{ \frac{1}{2} \partial_\mu \left(g^{\alpha\sigma} - \tilde{t}^{(\alpha\sigma)} + \tilde{t}^{2(\alpha\sigma)} \right) \partial_{\dot\beta} \left(g_{\dot\sigma\dot\alpha} + \underline{t}_{\dot\sigma\dot\alpha} + \underline{t}^2{}_{\dot\sigma\dot\alpha} \right) \\ &+ \frac{1}{2} \left(g^{\alpha\sigma} - \tilde{t}^{(\alpha\sigma)} + \tilde{t}^{2(\alpha\sigma)} \right) \partial_\mu \partial_{\dot\beta} \left(g_{\dot\sigma\dot\alpha} + \underline{t}_{\dot\sigma\dot\alpha} + \underline{t}^2{}_{\dot\sigma\dot\alpha} \right) \Bigg\} - \{ \text{lower indices } (\mu \leftrightarrow \alpha) \} \\ &+ \Bigg\{ \frac{1}{4} \left(g^{\alpha\sigma} - \tilde{t}^{(\alpha\sigma)} + \tilde{t}^{2(\alpha\sigma)} \right) \partial_{\dot\lambda} \left(g_{\dot\sigma\dot\mu} + \underline{t}_{\dot\sigma\dot\mu} + \underline{t}^2{}_{\dot\sigma\dot\mu} \right) \\ &\times \left(g^{\lambda\rho} - \tilde{t}^{(\lambda\rho)} + \tilde{t}^{2(\lambda\rho)} \right) \partial_{\dot\beta} \left(g_{\dot\rho\dot\alpha} + \underline{t}_{\dot\rho\dot\alpha} + \underline{t}^2{}_{\dot\rho\dot\alpha} \right) \Bigg\} - \{ \text{lower indices } (\mu \leftrightarrow \alpha) \} \Bigg] \\ &- \frac{c^3 \Lambda}{16\pi G} e |w_{\mathrm{V-A}}|, \end{aligned} \tag{20}$$

where we have put

$$\begin{aligned} &s_{\alpha\beta} = g_{\alpha\beta} + \underline{t}_{(\alpha\beta)} + \underline{t}^2{}_{(\alpha\beta)}, \qquad s^{\alpha\beta} = g^{\alpha\beta} - \tilde{t}^{(\alpha\beta)} + \tilde{t}^{2(\alpha\beta)}, \\ &\underline{t}_{(\mu\nu)} = t_{\mu\nu} + t_{\nu\mu}, \qquad \underline{t}^2{}_{(\mu\nu)} = t^\rho{}_\mu t_{\rho\nu}, \\ &\tilde{t}^{(\mu\nu)} = t^{\mu\nu} + t^{\nu\mu}, \qquad \tilde{t}^{2(\mu\nu)} = t^\mu{}_\rho t^{\rho\nu} + t^\nu{}_\rho t^{\rho\mu} + t^{\mu\rho} t^\nu{}_\rho, \end{aligned} \tag{21}$$

and the Christoffel symbols of the first kind of SGM space-time contained in (19) are abbreviated as

$$\begin{aligned} &\partial_{\dot\mu} g_{\dot\sigma\dot\nu} = \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\nu\mu}, \\ &\partial_{\dot\mu} \underline{t}_{\dot\sigma\dot\nu} = \partial_\mu \underline{t}_{(\sigma\nu)} + \partial_\nu \underline{t}_{(\mu\sigma)} - \partial_\sigma \underline{t}_{(\nu\mu)}, \\ &\partial_{\dot\mu} \underline{t}^2{}_{\dot\sigma\dot\nu} = \partial_\mu \underline{t}^2{}_{(\sigma\nu)} + \partial_\nu \underline{t}^2{}_{(\mu\sigma)} - \partial_\sigma \underline{t}^2{}_{(\nu\mu)}. \end{aligned} \tag{22}$$

By expanding the scalar curvature Ω in the power series of t which terminates with t^4, we have the following complete expression of two dimensional SGM,

$$\begin{aligned} L_{2d\mathrm{SGM}} = &-\frac{c^3 \Lambda}{16\pi G} e |w_{\mathrm{V-A}}| - \frac{c^3}{16\pi G} e |w_{\mathrm{V-A}}| \Bigg[R - 2\tilde{t}^{(\mu\nu)} R_{\mu\nu} + \frac{1}{2} \Big\{ g^{\mu\nu} \partial^\rho \partial_\rho \underline{t}_{(\mu\nu)} \\ &- \underline{t}^{(\mu\nu)} \partial^\rho \partial_\rho g_{\mu\nu} + g^{\mu\nu} \partial^\rho \underline{t}_{(\mu\sigma)} \partial^\sigma g_{\rho\nu} - 2 g^{\mu\nu} \partial^\rho \underline{t}_{(\mu\nu)} \partial^\sigma g_{\rho\sigma} - g^{\mu\nu} g^{\rho\sigma} \partial^\kappa \underline{t}_{(\rho\sigma)} \partial_\kappa g_{\mu\nu} \Big\} \\ &+ \tilde{t}^{2(\beta\mu)} R_{\beta\mu} + \tilde{t}^{(\beta\mu)} \tilde{t}^{(\alpha\sigma)} R_{\mu\alpha\sigma\beta} - \frac{1}{2} \tilde{t}^{(\beta\mu)} \Big\{ g^{\alpha\sigma} \partial_\mu \partial_\beta \underline{t}_{(\alpha\sigma)} - \partial^\sigma \partial_\beta \underline{t}_{(\sigma\mu)} \\ &+ \partial_\mu \tilde{t}^{(\alpha\sigma)} \partial_\beta g_{\sigma\alpha} - \partial_\mu g^{\alpha\sigma} \partial_\beta \underline{t}_{(\sigma\alpha)} + \partial_\alpha g^{\alpha\sigma} \partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\alpha \tilde{t}^{(\alpha\sigma)} \partial_\beta g_{\sigma\mu} + 2 \partial^\rho \underline{t}_{(\sigma\mu)} \partial^\sigma g_{\beta\rho} \\ &- 2 g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\mu)} \partial^\lambda g_{\alpha\beta} + g^{\alpha\sigma} g^{\lambda\rho} \partial_\mu \underline{t}_{(\lambda\sigma)} \partial^\beta g_{\rho\alpha} - 2 g^{\alpha\sigma} \partial^\rho \underline{t}_{(\sigma\alpha)} \partial_\beta g_{\rho\mu} \\ &+ g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \partial^\lambda g_{\mu\beta} \Big\} - g^{\beta\mu} \partial_\mu \left(g^{\alpha\sigma} \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} + \tilde{t}^{2(\alpha\sigma)} \partial_\beta g_{\sigma\alpha} - \tilde{t}^{(\alpha\sigma)} \partial_\beta \tilde{t}_{(\sigma\alpha)} \right) \end{aligned}$$

$$
\begin{aligned}
& - \tilde{t}^{(\alpha\sigma)}\left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right) + g^{\beta\mu}\partial_\alpha\Big\{ g^{\alpha\sigma}\left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \\
& + \tilde{t}^{2(\alpha\sigma)}\left(2\partial_\beta g_{\sigma\mu} - \partial_\sigma g_{\mu\beta}\right)\Big\} + 2\partial^\alpha g_{\lambda\mu} g^{\beta\lambda}\left(2\partial^\mu \underline{t}^2{}_{(\alpha\beta)} - \partial_\alpha \underline{t}^2{}_{(\beta\rho)} g^{\mu\rho}\right) \\
& + 2\tilde{t}^{2(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right) - 2\tilde{t}^{(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\alpha)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\beta\mu}\right) \\
& + \partial^\rho \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\alpha\sigma}\right) + \tilde{t}^{(\alpha\sigma)}\tilde{t}^{(\lambda\rho)}\Big\{\partial^\beta g_{\lambda\sigma}\partial_\beta g_{\rho\alpha} \\
& + 2\partial_\lambda g_{\sigma\mu} g^{\mu\beta}\left(\partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)\Big\} - 2\tilde{t}^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta} g^{\alpha\sigma}\right) \\
& - \partial^\rho g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)} g^{\beta\mu}\right) - \partial^\rho \underline{t}^2{}_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} g^{\sigma\alpha} - \partial^\lambda g_{\mu\beta} g^{\mu\beta}\right) \\
& - \tilde{t}^{2(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\mu\beta}\right) - \tilde{t}^{2(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial^\rho g_{\mu\beta} g^{\mu\beta}\right) \\
& - \tilde{t}^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) - \tilde{t}^{(\alpha\sigma)}\partial^\rho t_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
& + g^{\alpha\sigma}\partial^\rho \tilde{t}_{(\sigma\alpha)}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) + \tilde{t}^{(\alpha\sigma)}\tilde{t}^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
& - \tilde{t}^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) - \tilde{t}^{(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) \\
& + \frac{1}{2}\tilde{t}^{2(\beta\mu)}\Big\{ g^{\alpha\sigma}\partial_\mu\partial_\beta \underline{t}_{(\alpha\sigma)} - \partial^\sigma\partial_\beta \underline{t}_{(\sigma\mu)} + \partial_\mu \tilde{t}^{(\alpha\sigma)}\partial_\beta g_{\sigma\alpha} - \partial_\mu g^{\alpha\sigma}\partial_\beta \underline{t}_{(\sigma\alpha)} + \partial_\alpha g^{\alpha\sigma}\partial_\beta \underline{t}_{(\sigma\mu)} \\
& - \partial_\alpha \tilde{t}^{(\alpha\sigma)}\partial_\beta g_{\sigma\mu} + 2\partial^\rho \underline{t}_{(\sigma\mu)}\partial^\sigma g_{\beta\rho} - 2g^{\alpha\sigma}\partial_\lambda \underline{t}_{(\sigma\mu)}\partial^\lambda g_{\alpha\beta} + g^{\alpha\sigma} g^{\lambda\rho}\partial_\mu \underline{t}_{(\lambda\sigma)}\partial^\beta g_{\rho\alpha} \\
& - 2g^{\alpha\sigma}\partial^\rho \underline{t}_{(\sigma\alpha)}\partial_\beta g_{\rho\mu} + g^{\alpha\sigma}\partial_\lambda \underline{t}_{(\sigma\alpha)}\partial^\lambda g_{\mu\beta}\Big\} - \frac{1}{2}\tilde{t}^{(\beta\mu)}\Big\{\partial_\mu\left(g^{\alpha\sigma}\partial_\beta \underline{t}^2{}_{(\sigma\alpha)} - \tilde{t}^{(\alpha\sigma)}\partial_\beta \tilde{t}_{(\sigma\alpha)}\right) \\
& + \partial_\mu \tilde{t}^{2(\alpha\sigma)}\partial_\beta g_{\sigma\alpha} + \partial_\alpha\Big\{ g^{\alpha\sigma}\left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{\mu\beta}\right) - \tilde{t}^{(\alpha\sigma)}\left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right)\Big\} \\
& + \partial_\alpha \tilde{t}^{2(\alpha\sigma)}\left(2\partial_\beta g_{\sigma\mu} - \partial_\sigma g_{\mu\beta}\right) + 2\partial^\alpha g_{\lambda\mu} g^{\beta\lambda}\left(2\partial^\mu \underline{t}^2{}_{(\alpha\beta)} - \partial_\alpha \underline{t}^2{}_{(\beta\rho)} g^{\mu\rho}\right) \\
& + 2\tilde{t}^{2(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right) - 2\tilde{t}^{(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\alpha)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\beta\mu}\right) \\
& + \partial^\rho \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\alpha\sigma}\right) + \tilde{t}^{(\alpha\sigma)}\tilde{t}^{(\lambda\rho)}\Big\{\partial^\beta g_{\lambda\sigma}\partial_\beta g_{\rho\alpha} \\
& + 2\partial_\lambda g_{\sigma\mu} g^{\mu\beta}\left(\partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)\Big\} - 2\tilde{t}^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta} g^{\alpha\sigma}\right) \\
& - \partial^\rho g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)} g^{\beta\mu}\right) - \partial^\rho \underline{t}^2{}_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} g^{\sigma\alpha} - \partial^\lambda g_{\mu\beta} g^{\mu\beta}\right) \\
& - \tilde{t}^{2(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\mu\beta}\right) - \tilde{t}^{2(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial^\rho g_{\mu\beta} g^{\mu\beta}\right) \\
& - \tilde{t}^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) - \tilde{t}^{(\alpha\sigma)}\partial^\rho t_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
& + g^{\alpha\sigma}\partial^\rho \tilde{t}_{(\sigma\alpha)}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) + \tilde{t}^{(\alpha\sigma)}\tilde{t}^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
& - \tilde{t}^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) - \tilde{t}^{(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right)\Big\} \\
& + \partial^\beta\Big\{\left(\tilde{t}^{2(\alpha\sigma)}\partial_\beta \underline{t}_{(\sigma\alpha)} - \tilde{t}^{(\alpha\sigma)}\partial_\beta \underline{t}^2{}_{(\sigma\alpha)}\right)\Big\} \\
& - g^{\beta\mu}\partial_\alpha\Big\{\tilde{t}^{2(\alpha\sigma)}\left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right) - \tilde{t}^{(\alpha\sigma)}\left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\beta \underline{t}^2{}_{(\mu\beta)}\right)\Big\} \\
& + \frac{1}{2}\Big\{ g^{\alpha\sigma}\left(\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\mu\lambda}\right)\tilde{t}^{2(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
& + g^{\alpha\sigma}\left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right)\tilde{t}^{2(\lambda\rho)}\underline{t}^{(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right) \\
& - g^{\alpha\sigma}\left(\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\mu\lambda}\right)\tilde{t}^{(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
& - g^{\alpha\sigma}\left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right)\tilde{t}^{(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\mu\lambda}\right) \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda g_{\sigma\alpha} \tilde{t}^{2\lambda\rho} g^{\beta\mu} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{2(\lambda\rho)} \left(2\partial_\beta g_{\rho\mu} - \partial_\rho g_{\mu\beta}\right) \\
&+ g^{\alpha\sigma} \partial_\lambda g_{\sigma\alpha} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\mu)} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta g_{\rho\mu} - \partial_\rho g_{\mu\beta}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \partial_\lambda g_{\sigma\alpha} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&+ \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\alpha)} + \partial_\alpha g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\alpha\lambda)}\right) g^{\lambda\rho} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\mu)} + \partial_\mu \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \Big\} \\
&- \tilde{t}^{(\beta\mu)} \Big\{ \frac{1}{2} \Big\{ \partial_\mu \tilde{t}^{(\alpha\sigma)} \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} - \partial_\mu \tilde{t}^2_{(\alpha\sigma)} \partial_\beta \underline{t}_{(\sigma\alpha)} + \tilde{t}^{(\alpha\sigma)} \partial_\mu \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} \\
&- \partial_\alpha \tilde{t}^{(\alpha\sigma)} \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) + \partial_\alpha \tilde{t}^2_{(\alpha\sigma)} \left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \partial_\alpha \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \Big\} \\
&+ \frac{1}{4} \Big\{ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&+ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + \tilde{t}^{(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \Big\} \Big\} \\
&+ g^{\beta\mu} \Big\{ \frac{1}{2} \Big\{ \partial_\mu \tilde{t}^{2(\alpha\sigma)} \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} + \tilde{t}^{2(\alpha\sigma)} \partial_\mu \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} - \partial_\alpha \tilde{t}^{2(\alpha\sigma)} \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \\
&- \tilde{t}^{2(\alpha\sigma)} \partial_\alpha \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \Big\} \\
&+ \frac{1}{4} \Big\{ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&+ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{2(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&+ \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&+ \tilde{t}^{2(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\mu)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{2(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\mu)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&+ \tilde{t}^{(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) - \tilde{t}^{(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&- \tilde{t}^{2(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \Big\} \Big\} \Big],
\end{aligned}
\tag{23}
$$

where $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R are the curvature tensors of Riemann space and $|w_{\text{V–A}}| = \{1 + t^a{}_a + \frac{1}{2}(t^a{}_a t^b{}_b - t^a{}_b t^b{}_a)\}$ is V–A model in two dimensional flat space.

4 Discussions

We have shown that contrary to its simple expression (1) in unified SGM space-time the complete expansion of SGM action posesses the very complicated and rich structures describing the graviton-superon interactions in Riemann space-time, even in two dimensional space-time.

The SGM in four dimensional space-time has far much more complicated structures, which may be unavoidable features for a unified theory to describe the rationale of beings of all elementary particles. Note that the total number of elementary particles in SGM is at most a few hundreds and most of them are (heavy) massive.

Here we just emphasize that SGM action in SGM space-time is a nontrivial generalization of E–H action in Riemann space-time despite the liner relation $w^a{}_\mu = e^a{}_\mu + t^a{}_\mu$. In fact, by the redefinitions(variations) $e^a{}_\mu \to e^a{}_\mu + \delta e^a{}_\mu = e^a{}_\mu - t^a{}_\mu$ and $\delta e_a{}^\mu = -e_a{}^\nu e_b{}^\mu \delta e^b{}_\nu = +t^\mu{}_a$ the inverse $w_a{}^\mu = e_a{}^\mu - t^\mu{}_a + t^\rho{}_a t^\mu{}_\rho - t^\rho{}_a t^\sigma{}_\rho t^\mu{}_\sigma + t^\rho{}_a t^\sigma{}_\rho t^\kappa{}_\sigma t^\mu{}_\kappa$ does not reduce to $e_a{}^\mu$, i.e. the nonlinear terms in $t^\mu{}_a$ in the inverse $w_a{}^\mu$ can not be eliminated. Because $t^a{}_\mu$ is not a vierbein. Such a redefinition breaks the metric properties of $w^a{}_\mu$ and $w_a{}^\mu$. Note that SGM action posesses two inequivalent flat spaces, i.e. SGM-flat $w^a{}_\mu \to \delta^a{}_\mu$ and Riemann-flat $e^a{}_\mu \to \delta^a{}_\mu$. The expansion of SGM action in terms of $e^a{}_\mu$ and $t^a{}_\mu$ is a spontaneous breakdown of space-time from SGM space-time to Riemann space-time connecting with Riemann-flat space-time.

Concerning the above-mentioned two inequivalent flat-spaces (i.e. the vacuum of the gravitational energy) of SGM action we can interprete them as follows. SGM action (1) written by the vierbein $w_a{}^\mu(x)$ and metric $s^{\mu\nu}(x)$ of SGM space-time is invariant under (besides the ordinary local $GL(4,\mathbb{R})$) the general coordinate transformation [7] with a generalized parameter $i\kappa(\bar{\zeta}\gamma^\mu\psi(x))$ (originating from the global supertranslation in SGM space-time [2]). As proved for E–H action of GR (the positive definitness of Einstein–Hilbert actionwas proved by E. Witten [10]), the energy of SGM action of E–H type is expected to be positive (for positive Λ). Regarding the scalar curvature tensor Ω for the unified metric tensor $s^{\mu\nu}(x)$ as an analogue of the Higgs potential for the Higgs scalar, we can observe that (at least the vacuum of) SGM action (i.e. SGM-flat $w^a{}_\mu(x) \to \delta^a{}_\nu$ space-time), which allows Riemann space-time and has a positive energy density with the positive cosmological constant $\frac{c^3\Lambda}{16\pi G}$ indicating the spontaneous SUSY breaking, is unstable (i.e. degenerates) against the supertransformation (3) and (4) with the global spinor parameter ζ in SGM space-time and breaks down spontaneously to Riemann space-time $w^a{}_\mu(x) = e^a{}_\mu(x) + t^a{}_\mu(x)$ with N–G fermions *superons* corresponding to $\frac{\text{super}\,GL(4,\mathbb{R})}{GL(4,\mathbb{R})}$. (Note that SGM-flat space-time allows Riemann space-time.) Remarkably the observed Riemann space-time of EGRT and matter(superons) appear simultaneously from (the vacuum of) SGM action by the spontaneous SUSY breaking.

The investigation of the structures of the vacuum of Riemann-flat space-time (described by $N = 10$ V–A action with derivative terms like (17)) plays an important role to linearize SGM and to derive SM as the low energy effective theory of SGM, which remain to be challenged. Such (higher) derivative terms can be rewritten in the tractable forms similar to (17) up to the total derivative terms.

As for the linearization, the linearization of the flat-space $N = 1$ V–A model was already carried out [9]. They proved that the linear SUSY action of a scalar supermultiplet with SUSY breaking is equivalent to V–A action under SUSY invariant constraints obtained by the systematic arguments. Recently we have shown explicitly that the action of $U(1)$ vector supermultiplet with Feyet–Iliopoulos term is equivalent to $N = 1$ V–A model [11]. It is remarkable that the renormalizable low energy effective U(1) gauge theory is derived from the highly nonlinear theory by systematic arguments. While, in the linearization of SGM (i.e. V–A model in curved space-time) it should be taken into consideration further that the algebra (gauge symmetry) would be changed from (8) to broken $SO(10)$ SP symmetry.

From the physical point of view the linearization of the flat-space $N = 2$ V–A model is very important as a toy model, for it may be equivalent to the following Higgs–Kibble–Dirac Lagrangian (composed of $N = 2$ SP off-shell multiplet)

$$L_{\mathrm{HKD}} = \frac{1}{4}F^2{}_{\mu\nu} + \bar{\psi}\gamma_\mu D^\mu\psi + \frac{1}{2}(\partial_\mu\phi_i)^2 + 2g\phi_i\bar{\psi}\psi - 2gD\phi_i{}^2 + \frac{1}{2}\left(D^2 + |F|^2\right), \tag{24}$$

where $F_{\mu\nu}$ is a gauge field, ψ is a Dirac field, ϕ_i $(i = 1, 2)$ is a real scalar field and the fields D, and F are auxiliary fields. This, speculative so far, is remarkable, for the $(U(1))$ gauge field including the gauge coupling constant is expressed in terms of the superons and the the fundamental coupling constant of V–A model including the order parameter of the symmetry breaking. A nonlinear $N = 2$ SUSY equivalent to $N = 2$ SUSY Yang–Mills theory investigated by Seiberg and Witten [12] may be a realistic case. Furthermore the baryon abundance of the universe should be explained by the spontaneous symmetry breaking of the linearized (low energy) effective theory.

Finally we just mention the hidden symmetries characteristic to SGM. It is natural to expect that SGM action may be invariant under a certain exchange between $e^a{}_\mu$ and $t^a{}_\mu$, for they contribute equally to the unified SGM vierbein $w^a{}_\mu$ as seen in (10). In fact we find, as a simple example, that SGM action is invariant under the following exchange of $e^a{}_\mu$ and $t^a{}_\mu$ [13] (in 4 dimensional space-time).

$$e^a{}_\mu \longrightarrow 2t^a{}_\mu, \qquad t^a{}_\mu \longrightarrow e^a{}_\mu - t^a{}_\mu, \qquad e_a{}^\mu \longrightarrow e_a{}^\mu. \tag{25}$$

The physical meaning of such symmetries remains to be studied. Also SGM action has Z_2 symmetry $\psi^j \to -\psi^j$ but not $e^a{}_\mu \to -e^a{}_\mu$.

Beside the composite picture of SGM it is interesting to consider (elementary field) SGM with the extra dimensions and their compactifications. The compactification of $w^A{}_M = e^A{}_M + t^A{}_M$, $(A, M = 0, 1 \ldots, D-1)$ produces rich spectrum of particles and (hidden) internal symmetries and may give a new framework for the unification of space-time and matter.

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