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# Symmetries of Integro-Differential Equations

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The Ovsiannikov method of finding Lie symmetries is generalized to the case of point transformations of integro-differential equations. The new method is direct and applicable to practical cases, for instance to Vlasov–Maxwell equations of plasmas.

## 1 Introduction

We present a general and direct method of determination of symmetry groups of point transformations for integro-differential equations. The method is a natural generalization of the Ovsiannikov method for differential equations [1, 2, 3, 4, 5, 6].

We consider a system of integro-differential equations (IDE's) of the form

$$W\left(F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}), \int_X dx^1 \cdots dx^l f(x, y, \underset{1}{y}, \ldots, \underset{k}{y})\right) = 0, \tag{1}$$

where $n$, $m$, $k$, $l$ are arbitrary natural numbers ($l \leq n$), $x = (x^1, \ldots, x^n)$, functions $W$, $F$ and $f$ are arbitrary but sufficiently regular to secure the existence of solutions to (1), limits of integrations (region $X$) are also arbitrary. The symbol $\underset{m}{y}$ denotes the set of all partial derivatives of $m$-order:

$$\underset{m}{y} = \left\{ \frac{\partial^m y}{\partial x^{i_1} \cdots \partial x^{i_m}} \equiv \partial_{x^{i_1}} \cdots \partial_{x^{i_m}} y \equiv y_{i_1 \cdots i_m} \right\}.$$

The equation (1) reduces to a differential equation for $f = 0$, thus our method contains the Ovsiannikov method as a particular case.

Earlier approaches to investigations of symmetries of IDE's can be found in [7], in CRC Handbook [8], and in references therein. The lack of a general and universal method has led to many attempts using various methods which often constitute *ad hoc* means adapted for each case. For example, specific kinds of IDE's were chosen so that certain methods could be used effectively. In [9] the integral term of the IDE has the form of a square root of a differential operator. The method used there consists in finding a partial differential equation (PDE) with the space of solutions containing the solutions of the considered IDE. After finding symmetries of the auxiliary PDE by standard method the symmetries of IDE are found by inspection. In [10] the IDE with the integral in the form of a Fourier transform is considered. In this case the Lie derivative is found effectively and used for the determination of symmetries.

Methods called *indirect* methods form a separate class. They are based on a transformation of a given set of IDE's to an equivalent set of auxiliary equations for which symmetries are known or can be found by known methods. Then symmetries of the initial system of IDE's can be reconstructed. Usually, this auxiliary set of equations consists of PDE's as, for example, in Taranov's method [11]. He transformed the Vlasov–Maxwell equations for one-component plasma into an infinite chain of differential equations for the moments of a distribution function. Another indirect approach is based on an extension of the Harrison and Estabrook method [12] to the case of IDE's. A given set of equations is transformed to an equivalent set of differential forms. This method was used in [13] for IDE's invariant with respect to Galilei, Poincaré, Schrödinger and conformal groups, in [14] for the Boltzmann equation and in [15, 16] for IDE's of Hartree type. These methods are encumbered with the usual burden of indirect methods. They involve the necessary movement there and back with the crucial problem of equivalence and an interpretation of results. Moreover, quite often an auxiliary problem is more complicated than the initial one when our direct method is applied.

A direct method is presented in [5] and in Chapter 5 of Vol.3 of CRC Handbook [8]. It consists in assuming equal to zero the derivative with respect to the group parameter of a transformed IDE (depending on the parameter) at zero value of this parameter. When this condition is properly evaluated, that is when the dependence of limits of an integral on the group parameter is taken into account, then it leads to our criterion of symmetry of IDE's (8). However, this evaluation must be done every time when this condition is used. This may be suitable for a computer (see [17]) but not for a man. The dependence of limits of an integral (even constant limits!) on the group parameter is sometimes overlooked in certain papers. Moreover, it is more appropriate to consider a region of integration since the expression of the $n$-dimensional integral by the $n$-fold integral is not invariant with respect to point transformations. The problem disappears for Bäcklund symmetries in the form of vertical transformations because there is no transformation of independent variables in this case. The method was used in [18, 19] for finding symmetries of the Boltzmann equation of a special kind.

The general and sophisticated method of Vinogradov and Krasilshchik [20, 21] has arisen from a simple idea of elimination of integrals from IDE's by virtue of the fundamental theorem of calculus by further prolongation to *nonlocal* variables: the primitive functions of dependent variables. This is natural in the case of IDE's with variable limits of integrals, for example for the Volterra type of IDE's. However, the most important IDE's in physics, such as equations of kinetic theory, contain integrals with constant limits. Then, this construction is somewhat artificial and complicated. The method becomes indirect since it leads to the so called *boundary-differential* equations [21]. The method requires advanced and sophisticated mathematics, for example the theory of coverings of a system of differential equations and the prolongation procedure for boundary–differential equations. The method was used in [22, 23] for finding symmetries of the coagulation kinetic equation.

Since, in general, an integral structure of equations cannot be transformed into an algebraic one by admitting nonlocal variables, we stay in a jet space to deal with derivatives, as in the Ovsiannikov method, and find a new infinitesimal criterion of symmetry in our case of IDE's. This criterion is the essence of our direct method.

## 2 Extension of a group

We look for a Lie symmetry group of *point transformations*

$$\widetilde{x}^i = e^{\epsilon G} x^i = x^i + \epsilon \xi^i(x, y) + \mathcal{O}\left(\epsilon^2\right), \qquad \widetilde{y} = e^{\epsilon G} y = y + \epsilon \eta(x, y) + \mathcal{O}\left(\epsilon^2\right), \tag{2}$$

with the *infinitesimal generator* (summation over repeated indices is assumed)

$$G = \xi^i(x, y)\partial_{x^i} + \eta(x, y)\partial_y, \tag{3}$$

admitted by the system of IDE's (1). As in the Ovsiannikov method we extend the group of point transformations (2) to a jet space of independent and dependent variables and derivatives of dependent variables in the usual way [1, 2, 3, 4, 5, 6]

$$\begin{aligned}
&\widetilde{x}^i = e^{\epsilon G^{(m)}} x^i = x^i + \epsilon \xi^i(x,y) + \mathcal{O}\left(\epsilon^2\right), \\
&\widetilde{y} = e^{\epsilon G^{(m)}} y = y + \epsilon \eta(x,y) + \mathcal{O}\left(\epsilon^2\right), \\
&\widetilde{y}_i = e^{\epsilon G^{(m)}} y_i = y_i + \epsilon \eta_i(x,y,\underset{1}{y}) + \mathcal{O}\left(\epsilon^2\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\widetilde{y}_{i_1\cdots i_m} = e^{\epsilon G^{(m)}} y_{i_1\cdots i_m} = y_{i_1\cdots i_m} + \epsilon \eta_{i_1\cdots i_m}(x,y,\underset{1}{y},\ldots,\underset{m}{y}) + \mathcal{O}\left(\epsilon^2\right),
\end{aligned} \tag{4}$$

where the *extended generator* is of the form

$$G^{(m)} = G + \eta_i \partial_{y_i} + \cdots + \eta_{i_1\cdots i_m} \partial_{y_{i_1\cdots i_m}}. \tag{5}$$

The coefficients $\eta_i, \ldots, \eta_{i_1\cdots i_m}$, defining the extended group, are given by the recursion relations:

$$\begin{aligned}
&\eta_i = D_i \eta - y_j D_i \xi^j, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\eta_{i_1\cdots i_m} = D_{i_m} \eta_{i_1\cdots i_{m-1}} - y_{i_1\cdots i_{m-1}j} D_{i_m} \xi^j
\end{aligned} \tag{6}$$

and the total derivative $D_i$ is defined as follows

$$D_i = \partial_i + y_i \partial_y + y_{ij} \partial_{(y_j)} + \cdots + y_{i i_1\cdots i_n} \partial_{(y_{i_1\cdots i_n})} + \cdots .$$

# 3 Criterion of invariance of integro-differential equations

Invariance of an equation means invariance of the space of its solutions. Thus, point transformation (2) maps any solution $y(x)$ of the equation (1) into another solution $\widetilde{y}(\widetilde{x})$ of the equation. In our geometric language, where solutions $y(x)$ are represented by their graphs in a jet space, it means that the following implication holds

$$W(F,I) = 0 \quad \Longrightarrow \quad W(\widetilde{F},\widetilde{I}) = 0, \tag{7}$$

where $I$ means integral term in (1), $\widetilde{F} \equiv F(\widetilde{\cdot})$ and $\widetilde{I}$ are obtained by extended transformations (4).

According to the definition (7), we act on the integro-differential equation (1) by extended transformations (4) writing down explicitly only terms that are linear with respect to the parameter $\epsilon$. Next, by expanding functions $W$, $F$ and $f$ in their Taylor series and changing variables in the integral, we express the change of (1) in terms of the extended generator (5). From the definition of symmetry (7), this change must be equal to zero for all values of $\epsilon$. Thus, we obtain an infinitesimal criterion of invariance of the equation (1).

We restrict our considerations to the one scalar equation of the type (1) for the sake of simplicity of notation. For a system of equations with $p$ dependent variables $y = (y^1, \ldots, y^p)$ some minor changes are evident and the resulting criterion is to be applied to each equation of the system. Expanding the function $W$ in a Taylor series, we obtain

$$W(\widetilde{F},\widetilde{I}) = W(F,I) + \frac{\partial W}{\partial F}\Delta F + \frac{\partial W}{\partial I}\Delta I + \cdots .$$

The change $\Delta F$ of the differential term of (1) is calculated by expanding the function $F$ in a Taylor series

$$\begin{aligned}\Delta F &= F(\widetilde{x},\widetilde{y},\underset{1}{\widetilde{y}},\ldots,\underset{m}{\widetilde{y}}) - F(x,y,\underset{1}{y},\ldots,\underset{m}{y})\\ &= F\Big(x^1+\epsilon\xi^1+\mathcal{O}\left(\epsilon^2\right),\ldots,x^n+\epsilon\xi^n+\mathcal{O}\left(\epsilon^2\right),y+\epsilon\eta+\mathcal{O}\left(\epsilon^2\right),y_1+\epsilon\eta_1+\mathcal{O}\left(\epsilon^2\right),\\ &\qquad \ldots,y_n+\epsilon\eta_n+\mathcal{O}\left(\epsilon^2\right),y_{i_1\ldots i_m}+\epsilon\eta_{i_1\cdots i_m}+\mathcal{O}\left(\epsilon^2\right)\Big) - F(x,y,\underset{1}{y},\ldots,\underset{m}{y})\\ &= \epsilon\left[\xi^i\partial_{x^i}F+\eta\partial_y F+\eta_i\partial_{y_i}F+\cdots+\eta_{i_1\cdots i_m}\partial_{y_{i_1}}\cdots\partial_{y_{i_m}}F\right]+\mathcal{O}\left(\epsilon^2\right).\end{aligned}$$

Due to the definition of the extended generator (5) we can rewrite the above result in the form

$$\Delta F = \epsilon G^{(m)}F(x,y,\underset{1}{y},\ldots,\underset{m}{y})+\mathcal{O}\left(\epsilon^2\right).$$

Thus, the condition $\Delta F = 0$ leads to the Ovsiannikov infinitesimal criterion of invariance of differential equation $G^{(m)}F(x,y,\underset{1}{y},\ldots,\underset{m}{y})=0$.

Let us consider the change of an integral term in the equation (1)

$$\Delta I = \int_{\widetilde{X}} d\widetilde{x}^1\cdots d\widetilde{x}^l f(\widetilde{x},\widetilde{y},\underset{1}{\widetilde{y}},\ldots,\underset{k}{\widetilde{y}}) - \int_X dx^1\cdots dx^l f(x,y,\underset{1}{y},\ldots,\underset{k}{y})$$

under the extended transformations (4). To this end, we change variables in the first integral according to the transformations (4):

$$\left\{\widetilde{x}^1,\ldots,\widetilde{x}^l\right\}\mapsto\left\{x^1,\ldots,x^l\right\}.$$

By virtue of (4) the elements of Jacobi's matrix are equal

$$\frac{\partial\widetilde{x}^i}{\partial x^j} = \delta_{ij}+\epsilon\frac{\partial\xi^i}{\partial x^j}+\mathcal{O}\left(\epsilon^2\right),\qquad i,j=1,\ldots,l.$$

Because the off-diagonal elements of the matrix are of the order $\mathcal{O}\left(\epsilon^2\right)$, thus the linear contribution to the Jacobian comes only from the product of the diagonal elements:

$$\frac{\partial\left(\widetilde{x}^1\cdots\widetilde{x}^l\right)}{\partial\left(x^1\cdots x^l\right)} = \left(1+\epsilon\frac{\partial\xi^1}{\partial x^1}\right)\cdots\left(1+\epsilon\frac{\partial\xi^l}{\partial x^l}\right)+\mathcal{O}\left(\epsilon^2\right) = 1+\epsilon\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x^i}+\mathcal{O}\left(\epsilon^2\right).$$

We do not use the summation convention when summation goes over the range $1,\ldots,l\leq n$ only.

Consequently, the change $\Delta I$ of the integral term is equal

$$\begin{aligned}\int_X dx^1\cdots dx^l\Bigg[&\left(1+\epsilon\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x^i}\right)f\Big(x^1+\epsilon\xi^1+\mathcal{O}\left(\epsilon^2\right),\ldots,x^n+\epsilon\xi^n+\mathcal{O}\left(\epsilon^2\right),\\ &y+\epsilon\eta+\mathcal{O}(\epsilon^2),y_1+\epsilon\eta_1+\mathcal{O}\left(\epsilon^2\right),\ldots,y_n+\epsilon\eta_n+\mathcal{O}\left(\epsilon^2\right),\ldots,\\ &y_{i_1\cdots i_k}+\epsilon\eta_{i_1\cdots i_k}+\mathcal{O}\left(\epsilon^2\right)\Big) - f(x,y,\underset{1}{y},\ldots,\underset{k}{y})\Bigg]+\mathcal{O}\left(\epsilon^2\right).\end{aligned}$$

Expanding the function $f$ into a Taylor series we obtain

$$\Delta I = \epsilon\int_X dx^1\cdots dx^l\left[\xi^i\partial_{x_i}f+\eta\partial_y f+\eta_i\partial_{y_i}f+\cdots+\eta_{i_1\cdots i_k}\partial_{y_{i_1}}\cdots\partial_{y_{i_k}}f+f\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x^i}\right]+\mathcal{O}\left(\epsilon^2\right).$$

In view of the definition of the extended generator (5) we can rewrite the above result as follows

$$\Delta I = \epsilon \int_X dx^1 \cdots dx^l \left[ G^{(k)} f(x, y, \underset{1}{y}, \ldots, \underset{k}{y}) + f(x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] + \mathcal{O}\left(\epsilon^2\right).$$

From the calculations performed above we see that the implication (7) leads to the following infinitesimal *criterion of invariance* of integro-differential equations of the type (1) under the point transformations (2):

$$\frac{\partial W}{\partial F} G^{(m)} F + \frac{\partial W}{\partial I} \int_X dx^1 \cdots dx^l \left[ G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] = 0 \qquad \text{on solutions of (1).} \tag{8}$$

For a system of equations of the type (1) we apply the criterion (8) to each equation of the system as was mentioned earlier. Generalization to the case of more than one integral term

$$I_1 = \int_{X_1} dx^1 \cdots dx^l f(\,\cdot\,), \qquad I_2 = \int_{X_2} dx^1 \cdots dx^p g(\,\cdot\,), \qquad \ldots$$

in the equation (1) is simple. Then, in the resulting criterion we get the following summation

$$\frac{\partial W}{\partial I_1} \int_{X_1} dx^1 \cdots dx^l \left[ G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] + \frac{\partial W}{\partial I_2} \int_{X_2} dx^1 \cdots dx^p \left[ G^{(k)} g + g \sum_{i=1}^{p} \frac{\partial \xi^i}{\partial x^i} \right] + \cdots.$$

In the case of $W = F + I$, which corresponds to our example of the Vlasov–Maxwell equations, the criterion (8) takes the form

$$G^{(m)} F + \int_X dx^1 \cdots dx^l \left[ G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] = 0 \qquad \text{on solutions of (1).}$$

According to the criterion (8) we have to take into account the equation (1), which is now a constraint on extended variables. Using this equation we can eliminate some of them. Remaining variables are essentially independent, thus the equation (8) must be satisfied identically with respect to them. It means that the coefficients at independent expressions, involving these variables, must be equal to zero. This leads to the system of the so called *determining equations* for the integro-differential equation (1). They are homogeneous and *linear* integro-differential equations for coefficients $\xi^i$, $\eta$ determining the generator (3) and the point transformations (2). In applications, we have additional information in each particular case. Often, this information enables us to go to the integrands in integral determining equations by using the Lagrange lemma of variational calculus [24]. This leads to differential determining equations.

The criterion (8) is a *necessary* condition for symmetry of the equation (1), so it allows us to find all *possible* symmetry transformations of (1). The difficult task to obtain is to find a sufficient condition of symmetry. To this end we need a theorem on *global* existence and uniqueness of the solutions of the equation (1). The latter problem is far from being solved, see [25]. From a practical point of view the necessary condition is more important and useful than the sufficient one as the main task is to find symmetry transformations. A possible symmetry transformation of the equation (1) can be easily verified by inspection and this should be done anyway.

## 4 Symmetries of Vlasov–Maxwell equations

Let us consider the Vlasov–Maxwell system of equations for collisionless, multicomponent, one-dimensional plasmas with no magnetic field:

$$\partial_t f_\alpha + u\partial_x f_\alpha + \frac{q_\alpha}{m_\alpha} E\partial_u f_\alpha = 0,$$
$$\partial_t E + \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_{-\infty}^{\infty} du\, u f_\alpha = 0, \qquad \partial_x E - \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_{-\infty}^{\infty} du f_\alpha = 0, \tag{9}$$

where $E = E(t,x)$ is the $x$-component of electric vector field $\boldsymbol{E} = (E,0,0)$, $u$ is the $x$-component of vector velocity $\boldsymbol{v} = (u,0,0)$, $f_\alpha = f_\alpha(t,x,u)$ is the distribution function of $\alpha$-plasma component, $q_\alpha$, $m_\alpha$ are charge and mass of $\alpha$-particles, respectively and $\epsilon_0$ is electric permittivity of free space.

In this case, the generators (3) of point transformations (2) take the form

$$G = \tau\partial_t + \xi\partial_x + \rho\partial_u + \sum_\alpha \eta_\alpha \partial_{f_\alpha} + \zeta\partial_E. \tag{10}$$

Using the criterion (8) we obtain

$$0 = \partial_{f_\alpha}\tau = \partial_{f_\alpha}\xi = \partial_{f_\alpha}\rho = \partial_E\tau = \partial_E\xi = \partial_E\rho,$$

and the following determining equations (limits $\pm\infty$ of integrals are dropped):

$$0 = \partial_t\zeta = \partial_x\zeta = \partial_{f_\alpha}\zeta, \qquad 0 = u\partial_u\tau - \partial_u\xi,$$
$$0 = \partial_t\eta_\alpha + u\partial_x\eta_\alpha + \frac{q_\alpha}{m_\alpha}E\partial_u\eta_\alpha, \qquad 0 = u\partial_t\tau - \partial_t\xi + \rho + u^2\partial_x\tau - u\partial_x\xi,$$
$$0 = \sum_\beta E\left(\frac{q_\alpha}{m_\alpha} - \frac{q_\beta}{m_\beta}\right)(\partial_u f_\beta)\partial_{f_\beta}\eta_\alpha, \qquad 0 = \sum_\beta \frac{q_\beta}{\epsilon_0}\left(u\int du f_\beta - \int du\, u f_\beta\right)\partial_E\eta_\alpha,$$
$$0 = \frac{q_\alpha}{m_\alpha}\left(\partial_t\tau + u\partial_x\tau + \frac{q_\alpha}{m_\alpha}E\partial_u\tau - \partial_u\rho\right)E + \frac{q_\alpha}{m_\alpha}\zeta - \partial_t\rho - u\partial_x\rho,$$
$$0 = (\partial_t\tau - \partial_E\zeta)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u f_\beta - (\partial_t\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du f_\beta + \sum_\beta \frac{q_\beta}{\epsilon_0}\int du(\rho f_\beta + u\eta_\beta + u f_\beta\partial_u\rho),$$
$$0 = (\partial_E\zeta - \partial_x\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du f_\beta + (\partial_x\tau)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u f_\beta - \sum_\beta \frac{q_\beta}{\epsilon_0}\int du(\eta_\beta + f_\beta\partial_u\rho).$$

Except for the nonphysical case of a constant charge to mass ratio $q_\alpha/m_\alpha = \text{const}$ we easily find from the differential determining equations that

$$0 = \partial_u\tau = \partial_u\xi = \partial_t\eta_\alpha = \partial_x\eta_\alpha = \partial_u\eta_\alpha = \partial_E\eta_\alpha = \partial_{f_\beta}\eta_\alpha \quad \text{for} \quad \alpha \neq \beta, \qquad \zeta = \lambda_1 E.$$

Then, the last two integro-differential lead to

$$0 = \int du\,[f_\alpha(u\partial_u\tau - \lambda_1 u + \rho + u\partial_u\rho) + u\eta_\alpha], \qquad 0 = \int du[f_\alpha(\lambda_1 - \partial_x\xi + u\partial_x\tau - \partial_u\rho) + \eta_\alpha].$$

for every $\alpha$. We assume that the point transformations (2) are analytic functions of the point $(t,x,u,f_\alpha)$. In general, analyticity with respect to the parameter $\epsilon$ and infinite differentiability with respect to the point is assumed for Lie groups. However, the latter dependence is in fact also analytic due to a physical interpretation. Expanding $\eta_\alpha(f_\alpha)$ in the Taylor series, using the generalized mean value theorem and well known special solutions of the Vlasov–Maxwell equations (9), that is the stationary solutions depending only on velocity and BGK solutions, we find that the coefficients $\eta_\alpha$ can depend on $f_\alpha$ only linearly $\eta_\alpha = \lambda_2 f_\alpha$. Thus, we can apply the Lagrange lemma calculus of variations [24] and obtain *differential* equations for integrands.

Solutions of the determining equations are given by

$$\tau = -\frac{1}{3}(\lambda_1 + \lambda_2)t + \lambda_3, \qquad \xi = \frac{1}{3}(\lambda_1 - 2\lambda_2)x + \lambda_4 t + \lambda_5, \qquad \rho = \frac{1}{3}(2\lambda_1 - \lambda_2)u + \lambda_4,$$
$$\eta_\alpha = \lambda_2 f_\alpha, \qquad \zeta = \lambda_1 E,$$

where $\lambda_1, \ldots, \lambda_5$ are arbitrary parameters. Substituting the solutions into (10) and choosing all parameters equal to zero except one, which is assumed to be equal to 1, in each case, we derive the following five generators

$$G_1 = \partial_t, \qquad G_2 = \partial_x, \qquad G_3 = t\partial_x + \partial_u,$$
$$G_4 = -t\partial_t + x\partial_x + 2u\partial_u + 3E\partial_E, \qquad G_5 = -t\partial_t - 2x\partial_x - u\partial_u + 3\sum_\alpha f_\alpha \partial_{f_\alpha}, \tag{11}$$

which span the Lie algebra of the group of point symmetry transformations of the Vlasov–Maxwell equations (9). Non-vanishing commutators between these generators are given by

$$[G_1, G_3] = G_2, \qquad [G_1, G_4] = -G_1, \qquad [G_1, G_5] = -G_1, \qquad [G_2, G_4] = G_2,$$
$$[G_2, G_5] = -2G_2, \qquad [G_3, G_4] = 2G_3, \qquad [G_3, G_5] = -G_3.$$

The algebra is solvable.

Summing up the Lie series we obtain one-parameter subgroups of the symmetry group of transformations corresponding to the generators (11). For $G_1$ and $G_2$ we have translations in time and translations in space respectively. These symmetries follow from the fact, that coefficients of equation (9) do not depend on time and space variables, and lead to the conservation laws of energy and momentum respectively. For $G_3$ we have Galilean transformations. The above three kinetic symmetries are obvious as they express the geometric properties of space-time in nonrelativistic theory. The dynamical symmetries, which depend on details of interaction, are more interesting. In the case of the Vlasov–Maxwell equations they are generated by $G_4$ and $G_5$ and have the form of scaling transformations. We can construct a general symmetry transformation of the Vlasov–Maxwell equation (9) from the above one-parameter transformations.

Other approaches to the problem of finding symmetries of Vlasov–Maxwell equations can be found in papers [26, 27, 28] and in Chapter 16 of Vol.2 of CRC Handbook [8].

# 5 Conclusions

It has been shown that there is no need for a nonlocal extension of a symmetry group in the case of integro-differential equations. It is sufficient to stay in a jet space as in the case of differential equations. The generalization of the Ovsiannikov method consists in the change of the infinitesimal criterion of symmetry. The method has been successfully applied to significant integro-differential equations. In addition to the Vlasov–Maxwell equations we have also determined the symmetry group of the nonlocal NLS equation for modulated Langmuir waves in plasmas. In this last case a further generalization of the Ovsiannikov method to equations with delayed arguments is needed.

[1] Ovsiannikov L.V., Group analysis of differential equations, Boston, Academic Press, 1982.

[2] Ibragimov N.H., Transformation groups applied to mathematical physics, Dordrecht, Reidel, 1985.

[3] Olver P.J., Applications of Lie groups to differential equations, New York, Springer, 1986.

[4] Bluman G.W. and Kumei S., Symmetries and differential equations, New York, Springer, 1989.

[5] Stephani H., Differential equations. Their solutions using symmetries, Editor M. Mc Callum, Cambridge, Cambridge University Press, 1989.

[6] Zawistowski Z.J., On symmetries of integro-differential equations, *Bull. Polish Acad. Sci. Tech. Sci.*, 1998, V.46, N 2, 187–197.

[7] Fushchich V.I., Shtelen V.M. and Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993.

[8] Ibragimov N.H. (Editor), CRC handbook of Lie group analysis of differential equations, Boca Raton, Florida, CRC Press, Inc., Vol.1, 1994; Vol.2, 1995; Vol.3, 1996.

[9] Fushchich V.I., On a method of investigating the group properties of integro-differential equations, *Ukrain. Mat. Zh.*, 1981, V.33, N 6, 834–838 (in Russian).

[10] Selekhman N.A., Maximal algebra of symmetry of a system of integro-differential equations, in Algebra-Theoretic Investigations in Mathematical Physics, Kiev, Institute of Mathematics Akad. Nauk UkrSSR, 1981, 125–131 (in Russian).

[11] Taranov V.B., On symmetries of one-dimensional, high frequency movements in collisionless plasma, *ZTF*, 1976, V.46, N 6, 1271–1277 (in Russian).

[12] Harrison B.K. and Estabrook F.B., Geometric approach to invariance group and solution of partial differential system, *J. Math. Phys.*, 1971, V.12, N 4, 653–666.

[13] Fushchich V.I. and Selekhman N.A., Integro-differential equations that are invariant with respect to the Galilei, Poincaré, Schrödinger and conformal groups, *Dokl. Akad. Nauk Ukrain. SSR, ser. A*, 1983, N 5, 21–24 (in Ukrainian).

[14] Selekhman N.A., The group properties of the Boltzmann equation, in Algebra-Theoretic Investigations in Mathematical Physics, Kiev, Institute of Mathematics Akad. Nauk UkrSSR, 1983, 65–68 (in Russian).

[15] Stoigny V.I., The group properties of integro-differential equations of Hartree type, in Symmetry and Solutions of Equations of Mathematical Physics, Kiev, Institute of Mathematics Akad. Nauk UkrSSR, 1989, 82–83 (in Russian).

[16] Popovich R.E., Symmetry properties of an integro-differential of Hartree type, in Algebra-Theoretic Investigations in Mathematical Physics, Kiev, Institute of Mathematics Akad. Nauk UkrSSR, 1990, 50–53 (in Russian).

[17] Baumann G., Symmetry analysis of differential equations with *Mathematica*, New York, Springer, 2000.

[18] Grigoriev Yu.N. and Meleshko S.V., Group analysis of the Boltzmann integro-differential equation, *Dokl. Akad. Nauk SSSR*, 1987, V.297, N 2, 323–327 (in Russian).

[19] Grigoriev Yu.N. and Meleshko S.V., Group theoretical analysis of kinetic Boltzmann equation and its models, *Arch. Mech.*, 1990, V.42, N 6, 693–701.

[20] Vinogradov A.M. and Krasilshchik I.S., Nonlocal symmetries and the theory of coverings, *Acta Applicandae Mathematicae*, 1984, V.2, N 1, 79–96.

[21] Vinogradov A.M. and Krasilshchik I.S. (Editors), Symmetries and conservation laws for equations of mathematical physics, Moscow, Factorial, 1997 (in Russian); The English translation of the book: Translations of Mathematical Monographs, Vol.182, American Mathematical Society, Providence, Rhode Island, 1999.

[22] Chetverikov V.N. and Kudryavtsev A.G., A method for computing symmetries and conservation laws of integro-differential equations, *Acta Applicandae Mathematicae*, 1995, V.412, N 1, 45–56.

[23] Chetverikov V.N. and Kudryavtsev A.G., Modeling integro-differential equations and a method for computing their symmetries and conservation laws, in The Interplay Between Differential Geometry and Differential Equations, Editor V.V. Lychagin, Providence, Rhode Island, Amer. Math. Soc., *Amer. Math. Soc. Transl.*, Ser.2, 1995, V.167, 1–22.

[24] Gelfand I.M. and Fomin S.W., Calculus of variations, Englewood Cliff, Prentice-Hall, 1962.

[25] Glassey R.T., The Cauchy problem in kinetic theory, Philadelphia, SIAM, 1992.

[26] Kovalev V.N., Krivenko S.V. and Pustovalov V.V., Symmetry group of kinetic equation for collisionless plasma, *Pisma Zh. Exper. Teoret. Fiz*, 1992, V.55, N 4, 256–259 (in Russian); English transl. in *JEPT Lett.*, 1992.

[27] Kovalev V.N., Krivenko S.V. and Pustovalov V.V., Symmetry group of Vlasov–Maxwell equations in plasma theory, *J. Nonlin. Math. Phys.*, 1996, V.3, N 1–2, 175–180.

[28] Markov Yu.A., Rudykh G.A., Sidorov N.A. and Sinitsyn A.V., A family of solutions of the Vlasov–Maxwell system and their stability, *Mat. Model.*, 1990, V.2, N 12, 88–101 (in Russian).