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Jacobson Generators of (Quantum) $sl(n+1|m)$. Related Statistics

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A description of the quantum superalgebra $U_q[sl(n+1|m)]$ and hence (at $q=1$) of the special linear superalgebra $sl(n+1|m)$ via a new set of generators, called Jacobson generators, is given. It provides an alternative to the canonical description of $U_q[sl(n+1|m)]$ in terms of Chevalley generators. The Jacobson generators satisfy three linear supercommutation relations and define $U_q[sl(n+1|m)]$ as a deformed Lie supertriple system. Fock representations are constructed and the action of the Jacobson generators on the Fock basis is written down. The Jacobson generators and the Fock representations allow for an interpretation in terms of quantum statistics, and the properties of the underlying statistics are shortly discussed.

1 Introduction

The Lie superalgebra $sl(n+1|m)$ is one of the basic classical simple Lie superalgebras in Kac's classification [1]. It can be considered as the superanalogue of the special linear Lie algebra $sl(n+1)$. The quantum superalgebra $U_q[sl(n+1|m)]$ is a Hopf superalgebra deformation of the universal enveloping superalgebra $U[sl(n+1|m)]$ of $sl(n+1|m)$.

Usually, $U_q[sl(n+1|m)]$ is defined by its Chevalley generators e_i, f_i, h_i, $i=1,\ldots,n+m$, subject to the Cartan–Kac relations and the Serre relations [2, 3, 4]. Beside these defining relations, also the other Hopf superalgebra maps (comultiplication, co-unit and antipode) are part of the definition. In the present talk, however, we do not use these other Hopf superalgebra maps; so we shall concentrate on $U_q[sl(n+1|m)]$ as an associative superalgebra.

The definition in terms of Chevalley generators has the advantage that the comultiplication, co-unit and antipode are easy to give. Furthermore, certain representations can be constructed explicitly (e.g. for the essentially typical representations a Gelfand–Zetlin basis exist for which the action of the Chevalley generators is known [5]). Having certain physical applications in mind, however, it is sometimes more useful to work with a different set of generators for $U_q[sl(n+1|m)]$.

The different set of generators for $U_q[sl(n+1|m)]$ given here are the Jacobson generators (JGs) (denoted by a_i^+, a_i^- and H_i, with $i=1,\ldots,n+m$). For the case of $sl(n+1)$, such generators were originally introduced by Jacobson [6, 7]. The use of Jacobson generators has a number of advantages.

First of all, in certain applications it is necessary to have a complete basis of $U_q[sl(n+1|m)]$ (following from the Poincaré–Birkhoff–Witt theorem). Such a basis is given in terms of the

Cartan–Weyl elements. Although it is possible to express all Cartan–Weyl elements in terms of the Chevalley generators, such expressions soon become rather unmanageable. In terms of the Jacobson generators, the description of all Cartan–Weyl elements is very easy.

Secondly, the Jacobson generators allow for the definition of a simple class of representations, the Fock representations of $U_q[sl(n+1|m)]$. In these representations, the Jacobson generators a_i^+ and a_i^- share certain properties with ordinary creation and annihilation operators.

A disadvantage of the Jacobson generators compared to the Chevalley generators is that the explicit expressions for the other Hopf (super)algebra maps (comultiplication, co-unit and antipode) become very lengthy and complicated.

In Section 2 we define the Jacobson generators of $U_q[sl(n+1|m)]$ as a special subset of the Cartan–Weyl elements. The description of all Cartan–Weyl elements in terms of the Jacobson generators becomes very simple. In order to apply these results (e.g. in representations) one must have a list of all (super)commutation relations between these Cartan–Weyl elements; in terms of Jacobson generators, this means one has to determine certain triple relations. These are given in Theorem 2. In Section 3 we define Fock representations for $U_q[sl(n+1|m)]$, related to the Jacobson generators. The Fock representations are labeled by a number p; when p is a nonnegative integer, the Fock representation is finite-dimensional. These representations are further analyzed. Following conditions required in a physical context, it is determined when these Fock representations are unitary, see Theorem 4. In that case, an orthonormal basis of the Fock space is given, together with the action of the Jacobson generators on these basis elements. Finally, in Section 4 the Jacobson generators are interpreted as operators creating or annihilating a "particle", and the underlying quantum statistics is discussed.

2 Jacobson generators of $U_q[sl(n+1|m)]$

The Hopf superalgebra $U_q[sl(n+1|m)]$ is defined in the sense of Drinfeld [8], as a topologically free $\mathbb{C}[[h]]$ module. As a superalgebra, $U_q[sl(n+1|m)]$ is usually defined by means of its Chevalley generators, subject to the Cartan–Kac relations and the Serre relations [2, 3, 4]. Here, we present an alternative description of $U_q[sl(n+1|m)]$ in terms of the so-called Jacobson generators. The definition of JGs can be best presented in the framework of a set of Cartan–Weyl elements e_{ij}, $i,j = 0,\ldots,n+m$ of $U_q[gl(n+1|m)]$ [9]. The elements e_{ij} are the q-analogues of the defining basis of $gl(n+1|m)$; their grading is given by $\deg(e_{ij}) = \theta_{ij} = \theta_i + \theta_j$, where

$$\theta_i = \begin{cases} \bar{0} & \text{if} \quad i = 0,\ldots,n, \\ \bar{1} & \text{if} \quad i = n+1,\ldots,n+m. \end{cases}$$

We shall refer to e_{ij} as a positive root vector (resp. negative root vector) if $i < j$ (resp. $i > j$). For the formulation of the Poincaré–Birkhoff–Witt theorem, it is necessary to fix a total order for the set of elements e_{ij}. Among the positive root vectors, this order is given by

$$e_{ij} < e_{kl}, \quad \text{if} \quad i < k \quad \text{or} \quad i = k \quad \text{and} \quad j < l; \tag{1}$$

for the negative root vectors e_{ij} one takes the same rule (1), and total order is fixed by choosing

$$\text{positive root vectors} < \text{negative root vectors} < e_{ii}.$$

The difference between $U_q[sl(n+1|m)]$ and $U_q[gl(n+1|m)]$ is in the elements of the Cartan subalgebra. For $U_q[gl(n+1|m)]$ the Cartan subalgebra is generated by e_{ii} ($i = 0,\ldots,n+m$). For $U_q[sl(n+1|m)]$ the Cartan subalgebra is generated by the elements H_i, with

$$H_i = e_{00} - (-1)^{\theta_i} e_{ii}, \qquad i = 1,\ldots,n+m. \tag{2}$$

We will use also the elements L_i and $\bar{L}_i$, where

$$L_i = q^{H_i}, \qquad \bar{L}_i = q^{-H_i}, \qquad i = 1, \dots, n+m. \tag{3}$$

The Cartan–Weyl elements of $U_q[sl(n+1|m)]$ are now given by $\{H_i; i = 1, \dots, n+m\} \cup \{e_{ij}; i \neq j = 0, \dots, n+m\}$. The complete set of supercommutation relations between these Cartan–Weyl elements is given by

$$[H_i, H_j] = 0; \tag{4}$$

$$[H_i, e_{jk}] = (\delta_{0j} - \delta_{0k} - (-1)^{\theta_i}(\delta_{ij} - \delta_{ik}))e_{jk}; \tag{5}$$

for two positive root vectors $e_{ij} < e_{kl}$:

$$[\![e_{ij}, e_{kl}]\!]_{q^{(-1)^{\theta_j}\delta_{jl} - (-1)^{\theta_j}\delta_{jk} + (-1)^{\theta_i}\delta_{ik}}} = \delta_{jk}e_{il} + \left(q - q^{-1}\right)(-1)^{\theta_k}\theta(l > j > k > i)e_{kj}e_{il}; \tag{6}$$

for two negative root vectors $e_{ij} > e_{kl}$:

$$[\![e_{ij}, e_{kl}]\!]_{q^{-(-1)^{\theta_j}\delta_{jl} + (-1)^{\theta_j}\delta_{jk} - (-1)^{\theta_i}\delta_{ik}}} = \delta_{jk}e_{il} - \left(q - q^{-1}\right)(-1)^{\theta_k}\theta(i > k > j > l)e_{kj}e_{il}; \tag{7}$$

and finally for a positive root vector e_{ij} and a negative root vector e_{kl}:

$$\begin{aligned}
[\![e_{ij}, e_{kl}]\!] &= \frac{\delta_{il}\delta_{jk}}{q - q^{-1}}\left(L_j^{(-1)^{\theta_i}}\bar{L}_i^{(-1)^{\theta_i}} - \bar{L}_j^{(-1)^{\theta_i}}L_i^{(-1)^{\theta_i}}\right) \\
&+ \Big(\left(q - q^{-1}\right)\theta(j > k > i > l)(-1)^{\theta_k}e_{kj}e_{il} - \delta_{il}\theta(j > k)(-1)^{\theta_{kl}}e_{kj} + \delta_{jk}\theta(i > l)e_{il}\Big)L_i\bar{L}_k \\
&+ L_j\bar{L}_l\Big(-\left(q - q^{-1}\right)\theta(k > j > l > i)(-1)^{\theta_j}e_{il}e_{kj} - \delta_{il}\theta(k > j)(-1)^{\theta_{ij}}e_{kj} + \delta_{jk}\theta(l > i)e_{il}\Big),
\end{aligned} \tag{8}$$

where

$$[a, b]_x = ab - xba, \qquad \{a, b\}_x = ab + xba, \qquad [\![a, b]\!]_x = ab - (-1)^{\deg(a)\deg(b)}xba,$$

$$\theta(i_1 > i_2 > \ldots > i_r) = \begin{cases} 1, & \text{if } i_1 > i_2 > \ldots > i_r, \\ 0, & \text{otherwise.} \end{cases}$$

Define the Jacobson generators of $U_q[sl(n+1|m)]$ to be the following Cartan–Weyl vectors:

$$a_i^- = e_{0i}, \qquad a_i^+ = e_{i0}, \qquad H_i, \qquad i = 1, \dots, n+m. \tag{9}$$

Then from (8) one obtains:

$$[\![a_i^-, a_j^+]\!] = -(-1)^{\theta_i}L_ie_{ji}, \quad (i < j); \qquad [\![a_i^-, a_j^+]\!] = -(-1)^{\theta_j}e_{ji}\bar{L}_j, \quad (i > j). \tag{10}$$

In terms of the JGs the definition of $U_q[sl(n+1|m)]$ reads

Theorem 1. *$U_q[sl(n+1|m)]$ is a unital associative algebra with generators $\{H_i,\ a_i^\pm\}_{i=1,\dots,n+m}$ and relations*

$$\begin{aligned}
&[H_i, H_j] = 0, \qquad [H_i, a_j^\pm] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^\pm, \\
&[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}}, \qquad L_i = q^{H_i}, \qquad \bar{L}_i \equiv L_i^{-1} = q^{-H_i}, \qquad \bar{q} \equiv q^{-1}, \\
&[\![[\![a_i^\eta, a_{i+\xi}^{-\eta}]\!], a_k^\eta]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} = \eta^{\theta_k}\delta_{k,i+\xi}L_k^{-\xi\eta}a_i^\eta, \\
&[\![a_1^\xi, a_2^\xi]\!]_q = 0, \qquad [\![a_1^\xi, a_1^\xi]\!] = 0, \qquad \xi, \eta = \pm \quad \textit{or} \quad \pm 1.
\end{aligned} \tag{11}$$

The set of relations (11) is the minimal one defining the algebra $U_q[sl(n+1|m)]$. This description of $U_q[sl(n+1|m)]$ (resp. $sl(n+1|m)$) is somewhat similar to the Lie triple system description of Lie algebras, initiated by Jacobson [6, 7] and generalized to Lie superalgebras by Okubo [10]. Therefore we have defined $U_q[sl(n+1|m)]$ (resp. $sl(n+1|m)$) as a (deformed) Lie supertriple system.

In order to be able to reorder the Cartan–Weyl elements, which appear when computing the transformations of the Fock spaces, it is convenient to write down all triple relations between the JGs (which certainly follow from the relations (11)).

Theorem 2. *A set of Cartan–Weyl elements of $U_q[sl(n+1|m)]$ is given by H_i, $a_i^\pm$, $[\![a_i^+, a_j^-]\!]$ $(i \neq j = 1, \dots, n+m)$. A complete set of supercommutation relations between these elements is given by:*

$$[H_i, H_j] = 0; \qquad [H_i, a_j^\pm] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^\pm; \tag{12}$$

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - q^{-1}}; \tag{13}$$

$$[\![a_i^\eta, a_j^\eta]\!]_q = 0 \quad (i < j); \qquad (a_i^\pm)^2 = 0 \quad (i = n+1, \dots, n+m); \tag{14}$$

$$\begin{aligned}
[\![[\![a_i^\eta, a_j^{-\eta}]\!], a_k^\eta]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} &= \eta^{\theta_j}\delta_{jk}L_k^{-\xi\eta}a_i^\eta + (-1)^{\theta_k}\epsilon(j,k,i)(q - \bar{q})[\![a_k^\eta, a_j^{-\eta}]\!]a_i^\eta \\
&= \eta^{\theta_j}\delta_{jk}L_k^{-\xi\eta}a_i^\eta + (-1)^{\theta_k\theta_j}\epsilon(j,k,i)q^\xi(q - \bar{q})a_i^\eta[\![a_k^\eta, a_j^{-\eta}]\!],
\end{aligned} \tag{15}$$

where $(j-i)\xi > 0$, ξ, $\eta = \pm$ and

$$\epsilon(j,k,i) = \begin{cases} 1, & \textit{if } j > k > i; \\ -1, & \textit{if } j < k < i; \\ 0, & \textit{otherwise,} \end{cases}$$

and we have used the notation $\bar{q} = q^{-1}$.

3 Fock representations

We construct the Fock modules using the induced module procedure. $G = U_q[sl(n+1|m)]$, with Cartan–Weyl elements H_i, $a_i^\pm$ and $[\![a_i^+, a_j^-]\!]$ $(i \neq j = 1, \dots, n+m)$, has a subalgebra $A = U_q[gl(n|m)]$ with Cartan–Weyl elements H_i and $[\![a_i^+, a_j^-]\!]$ $(i \neq j = 1, \dots, n+m)$. Define a trivial one-dimensional A module as follows:

$$[\![a_i^-, a_j^+]\!]|0\rangle = 0, \qquad (i \neq j = 1, \dots, n+m), \tag{16}$$

$$H_i|0\rangle = p|0\rangle, \tag{17}$$

where p is any complex number. Let P be the (associative) subalgebra of $G = U_q[sl(n+1|m)]$ generated by the elements of A and $\{a_i^-; i = 1, \dots, n+m\}$. The one-dimensional module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional P module by requiring:

$$a_i^-|0\rangle = 0, \qquad i = 1, \dots, n+m. \tag{18}$$

Now the G module $\bar{W}_p$ is defined as

$$\bar{W}_p = \mathrm{Ind}_P^G\, \mathbb{C}|0\rangle.$$

Clearly $\bar{W}_p$ is freely generated by the generators a_i^+ $(i = 1, \dots, n+m)$ acting on $|0\rangle$. Therefore a basis for $\bar{W}_p$ is given by

$$|p; r_1, r_2, \dots, r_{n+m}\rangle \equiv (a_1^+)^{r_1}(a_2^+)^{r_2}\cdots(a_n^+)^{r_n}(a_{n+1}^+)^{r_{n+1}}(a_{n+2}^+)^{r_{n+2}}\cdots(a_{n+m}^+)^{r_{n+m}}|0\rangle, \tag{19}$$

where $r_i \in \mathbb{Z}_+$ for $i = 1, \dots, n$ and $r_i \in \{0, 1\}$ for $i = n+1, \dots, n+m$.

Theorem 3. *The transformation of the basis (19) of $\bar{W}_p$ under the action of the JGs reads:*

$$H_i|p;r_1,r_2,\ldots,r_{n+m}\rangle = \left(p-(-1)^{\theta_i}r_i-\sum_{j=1}^{n+m}r_j\right)|p;r_1,r_2,\ldots,r_{n+m}\rangle, \tag{20}$$

$$\begin{aligned} a_i^-|p;r_1,r_2,\ldots,r_{n+m}\rangle &= (-1)^{\theta_1 r_1+\theta_2 r_2+\cdots+\theta_{i-1}r_{i-1}}q^{r_1+\cdots+r_{i-1}}[r_i]\left[p-\sum_{j=1}^{n+m}r_j+1\right] \\ &\quad \times|p;r_1,r_2,\ldots,r_{i-1},r_i-1,r_{i+1},\ldots,r_{n+m}\rangle, \end{aligned} \tag{21}$$

$$\begin{aligned} a_i^+|p;r_1,r_2,\ldots,r_{n+m}\rangle &= (-1)^{\theta_1 r_1+\theta_2 r_2+\cdots+\theta_{i-1}r_{i-1}}\bar{q}^{r_1+\cdots+r_{i-1}}(1-\theta_i r_i) \\ &\quad \times|p;r_1,r_2,\ldots,r_{i-1},r_i+1,r_{i+1},\ldots,r_{n+m}\rangle, \end{aligned} \tag{22}$$

where $i=1,\ldots,n+m$.

Proof. We sketch the proof. Equation (20) is an immediate consequence of $[H_i,a_j^+]=-(1+(-1)^{\theta_i}\delta_{ij})a_j^+$, which is one of the last relations in (12). Also the action of a_i^+ on the basis vectors is easy: (22) follows directly from (14). The proof of (21) follows from the following relations [11]:

- $$\begin{aligned} &[\![A,B_1B_2\cdots B_{i-1}B_iB_{i+1}\cdots B_j]\!]_{q^{b_1+b_2+\cdots+b_j}} \\ &= \sum_{i=1}^{j}q^{b_1+b_2+\cdots+b_{i-1}}(-1)^{\alpha(\beta_1+\cdots+\beta_{i-1})}B_1B_2\cdots B_{i-1}[\![A,B_i]\!]_{q^{b_i}}B_{i+1}\cdots B_j, \\ &\qquad \text{where}\quad \alpha=\deg(A)\quad\text{and}\quad \beta_i=\deg(B_i); \end{aligned} \tag{23}$$
- $$[\![a_i^-,(a_j^+)^r]\!] = \begin{cases} \dfrac{\bar{q}^{2r}-1}{\bar{q}^2-1}(a_j^+)^{r-1}[\![a_i^-,a_j^+]\!] & \text{when } i<j, \\ \dfrac{q^{2r}-1}{q^2-1}(a_j^+)^{r-1}[\![a_i^-,a_j^+]\!] & \text{when } i>j; \end{cases} \tag{24}$$
- $$[\![a_i^-,(a_i^+)^r]\!] = \frac{(a_i^+)^{r-1}}{q-\bar{q}}\left(\frac{\bar{q}^{2r}-1}{\bar{q}^2-1}L_i-\frac{q^{2r}-1}{q^2-1}\bar{L}_i\right); \tag{25}$$
- $$[\![[\![a_i^-,a_j^+]\!],(a_i^+)^r]\!]_{q^r} = -(-1)^{\theta_j}\frac{\bar{q}^{2r}-1}{\bar{q}^2-1}\bar{L}_ia_j^+(a_i^+)^{r-1}, \quad i>j, \tag{26}$$
- $$[\![[\![a_i^-,a_j^+]\!],(a_k^+)^r]\!]_{q^r} = (-1)^{\theta_j}(q^{2r}-1)a_j^+(a_k^+)^{r-1}[\![a_i^-,a_k^+]\!], \quad i>k>j, \tag{27}$$
- $$\begin{aligned} &[\![a_i^-,a_1^+]\!](a_2^+)^{r_2}\cdots(a_{n+m}^+)^{r_{n+m}}|0\rangle \\ &= -(-1)^{\theta_1+\theta_2r_2+\theta_3r_3+\cdots+\theta_{i-1}r_{i-1}}q^{2r_2+\cdots+2r_{i-1}+r_i+\cdots+r_{n+m}-p}[r_i] \\ &\times a_1^+(a_2^+)^{r_2}\ldots(a_{i-1}^+)^{r_{i-1}}(a_i^+)^{r_i-1}(a_{i+1}^+)^{r_{i+1}}\cdots(a_{n+m}^+)^{r_{n+m}}|0\rangle, \quad i>1. \end{aligned} \tag{28}$$

■

The action of the elements H_i and $a_i^\pm$ $(i=1,\ldots,n+m)$ on the basis vectors of $\bar{W}_p$, determined in Theorem 3, imply that $\bar{W}_p$ has an invariant submodule when p is a nonnegative integer. From now on we shall assume that $p\in\mathbb{Z}_+$. Then we have

Corollary 1. *The $U_q[sl(n+1|m)]$ module $\bar{W}_p$ has an invariant submodule V_p with basis vectors*

$$|p;r_1,r_2,\ldots,r_{n+m}\rangle, \qquad \textit{with} \quad \sum_{i=1}^{n+m}r_i>p.$$

The quotient module $W_p = \bar{W}_p/V_p$ is an irreducible representation for $U_q[sl(n+1|m)]$. The basis vectors of W_p are given by (the representatives of)

$$|p; r_1, r_2, \ldots, r_{n+m}\rangle, \qquad \textit{with} \quad \sum_{i=1}^{n+m} r_i \leq p. \tag{29}$$

Now we select a class of Fock modules important for physical applications. These are the ones for which the standard Fock metric is positive definite, and for which the representatives of $a_i^\pm$ and H_i $(i = 1, \ldots, n+m)$ satisfy the Hermiticity conditions:

$$(a_i^+)^\dagger = a_i^-, \qquad (a_i^-)^\dagger = a_i^+, \qquad (H_i)^\dagger = H_i. \tag{30}$$

For the Fock representation W_p, we can define a Hermitian form $(\,,\,)$ by requiring

$$(|0\rangle\,,\,|0\rangle) = \langle 0|0\rangle = 1, \tag{31}$$

and by postulating that the Hermiticity conditions (30) should be satisfied, i.e.

$$(a_i^\pm v, w) = (v, a_i^\mp w), \qquad \forall\, v, w \in W_p. \tag{32}$$

Then any two vectors $|p; r_1, r_2, \ldots, r_{n+m}\rangle$ and $|p; r_1', r_2', \ldots, r_{n+m}'\rangle$ with $(r_1, r_2, \ldots, r_{n+m}) \neq (r_1', r_2', \ldots, r_{n+m}')$ are orthogonal and

$$(|p; r_1, r_2, \ldots, r_{n+m}\rangle, |p; r_1, r_2, \ldots, r_{n+m}\rangle) = \frac{[p]!}{[p-R]!}\prod_{i=1}^{n+m}[r_i]! = \frac{[p]!}{[p-R]!}\prod_{i=1}^{n}[r_i]!, \tag{33}$$

where $R = r_1 + r_2 + \cdots + r_{n+m}$. The straightforward computations show that Hermiticity conditions hold if q is a phase, i.e.

$$q = e^{i\phi} \qquad (-\pi < \phi < \pi). \tag{34}$$

Let us now further investigate when the Hermitian form $(\,,\,)$ is an inner product. This means that for every $(r_1, \ldots, r_{n+m})$ with $0 \leq R \leq p$, the value in (33) should be positive. In particular, this implies that all the numbers

$$[p],\ [p-1],\ [p-2],\ \ldots,\ [2],\ [1]$$

should be positive. However, since $q = e^{i\phi}$ is a phase, we have

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sin(k\phi)}{\sin(\phi)}.$$

The common domain where all functions

$$\frac{\sin(2\phi)}{\sin(\phi)},\ \frac{\sin(3\phi)}{\sin(\phi)},\ \ldots,\ \frac{\sin(p\phi)}{\sin(\phi)}$$

are positive is

$$-\frac{\pi}{p} < \phi < \frac{\pi}{p}.$$

Thus we have

Theorem 4. *The irreducible Fock module W_p $(p \geq 2)$ is unitary if and only if q is a phase, i.e. $q = e^{i\phi}$, with $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$.*

Observe that whether q is a root of unity or not does not have any effect on the irreducibility or unitarity of the Fock module W_p, as long as the conditions of Theorem 4 are satisfied. Indeed, suppose that $q = e^{i\phi}$ is a root of unity with ϕ a rational multiple of π and $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$. Then the smallest integer N for which $q^N = -1$ is greater than p. As a consequence, the rhs in (33) is never zero. This implies that there are no singular vectors among the weight vectors $|p; r_1, \ldots, r_{n+m}\rangle$, and thus irreducibility holds.

Under the conditions of Theorem 4, we can define an orthonormal basis of W_p :

$$|p; r_1, r_2, \ldots, r_{n+m}) = \sqrt{\frac{[p - \sum\limits_{l=1}^{n+m} r_l]!}{[p]![r_1]! \cdots [r_{n+m}]!}} |p; r_1, r_2, \ldots, r_{n+m}\rangle, \tag{35}$$

where $0 \leq \sum\limits_{l=1}^{n+m} r_l \leq p$. In the new basis (35) the transformation formulas (20)–(22) read $(i = 1, \ldots, n+m)$:

$$H_i |p; r_1, r_2, \ldots, r_{n+m}) = \left(p - (-1)^{\theta_i} r_i - \sum_{j=1}^{n+m} r_j \right) |p; r_1, r_2, \ldots, r_{n+m}), \tag{36}$$

$$\begin{aligned} a_i^- |p; r_1, \ldots, r_{n+m}) &= (-1)^{\theta_1 r_1 + \cdots + \theta_{i-1} r_{i-1}} \\ &\times q^{r_1 + \cdots + r_{i-1}} \sqrt{[r_i] \left[p - \sum_{l=1}^{n+m} r_l + 1 \right]} |p; r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}), \end{aligned} \tag{37}$$

$$\begin{aligned} a_i^+ |p; r_1, \ldots, r_{n+m}) &= (-1)^{\theta_1 r_1 + \cdots + \theta_{i-1} r_{i-1}} \bar{q}^{r_1 + \cdots + r_{i-1}} (1 - \theta_i r_i) \\ &\times \sqrt{[r_i + 1] \left[p - \sum_{l=1}^{n+m} r_l \right]} |p; r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_{n+m}). \end{aligned} \tag{38}$$

4 Properties of the underlying statistics

In the present section we indicate that each $U_q[sl(n+1|m)]$ module W_p can be considered as a state space, where a_i^+ (resp. a_i^-) can be interpreted as operators creating (resp. annihilating) "particles" with, say, energy ε_i. To this end consider a "free" Hamiltonian

$$H = \sum_{i=1}^{n+m} \varepsilon_i e_{ii}. \tag{39}$$

Then

$$[H, a_i^{\pm}] = \pm \varepsilon_i a_i^{\pm}. \tag{40}$$

This result together with equations (37)–(38) allows one to interpret a_i^+ as an operator creating a particle with energy ε_i, or more precisely, creating a particle on the i-th orbital. The operator a_i^- annihilates a particle with energy ε_i, or equivalently annihilates a particle on the i-th orbital. On every orbital i with $i = n+1, \ldots, n+m$ there cannot be more than one particle since $(a_i^+)^2 = 0$ for $i = n+1, \ldots, n+m$, whereas such a restriction does not hold for the first n orbitals. These are Fermi like (resp. Bose like) properties. There is however one essential difference. If the corresponding Fock module is characterized by p then no more than p "particles" can be accommodated in the system, $\sum\limits_{i=1}^{n+m} r_i \leq p$. Hence the number of particles that can be accommodated on a given orbital, keeping the number of particles on all other orbitals fixed, depends on how many particles have already been accommodated in the system. If $\sum_{i=1}^{n+m} r_i < p$ the particles behave similar to bosons and fermions, but are neither bosons nor fermions since the maximum number of the particles in the system cannot exceed p. This condition together with the restrictions for the orbitals with $i = n+1, \ldots, n+m$ is the analogue of the Pauli principle for this statistics.

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[1] Kac V.G., Lie superalgebras, *Adv. Math.*, 1977, V.26, 8–96.

[2] Floreanini R., Leites D.A. and Vinet L., On the defining relations of quantum superalgebras, *Lett. Math. Phys.*, 1991, V.23, 127–132.

[3] Khoroshkin S.M. and Tolstoy V.N., Universal R-matrix for quantized (super)algebras, *Commun. Math. Phys.*, 1991, V.141, 599–617.

[4] Scheunert M., Serre type relations for special linear Lie superalgebras, *Lett. Math. Phys.*, 1992, V.24, 173–182.

[5] Palev T.D., Stoilova N.I. and Van der Jeugt J., Finite-dimensional representations of the quantum superalgebra $U_q[gl(n/m)]$ and related q-identities, *Commun. Math. Phys.*, 1994, V.166, 367–378.

[6] Jacobson N., Lie and Jordan triple systems, *Amer. J. Math.*, 1949, V.71, 149–170.

[7] Palev T.D. and Van der Jeugt J., Jacobson generators, Fock representations and statistics of $sl(n+1)$, hep-th/0010107.

[8] Drinfeld V.G., Hopf algebras and the quantum Yang–Baxter equation, *DAN SSSR*, 1985, V.283, 1060–1064 (*Sov. Math. Dokl.*, 1985, V.32, 254–258).

[9] Palev T.D. and Tolstoy V.N., Finite-dimensional irreducible representations of the quantum superalgebra $U_q[gl(n/1)]$, *Commun. Math. Phys.*, 1991, V.141, 549–558.

[10] Okubo S., Parastatistics as Lie-supertriple systems, *J. Math. Phys.*, 1994, V.35, 2785–2803.

[11] Palev T.D., Stoilova N.I. and Van der Jeugt J., Jacobson generators of the quantum superalgebra $U_q[sl(n+1|m)]$ and Fock representations, preprint RGD/05/01, Technical University of Clausthal; *J. Math. Phys.*, to appear.