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The Construction of Alternative Modified KdV Equation in $(2+1)$ Dimensions

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A typical and effective way to construct a higher dimensional integrable equation is to extend the Lax pair for a $(1+1)$ dimensional equation known as integrable to higher dimensions. Here we construct an alternative modified KdV equation in $(2+1)$ dimensions by the higher-dimensional extension of a Lax pair. And it is shown that this higher dimensional modified KdV equation passes the Painlevé test (WTC method).

1 Introduction

A central and so active topic in the theory of integrable systems is to construct as many higher dimensional integrable systems as possible. The Lax representation is a powerful tool for constructing integrable equations in $(2+1)$ dimensions. In this paper we will derive a $(2+1)$ dimensional equation of the modified KdV (mKdV) equation. Let us first recall here that the mKdV equation in $(1+1)$ dimensions reads

$$v_t + \frac{1}{4}v_{xxx} + \frac{3}{2}v^2 v_x = 0. \tag{1}$$

Higher dimensional integrable equations are not usually unique, in the sense that there exist several equations that reduce to a given one under dimensional reduction. It is widely known, for instance, that

$$v_t + \frac{1}{4}v_{xxz} + v^2 v_z + \frac{1}{2}v_x \partial_x^{-1}\left(v^2\right)_z = 0 \tag{2}$$

and

$$v_t + \frac{1}{4}v_{xxx} + \frac{3}{4}v_x \partial_z^{-1}\left\{v\left(\partial_z^{-1}v_x\right)_x\right\} + \frac{3}{4}\left(\partial_z^{-1}v_x\right)\left(v\partial_z^{-1}v_x\right)_x - \frac{3}{4}v_x\left(\partial_z^{-1}v_x\right)^2 = 0 \tag{3}$$

are the higher-dimensional mKdV equations [1, 2, 3]. It is easy to check equation (2) and (3) are reduced to equation (1), setting $z = x$. Our goal in this paper is to add into them an alternative one derived from the higher-dimensional extension of a Lax pair.

It is well-known that the Lax representation [4] describes $(1+1)$ dimensional integrable equations as follows. Consider two operators L and T which are called the Lax pair and given by

$$L = L_0 - \lambda, \tag{4}$$

$$T = \partial_x(L_0) + T_0' + \partial_t, \tag{5}$$

with λ being a spectral parameter independent upon t. Then the commutator

$$[L, T] = 0 \tag{6}$$

contains a nonlinear evolution equation for suitably chosen L and T. Equation (6) is so-called the Lax equation. For example if we take

$$L_0 = L_{\mathrm{mKdV}} = \partial_x^2 + 2\sigma v \partial_x, \tag{7}$$

$$T_0' = T_{\mathrm{mKdV}}' = \sigma v \partial_x^2 - \left(\frac{3}{2}v^2 + \frac{1}{2}\sigma v_x\right)\partial_x, \tag{8}$$

with $\sigma = \pm i$, then L_{mKdV} and T_{mKdV}' satisfy the Lax equation (6) provided that $v(x,t)$ satisfies the mKdV equation (1). By operator L_{mKdV1} (7), the mKdV equation (1) can be extended to the higher-dimensional ones (2) and (3). In this paper we will extend the mKdV equation (1) to an alternative $(2+1)$ dimensional one by taking a different L_0 operator

$$L_0 = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x. \tag{9}$$

This paper is organised as follows. In Section 2, we shall begin with verifying a different L_0 operator (9) gives the mKdV equation in $(1+1)$ dimensions. In the process, we use the Painlevé test. Next an alternative mKdV equation in $(2+1)$ dimensions is introduced by the extension of T operator of the Lax pair. In this process, we also need to perform the Painlevé test. Section 5 contains our summary.

2 The modified KdV equation

In this section, let us show the mKdV equation (1) can be constructed by the operator L_0

$$L_0 = L_{\mathrm{mKdV}'} = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x. \tag{10}$$

The Lax pair (4) and (5) are given by

$$L = L_{\mathrm{mKdV}'} - \lambda, \tag{11}$$

$$T = \partial_x(L_{\mathrm{mKdV}'}) + T_{\mathrm{mKdV}'}' + \partial_t, \tag{12}$$

where $T_{\mathrm{mKdV}'}'$ is an unknown operator. And then the Lax equation (6) gives

$$[L_{\mathrm{mKdV}'} - \lambda, \partial_x(L_{\mathrm{mKdV}'}) + \partial_t] + [L_{\mathrm{mKdV}'} - \lambda, T_{\mathrm{mKdV}'}'] = 0. \tag{13}$$

The first term in the left-hand side of equation (13) gives

$$\begin{aligned}[L_{\mathrm{mKdV}'} - \lambda, \partial_x(L_{\mathrm{mKdV}'}) + \partial_t] &= -v_x\partial_x^3 - \left(\frac{3}{2}vv_x + \frac{1}{2}v_{xx}\right)\partial_x^2 \\ &- \left(\frac{3}{2}v_x^2 + \frac{3}{4}v^2 v_x + \frac{1}{2}v_{xx} + v_t\right)\partial_x + \left(\frac{3}{4}v_x v_{xx} + \frac{1}{2}vv_{xxx} + \cdots\right),\end{aligned} \tag{14}$$

where note that $\partial_x(L_{\mathrm{mKdV}'}) = \partial_x^3 + v\partial_x^2 + \left(\frac{1}{4}v^2 + \frac{3}{2}v_x\right)\partial_x + \frac{1}{2}vv_x + \frac{1}{2}v_{xx}$. So we choose here the form of the operator $T_{\mathrm{mKdV}'}'$ so that it involves, at least, a second-order differential operator,

$$T_{\mathrm{mKdV}'}' = U\partial_x^2 + V\partial_x + W, \tag{15}$$

where U, V and W are functions of x and t. Then the second term in the left-hand side of equation (13) gives

$$\begin{aligned}[L_{\mathrm{mKdV}'} - \lambda, T_{\mathrm{mKdV}'}')] &= 2U_x\partial_x^3 + \left(U_{xx} + 2V_x + vU_x - 2Uv_x\right)\partial_x^2 \\ &+ \left(V_{xx} + 2W_x + vV_x - Vv_x - Uvv_x - 2Uv_{xx}\right)\partial_x + \left(W_{xx} + vW_x + \cdots\right).\end{aligned} \tag{16}$$

By comparing the first term (14) to the second one (16),

$$U = \frac{v}{2}, \tag{17}$$

$$V = \frac{v^2}{2}, \tag{18}$$

$$2W_x = v_t + \frac{1}{2}vv_{xx} + \frac{1}{2}v_x^2 + \frac{3}{4}v^2v_x \tag{19}$$

and

$$W_{xx} + vW_x + \frac{3}{4}v_xv_{xx} + \frac{1}{2}vv_{xxx} + \cdots = 0, \tag{20}$$

where equation (20) is an identity by U, V and W_x. The exact forms of U and V have obtained and one of W has not yet.

Now let us get it by applying the Painlevé test in the sense of Weiss–Tabor–Carnevale (WTC) method [5]. For that, let us compute the degree of variables in equation (19). Equation (19) demands that, if taking $[\partial_x] = 1$,

$$[v] = 1, \tag{21}$$

$$[\partial_t] = 3, \tag{22}$$

$$[W_x] = 4, \tag{23}$$

where $[*]$ means the degree of a variable $*$. These degrees lead us to take as unknown function W_x

$$-2W_x = \alpha v_{xxx} + \beta vv_{xx} + \gamma v_x^2 + \delta v^2v_x, \tag{24}$$

where α, β, γ and δ are real constants. Equation (19) reads

$$v_t + \alpha v_{xxx} + \left(\beta + \frac{1}{2}\right)vv_{xx} + \left(\gamma + \frac{1}{2}\right)v_x^2 + \left(\delta + \frac{3}{4}\right)v^2v_x. \tag{25}$$

Now we show four constants in equation (25) are obtained such as passing the Painlevé test (WTC method). The solution to equation (1) has the form

$$v \sim v_0\phi^\eta. \tag{26}$$

Here ϕ is single valued about an arbitrary movable singular manifold. In η is a negative integer (leading order). By using leading order analysis, we obtain

$$\eta = -1. \tag{27}$$

Substituting

$$v(x,t) = \sum_{j=0} v_j(x,t)\phi(x,t)^{j-1} \tag{28}$$

leads to the resonances, after trivial algebra,

$$j = -1,\ 3,\ 4, \tag{29}$$

in the condition

$$\beta = \gamma = -\frac{1}{2}. \tag{30}$$

To simplify the calculations, we use the reduced manifold ansatz of Kruskal:

$$\phi(x,t) = x + \rho(t), \tag{31}$$

$$v_j(x,t) = v_j(t). \tag{32}$$

The resonance $j = -1$ in (29) corresponds to the arbitrary singularity manifold ϕ. We used *MATHEMATICA* to handle the computation for the existence of arbitrary functions corresponding to the resonances except $j = -1$. We find that v_3 and v_4 are arbitrary for equation (25). Thus the general solution v to equation (25) admits the sufficient number of arbitrary functions, thus passing the Painlevé test with the condition (30). Then equation (25) is reduced to the mKdV equation

$$v_t + \alpha v_{xxx} + \left(\delta + \frac{3}{4}\right) v^2 v_x = 0, \tag{33}$$

where α and δ are still arbitrary. Hereafter let us choose

$$\alpha = \frac{1}{4} \qquad \text{and} \qquad \delta = \frac{3}{4}. \tag{34}$$

This choice, of course, is meaningless. From condition (30) and (34), W and $T'_{\mathrm{mKdV}'}$ are given, respectively, by

$$W = -\frac{1}{8}v_{xx} + \frac{1}{4}vv_x - \frac{1}{8}v^3, \tag{35}$$

$$T'_{\mathrm{mKdV}'} = \frac{1}{2}v\partial_x^2 + \frac{1}{2}v^2\partial_x - \frac{1}{8}v_{xx} + \frac{1}{4}vv_x - \frac{1}{8}v^3. \tag{36}$$

Namely it has been shown that the operator $L_{\mathrm{mKdV}'}$ can give the mKdV equation (1) by the Lax equation (6) and the Painlevé test.

3 An extension the modified KdV equation to (2 + 1) dimensions

It is well known that the Lax differential operator plays a key role in constructing higher dimensional equations from lower dimensional ones. We extend only T operator to $(2+1)$ dimensions as follows [1, 6, 7]

$$T = \partial_z(L_{\mathrm{mKdV}'}) + \widetilde{T}_{\mathrm{mKdV}'} + \partial_t. \tag{37}$$

Here z is a new spatial coordinate. Then the Lax pair is given by

$$L = L_{\mathrm{mKdV}'} - \lambda, \tag{38}$$

$$T = \partial_z\left(L_{\mathrm{mKdV}'}\right) + \widetilde{T}_{\mathrm{mKdV}'} + \partial_t, \tag{39}$$

where note that $\partial_z(L_{\mathrm{mKdV}'}) = \partial_x^2\partial_z + v\partial_x\partial_z + v_z\partial_x + \left(\frac{1}{4}v^2 + \frac{1}{2}v_x\right)\partial_z + \frac{1}{2}vv_z + \frac{1}{2}v_{xz}$ and $\widetilde{T}_{\mathrm{mKdV}'}$ is an unknown operator. So we obtain

$$[L_{\mathrm{mKdV}'} - \lambda, \partial_z(L_{\mathrm{mKdV}'}) + \partial_t] + [L_{\mathrm{mKdV}'} - \lambda, \widetilde{T}_{\mathrm{mKdV}'}] = 0, \tag{40}$$

from the Lax equation (6) of the pair (38) and (39). The first term in the left-hand side of equation (40) gives

$$[L_{\mathrm{mKdV}'} - \lambda, \partial_z(L_{\mathrm{mKdV}'}) + \partial_t] = -v_z\partial_x^3 - \left(\frac{3}{2}vv_z + \frac{1}{2}v_{xz}\right)\partial_x^2$$
$$- \left(\frac{3}{2}v_xv_z + \frac{3}{4}v^2v_z + \frac{1}{2}v_{xz} + v_t\right)\partial_x + \left(\frac{3}{4}v_xv_{xz} + \frac{1}{2}vv_{xxz} + \cdots\right). \tag{41}$$

As in $(1+1)$ dimensions, we assume the form of the operator $\widetilde{T}_{\mathrm{mKdV}'}$

$$\widetilde{T}_{\mathrm{mKdV}'} = U\partial_x^2 + V\partial_x + W, \tag{42}$$

where U, V and W are functions of x, z and t. Then the second term in the left-hand side of equation (40) gives

$$\begin{aligned}[][L_{\mathrm{mKdV}'} - \lambda, \widetilde{T}_{\mathrm{mKdV}'})] &= 2U_x\partial_x^3 + (U_{xx} + 2V_x + vU_x - 2Uv_x)\,\partial_x^2 \\ &\quad + (V_{xx} + 2W_x + vV_x - Vv_x - Uvv_x - 2Uv_{xx})\,\partial_x + (W_{xx} + vW_x + \cdots)\,.\end{aligned} \tag{43}$$

By comparing the first term (41) to the second one (43),

$$U = \frac{1}{2}\partial_x^{-1}v_z, \tag{44}$$

$$V = \frac{1}{2}v\left(\partial_x^{-1}v_z\right), \tag{45}$$

$$2W_x = v_t + \frac{1}{4}v^2v_z + \frac{1}{2}vv_x\left(\partial_x^{-1}v_z\right) + \frac{1}{2}v_xv_z + \frac{1}{2}v_{xx}\left(\partial_x^{-1}v_z\right) \tag{46}$$

and

$$W_{xx} + vW_x + \frac{3}{4}v_xv_{xz} + \frac{1}{2}vv_{xxz} + \cdots = 0, \tag{47}$$

where equation (47) is an identity by U, V and W_x. The exact forms of U and V have obtained and one of W has not yet as in $(1+1)$ dimensions.

Let us compute the degree of variables in equation (46), if taking $[\partial_x] = 1$,

$$[v] = 1, \tag{48}$$

$$[\partial_t] = 2 + [\partial_z], \tag{49}$$

$$[W_x] = 3 + [\partial_z], \tag{50}$$

with $[\partial_z]$ being arbitrary. These degrees lead us to take as unknown function W_x

$$\begin{aligned}-2W_x &= av_{xxz} + bv_{xx}\left(\partial_x^{-1}v_z\right) + cvv_{xz} + dv_xv_z \\ &\quad + evv_x\left(\partial_x^{-1}v_z\right) + fv^2v_z + gv^3\left(\partial_x^{-1}v_z\right),\end{aligned} \tag{51}$$

where all from a to g is real constant. Substituting W_x into equation (46) gives

$$\begin{aligned}&v_t + av_{xxz} + \left(b + \frac{1}{2}\right)v_{xx}\left(\partial_x^{-1}v_z\right) + cvv_{xz} + \left(d + \frac{1}{2}\right)v_xv_z \\ &\quad + \left(e + \frac{1}{2}\right)vv_x\left(\partial_x^{-1}v_z\right) + \left(f + \frac{1}{4}\right)v^2v_z + gv^3\left(\partial_x^{-1}v_z\right) = 0.\end{aligned} \tag{52}$$

Here we perform the Painlevé test for equation (52) to get real constants in it. For that, we need to rewrite equation (52) for taking away the term of ∂_x^{-1}. That exact form, however, is very complicated for writing down here. We would like to write down the result. That is,

$$\text{leading order}: \quad -1 \tag{53}$$

$$\text{resonances}: \quad -1,\ 1,\ 3,\ 4 \tag{54}$$

$$\text{real constants}: \quad b = d = -\frac{1}{2}, \quad c = g = 0, \tag{55}$$

and other constants are arbitrary.

Thus equation (52) gives

$$v_t + \frac{1}{4}v_{xxz} + \left(\frac{1}{2} + e\right) vv_x \left(\partial_x^{-1} v_z\right) + (1-e)\, v^2 v_z = 0, \tag{56}$$

where we take $a = \frac{1}{4}$ and $f = \frac{3}{4} - e$ for being reduced to the mKdV equation (1) setting $z = x$. This equation (56) is quite different from the higher dimensional mKdV equation (2) and (3).

That is, the alternative mKdV equation (56) in $(2+1)$ dimensions was given by the Lax equation (6) and the Painlevé test.

4 Summary

A natural problem in the integrable systems is whether we can find new $(2+1)$ dimensional integrable equations from already known $(1+1)$ dimensional integrable ones. The Lax representation is a powerful tool to do so. The method used in this paper is based on works by Calogero et al.

Our results in this paper are as follows.

(i) The $(1+1)$ dimensional mKdV equation (1) has been obtained by the Lax pair

$$L = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x - \lambda, \tag{57}$$

$$T = \partial_x^3 + \frac{3}{2}v\partial_x^2 + \left(\frac{3}{4}v^2 + \frac{3}{2}v_x\right)\partial_x + \frac{3}{4}vv_x + \frac{3}{8}v_{xx} - \frac{1}{8}v^3 + \partial_t. \tag{58}$$

(ii) By extending the Lax pair (57) and (58) to $(2+1)$ dimensions, the higher dimensional mKdV equation (56) has been introduced. And then the Lax pair is given by

$$L = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x - \lambda, \tag{59}$$

$$\begin{aligned} T = {} & \partial_x^2\partial_z + \frac{1}{2}\left(\partial_x^{-1}v_z\right)\partial_x^2 + v\partial_x\partial_z + \left(v_z + \frac{1}{2}v\partial_x^{-1}v_z\right)\partial_x + \left(\frac{1}{4}v^2 + \frac{1}{2}v_x\right)\partial_z + \frac{3}{8}v_{xz} \\ & + \frac{1}{4}v_x\partial_x^{-1}v_z + \frac{1}{2}vv_z - \frac{e}{4}v^2\partial_x^{-1}v_z + \left(\frac{3e}{4} - \frac{3}{8}\right)\partial^{-1}\left(v^2 v_z\right) + \partial_t \end{aligned} \tag{60}$$

This equation is integrable in the sense of the existence of the Lax pair and passing the Painlevé test.

Next let us mention our further works.

(i) The higher dimensional mKdV equations (2) and (3) have various exact solutions (soliton solution and so on) [2, 3]. They constructed via Bilinear approach or Hirota method. We have not been able to constructed exact solutions to equation (56) yet.

(ii) We will extend the Lax pair (57) and (58) to (2+1) dimensions by using other method [7, 8].

We believe that higher dimensional integrable equations can be obtained from lower dimensional integrable ones by extending the Lax pairs to higher dimensions. We have a dream such as constructing $(3+1)$ dimensional integrable equations (if there exist). Further study on this topic continues.

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