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General Even and Odd Coherent States as Solutions of Discrete Cauchy Problems

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The explicit and exact solutions of the linear homogeneous difference equation with initial conditions (Cauchy problem) are constructed. The approach is quite general and relies on a novel and successful treatment of the linear recursion appropriately cast in matrix form. Our approach is exploited to solve the eigenvalues problem of a special set of non-Hermitian operators. A new class of generalized even and odd coherent states of a quantum harmonic oscillator are defined.

The occurrence of linear or nonlinear difference equations is ubiquitous in applied sciences. The exact treatment of many important problem in physics, chemistry, biology, economy, psychology and so on, depend on our ability to solve recursive relations of various kind. The importance of this particular chapter of mathematics may be for instance appreciated taking into account the close relation existing between difference and differential equations. Systematic methods for approximating intractable ordinary or partial differential equations by easier-to manage appropriate difference equations, are currently and successfully used in many contexts of applied sciences [1, 2, 3]. By definition a nth-order linear discrete Cauchy problem consists of a linear normal nth-order difference equation associated to given initial conditions. It is well known that when the vectors of the initial conditions defining n different discrete Cauchy problems relative to the same nth-order recursive equation are independent, then the n corresponding solutions constitute a fundamental set of solutions. In this paper we construct the explicit and exact solution of the following discrete Cauchy problems

$$\begin{aligned} &y_{k+n} = f_1(k)y_{k+n-1} + \cdots + f_{n-1}(k)y_{k+1} + f_n(k)y_k, \\ &y_0 = y_1 = y_2 = \cdots = y_{k-1} = 0, \quad y_k = 1, \quad y_{k+1} = y_{k+2} = \cdots = y_{n-1} = 0, \end{aligned} \tag{1}$$

where $f_i : S \to \mathbb{R}\ \forall\ i = 1, 2, \ldots, n$ and $f_n(k) \neq 0\ \forall\ k \in S = \{0, 1, 2, \ldots\}$. Our treatment is new and leads to a resolutive formula whose usefulness is vividly illustrated by an application to the physics of the quantum harmonic oscillator. To this end we transform equation (1) into the following homogeneous, linear, matrix, first order equation

$$Z_{k+1} = A^{(k)} Z_k, \qquad k = 0, 1, 2, \ldots \tag{2}$$

with $A^{(k)}$ $n \times n$ matrix defined by $A_{1j} = \delta_{jn}$, $A^{(k)}_{21} = f_n(k)$, $A^{(k)}_{2j} = f_{j-1}(k)$ $(j = 2, \ldots, n)$, $A_{rj} = \delta_{r-1,j}$ for $r = 3, \ldots, n$ and

$$Z_k^T = (y_k \ \ y_{k+n-1} \ \ \cdots \ \ y_{k+1}), \tag{3}$$

where superscript T denotes the transposition of column vector Z_k. Its formal solution has the form

$$Z_k = P^{(k)} Z_0, \tag{4}$$

where $P^{(0)} = I$ and $P^{(k)} = A^{(k-1)}A^{(k-2)} \cdots A^{(0)}$, $k \geq 1$. It is easy to verify that $P_{1j}^{(0)} = \delta_{1j}$, $P_{1j}^{(1)} = \delta_{nj}$ and, for any $k > 1$

$$P_{1j}^{(k)} = \sum_{h_0,h_1,\ldots,h_{k-2}} A_{1h_{k-2}}^{(k-1)} A_{h_{k-2}h_{k-1}}^{(k-2)} \cdots A_{h_1h_0}^{(1)} A_{h_0j}^{(0)}, \tag{5}$$

where h_i $(i = 0, 1, \ldots, k-2)$ runs from 1 to n. Consider first $1 \leq k \leq n-1$. In this case, in view of equation (5), $P_{1j}^{(k)}$ does not vanish only if $k = n-j+1$. In fact only this condition ensures the existence of a not vanishing contribution to $P_{1j}^{(k)}$ in the form of the following product of matrix elements

$$A_{1n}^{(k-1)} A_{nn-1}^{(k-2)} \cdots A_{j+1j}^{(k+j-n-1)} = 1. \tag{6}$$

Thus we arrive at the conclusion that $P_{1j}^{(k)} = \delta_{n+1-k,j}$. Moreover $P_{1j}^{(n)} = A_{2j}^{(0)}$ and $P_{1j}^{(n+1)} = A_{2j}^{(0)} A_{22}^{(1)} + (1 - \delta_{1,j}) A_{2j+1}^{(1)}$ provided we consistently put $A_{2\,n+1}^{(k)} \equiv A_{21}^{(k)}$. Observe that

$$\sum_{h_0,h_1,\ldots,h_{k-2}} A_{1h_{k-2}}^{(k-1)} A_{h_{k-2}h_{k-1}}^{(k-2)} \cdots A_{h_1h_0}^{(1)} A_{h_0j}^{(0)} = \sum_{h_0,h_1,\ldots,h_{k-n-1}} A_{2h_{k-n-1}}^{(k-n)} \cdots A_{h_1h_0}^{(1)} A_{h_0j}^{(0)}, \tag{7}$$

where the $n-1$ indices $h_{k-2}, h_{k-1}, \ldots, h_{k-n}$ have been eliminated with the help of relations essentially similar to that expressed by equation (6). The expression (5) for $P_{1j}^{(k)}$ with $k > n+1$ and $j = 1, 2, \ldots, n$, may be put in the following form

$$P_{1j}^{(k)} = \sum_{r=j}^{r^*} (\delta_{1,r} + \vartheta(j-1)) A_{2r}^{(r-j)} \sum_{h_{r-j+1},\ldots,h_{k-n-1}} A_{2h_{k-n-1}}^{(k-n)} \cdots A_{h_{r-j+1}2}^{(r-j+1)} + f_n(k,j), \tag{8}$$

where

$$\begin{aligned} f_n(k,j) = {} & (1 - \delta_{1,j})(1 - \vartheta(k - 2n + j - 2)) \\ & \times \left[(1 - \delta_{k-n,n-j+2}) A_{2j+k-n}^{(k-n)} + A_{22}^{(k-n)} A_{21}^{(k-n-1)} \delta_{k-n,n-j+2} \right], \end{aligned} \tag{9}$$

$\vartheta(x)$ is the Heaviside step function such that $\vartheta(0) = 0$ and $r^* = \min\{(n+1), k+j-n-2\}$. Equation (8) expresses $P_{1j}^{(k)}$ in terms of finite sums like

$$\sum_{h_{r-j+1},\ldots,h_{k-n-1}} A_{2h_{k-n-1}}^{(k-n)} \cdots A_{h_{r-j+1}2}^{(r-j+1)} \tag{10}$$

which may be further simplified exploiting the structural presence of "1" and "0" in the characteristic matrices $\left\{A^{(k)}\right\}$. To this end we note that there are only n not vanishing products of matrix elements beginning in the second row and ending in the second column:

$$\begin{aligned} & A_{21}^{(k-n)} A_{1n}^{(k-n-1)} \cdots A_{32}^{(k-2n+1)} = A_{21}^{(k-n)}, \\ & A_{22}^{(k-n)} = A_{22}^{(k-n)}, \\ & A_{23}^{(k-n)} A_{32}^{(k-n-1)} = A_{23}^{(k-n)}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & A_{2j}^{(k-n)} A_{jj-1}^{(k-n-1)} \cdots A_{32}^{(k-n)} = A_{2j}^{(k-n)}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & A_{2n}^{(k-n)} A_{nn-1}^{(k-n-1)} \cdots A_{32}^{(k-n)} = A_{2n}^{(k-n)}. \end{aligned} \tag{11}$$

To take advantage from equation (11) let us introduce the function $g : \{1, 2, \ldots, n\} \to \mathbb{N} \times \mathbb{N}$ defined putting $g(1) = (2, 2)$; $g(j) = (2, j + 1)$, $2 \le j \le n - 1$, $(n > 2)$ and $g(n) = (2, 1)$ which helps in writing down $P_{1j}^{(k)}$ as given by equation (8) in a conveniently irreducible form. In what follows we shall use the following symbols

$$A_{g(1)}^{(p)} \equiv A_{22}^{(p)}, \quad \ldots, \quad A_{g(j)}^{(p)} \equiv A_{21}^{(p)}. \tag{12}$$

Looking at equation (11) we see that the length of the sequence beginning with $A_{21}^{(k-n)}$ $\left(A_{2j}^{(k-n)}\right)$ (that is the number of factors) is $n(j-1)$. It is easy to convince oneself that each not vanishing contribution to the sum expressed by equation (10) may be subdivided into products of sequences of different length explicitly written down in equation (11). This circumstance provides the key for defining a useful algorithm to cast $P_{1j}^{(k)}$ into an explicit form where only elements of the second row of the matrices $\{A^{(k)},\ k = 1, 2, \ldots\}$ are present. For this purpose we find convenient to put the following definitions. Let h, n and p be positive integers. We say that a sequence of integers has order h and high n if it is constructed by h eventually repeated positive integers not exceeding n. A generic sequence of order h and high n is denoted by $(r_1, \ldots, r_h)_n$ and the set of all such sequences by $I_n(h)$. For each prefixed integer p such that $h \le p \le nh$, we say that $(r_1, \ldots, r_h)_n \in I_n(h)$ represents a $p-$sequence when it satisfies the additional condition to be also a partition of the integer p, that is $\sum\limits_{\nu=1}^{h} r_\nu = p$. A generic $p-$sequence of order h and high n is denoted by $(r_1, \ldots, r_h)_n^p$ and the certainly not empty subset of all such p-sequences by $I_n^p(h)$. Finally we put $I_n^p = \bigcup\limits_{h=1}^{p} I_n^p(h)$, that is I_n^p is the set of all the p-sequences of high n in correspondence with all the possible orders.

For instance if $n = 5$ and $p = 3$, I_5^3 has 4 elements: $1{+}1{+}1 = 2{+}1 = 1{+}2 = 3$, and for $p = 6$ I_5^6 has 31 elements: $1{+}1{+}1{+}1{+}1{+}1 = 2{+}1{+}1{+}1{+}1 = 1{+}2{+}1{+}1{+}1 = \cdots = 1{+}5 = 5{+}1 = 6$. It is possible to convince oneself that for any $k > n{+}1$ and $1 \le j \le n$ the expression (10) appearing in equation (8) may be cast in the following form

$$\sum_{(r_1,\ldots,r_h)_n^p \in I_n^p} A_{g(r_1)}^{(k-n-r_0)} A_{g(r_2)}^{\left(k-n-\sum\limits_{t=0}^{1} r_t\right)} \cdots A_{g(r_h)}^{\left(k-n-\sum\limits_{t=0}^{h-1} r_t\right)}, \tag{13}$$

where $r_0 = 0 \le h \le p$, $p = (k - n) - (r - j)$.

The important difference between the two expression (10) and (13) is of course that the latter equation contains only matrix elements of the second rows of the (at most) $(k - n)$ matrices $A^{(1)}, A^{(2)}, \ldots, A^{(k-n)}$ and therefore is irreducible. We wish to point out that $k - n - \sum\limits_{t=1}^{h-1} r_t > r - j + 1$ as it should be. Inserting equation (13) into equation (8) we are now in position to write down the definite expression of $P_{1j}^{(k)}$ as

$$\begin{aligned}
&\delta_{1,j}, \quad k = 0, \qquad \delta_{n+1-k,j}, \quad 1 \le k \le n, \qquad A_{21}^{(0)}, \quad k = n,\\
&A_{2j}^{(0)} A_{2j}^{(1)} + (1 - \delta_{1j}) A_{2j}^{(1)}, \quad k = n + 1,\\
&\sum_{r=j}^{r^*} [\delta_{1,r} + \vartheta(j-1)] A_{2r}^{(r-j)}\\
&\qquad \times \sum_{(r_1,\ldots,r_h)_n^p \in I_n^p} A_{g(r_1)}^{(k-n-r_0)} \cdots A_{g(r_h)}^{\left(k-n-\sum\limits_{t=0}^{h-1} r_t\right)} + f_n(k, j), \quad k > n + 1.
\end{aligned} \tag{14}$$

When j runs from 1 to n, $P_{1j}^{(k)}$ defines n independent solutions of equation (1). Introduce the non-hermitian operators

$$C_n = a^n e^{i\frac{2\pi}{n}a^\dagger a}, \qquad n = 1, 2, \ldots \tag{15}$$

The eigenstates of C_1 are the coherent states and the eigenstates of C_2 pertaining to a generic not null eigenvalue are the even and odd coherent states. The eigenvalue problem for C_n, formulated to construct generalizations of the even and odd coherent states,

$$C_n \sum_{k=0}^{\infty} b_k|k\rangle = \lambda \sum_{k=0}^{\infty} b_k|k\rangle, \qquad \sum_{k=0}^{\infty} |b_k|^2 < \infty, \qquad \lambda \in \mathbb{C} \setminus \{0\} \tag{16}$$

may be easily reduced to the resolution of the following linear discrete Cauchy problem:

$$\begin{aligned} &b_{k+n} = \lambda \sqrt{\frac{k!}{(k+n)!}} e^{-i\frac{2\pi}{n}k} b_k, \\ &b_k = 1, \qquad b_0 = b_1 = \cdots = b_{k-1} = b_{k+1} = \cdots = b_{n-1}, \end{aligned} \tag{17}$$

where k runs from 0 to $n-1$. The Fock states $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ are eigenstates of C_n, with eigenvalue 0. If normalizable solutions of the linear difference equation (17) of order n and with variable coefficients exist in correspondence to such initial conditions, then, in view of equation (17), the relative eigensolutions of C_n satisfy the property that the distance between two successive Fock states of their number representations is fixed and equal to n. Thus the n different eigenstates of C_n correspond to the n initial conditions, if normalizable, provide possible generalizations of the even and odd coherent states. We now solve equation (17) exploiting the formula (14) deduced in this paper. A comparison between equation (17) and (1) yields

$$A_{21}^{(k)} = \lambda \sqrt{\frac{k!}{(k+n)!}} e^{-i\frac{2\pi}{n}k}, \qquad A_{2j}^{(k)} = 0, \qquad j = 2, \ldots, n. \tag{18}$$

As a consequence, we immediately deduce that only when the choice $g(r_1) = g(r_2) = \cdots = g(r_h) = g(n)$ is compatible with a prefixed value of p, that is $p = nh$, the expression (12) does not vanish. This fact implies that the sum over r appearing in equation (16), contributes for $j = 1$ with the $(r = 1)$-term only and with the $(r = n+1)$-term only, if $j > 1$ and $k > 2n-j+2$. Thus exploiting equation (16) the general expression of $P_{11}^{(k)}$ in our case may be cast as follows:

$$\begin{aligned} P_{11}^{(k)} &= A_{21}^{(h-1)n} A_{21}^{(h-2)n} \cdots A_{21}^{(n)} A_{21}^{(0)} = \frac{\lambda^h}{\sqrt{(hn)!}}, \qquad \forall\, k = hn, \quad h = 1, 2, \ldots, \\ &0, \qquad \text{otherwise.} \end{aligned} \tag{19}$$

$P_{11}^{(k)}$ is the solution of equation (17) in correspondence to the initial condition $b_0 = 1$, $b_1 = b_2 = \cdots = b_{n-1} = 0$. The corresponding normalized eigenstate of C_n may be written down as

$$|\psi_0^{(n)}\rangle = N_0 \sum_{h=0}^{\infty} \frac{\lambda^h}{\sqrt{(hn)!}} |hn\rangle, \tag{20}$$

where

$$N_0 = n e^{-\frac{1}{2}|\beta|^2} \left\{ n + 2 \sum_{\nu=1}^{n-1} (n-\nu) e^{-2|\beta|^2 \sin^2\left(\frac{\pi}{n}\nu\right)} \cos\left(|\beta|^2 \sin\left(\frac{2\pi}{n}\nu \right) \right) \right\}^{-\frac{1}{2}} \tag{21}$$

and the relative eigenvalue is of course λ. The solution of equation (17) relative to the initial condition $b_{n-j+1} = 1$, $b_0 = b_1 = \cdots = b_{n-j} = b_{n-j+2} = \cdots = b_{n-1} = 0$ for $j > 1$ is $P_{1j}^{(k)}$ and corresponds to the following eigenstate of C_n

$$|\psi_k^{(n)}\rangle = N_k \sum_{h=0}^{\infty} \frac{\lambda^h}{\sqrt{(hn+k)!}} |hn+k\rangle, \qquad k = 0, \ldots, n-2, \tag{22}$$

where N_k is an appropriate normalization constant explicitly calculable.

It is possible to demonstrate that $|\psi_k^{(n)}\rangle$ can be expressed as linear combination of n equal-amplitude coherent states. In particular,

(a) $|\psi_0^{(n)}\rangle$ may be represented as the equal right linear combination of all the eigenstates of a^n pertaining to the same eigenvalue λ. It therefore generalizes the even coherent state;

(b) $|\psi_{\frac{n}{2}}^{(n)}\rangle$ (n are *even*) can also be expanded in terms of the same set of coherent states as before, with the difference that now the ratio between successive coefficients is -1. Appropriately adjusting its global phase, we may therefore state that it generalizes the odd coherent state.

In this paper we have derived a new way of representing the general solution of an arbitrary homogeneous linear difference equation. Our resolutive keys of this problem are two. The first one is the choice of the fundamental set of solutions used. The second one is the algorithm by which we succeed to express in the (best possible) closed form the first row of the product of an arbitrary number of the noncommutating matrices. Our resolutive formula (14) has been applied to solve the eigenvalue problem of a particular non-Hermitian operator building up a new class of states of a quantum harmonic oscillator. These states should attract interest in quantum optician community, for instance, in view of the fact that these generalized even and odd coherent states might exhibit remarkable non-classical features.

Concluding we wish to emphasize that the material presented in this paper provides a concrete stimulus toward other interesting applicable developments both in physics and in mathematics.

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