Brane Evolution -
Y Nambu Mechanics

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Why *not* Nambu mechanics?

The organization of the talk is as follows.

- Brane evolution as motivation
- Quantum Nambu brackets: Even and Odd
- Odd brackets, especially 3
- Bracket reductions: $4 \Rightarrow 3$
- Bracket equivalences
- Solenoidal flow
- Parameterization and interpretation
- Propagators

Classical and quantum Nambu mechanics are discussed in
T Curtright and C Zachos, Phys Rev **D68** (2003) 085001.
This paper provides an extensive guide to the literature.

[1]Back in Miami, we write that as: Brana Evolución Y Nambu Mecánica

1 Brane motivator

Suppose

$$dz_j = v_j\left[\boldsymbol{z}\right] d\tau \ .$$

What about

$$\boldsymbol{z}\left(\tau, \sigma_1, \sigma_2, \cdots, \sigma_d\right) \ ?$$

Nambu mechanics suggests an interesting answer. If

$$v_j\left[\boldsymbol{z}\right] = \varepsilon_{j_1 \cdots j_n} \frac{\partial}{\partial z_{j_1}} z_j \frac{\partial}{\partial z_{j_2}} I_1\left[\boldsymbol{z}\right] \cdots \frac{\partial}{\partial z_{j_n}} I_{n-1}\left[\boldsymbol{z}\right]$$

$$\equiv \left\{z_j, I_1\left[\boldsymbol{z}\right], \cdots, I_{n-1}\left[\boldsymbol{z}\right]\right\}_{\mathrm{NB}}$$

then

$$\mathbb{A} = \int dz_1 \wedge \cdots \wedge dz_n - d\tau \wedge dI_1 \wedge \cdots \wedge dI_{n-1} \ .$$

That is,

$$\Omega = \left(dz_1 - v_1\left[\boldsymbol{z}\right] d\tau\right) \wedge \cdots \wedge \left(dz_n - v_n\left[\boldsymbol{z}\right] d\tau\right)$$

$$= dz_1 \wedge \cdots \wedge dz_n - d\tau \wedge dI_1 \wedge \cdots \wedge dI_{n-1}$$

and

$$\mathbb{A} = \int_{M_n} \Omega$$

$$= \int_{\substack{\partial M_n \\ \text{world volume} \\ \text{for } d=n-2 \text{ brane}}} \left[\begin{array}{c} \frac{1}{n}\, \varepsilon^{j_1 \cdots j_n}\ z_{j_1}\ \partial_\tau z_{j_2}\ \partial_{\sigma_1} z_{j_3} \cdots\ \partial_{\sigma_{n-2}} z_{j_n} \\ + \frac{1}{n-1}\, \varepsilon^{k_1 \cdots k_{n-1}}\ I_{k_1}\ \partial_{\sigma_1} I_{k_2} \cdots\ \partial_{\sigma_{n-2}} I_{k_{n-1}} \end{array} \right] d\tau d\sigma_1 \cdots d\sigma_{n-2} \ .$$

The “d-brane” of initial data is evolved as a whole by this action.

Note that Nambu flow is always solenoidal[2], as is the more familiar Hamiltonian flow. That is,

$$\frac{\partial}{\partial z_j} v_j[\boldsymbol{z}] = \frac{\partial}{\partial z_j} \{z_j, I_1[\boldsymbol{z}], \cdots, I_{n-1}[\boldsymbol{z}]\}_{\mathrm{NB}} = 0$$

But, Nambu flow is not necessarily Hamiltonian. (More on this later.)

As the simplest possible example, consider phase-space for a unit mass free particle on the plane.

Time evolution of this system can be expressed either as Hamiltonian flow, or as Nambu flow.

$$z_j \in \; x, p_x, y, p_y \qquad \text{so} \qquad n = 4.$$

$$\frac{d}{dt} z_j = v_j = \{z_j, H\}_{\mathrm{PB}}$$

$$H = \left(p_x^2 + p_y^2\right)/2$$

$$v_j = \frac{\partial\left(z_j, L, p_x, p_y\right)}{\partial\left(x, p_x, y, p_y\right)}$$

$$L = x p_y - y p_x$$

Then the previous action would describe the evolution of a 2-surface or "membrane" of *phase-space* data for this system.

Other classical bracket systems are nearly as transparent. Although in general, Nambu brackets set different time scales on different dynamical sectors of a theory.

[2] A necessary condition for Ω to be exact is that it be closed. This implies the flow should be solenoidal, $\nabla \cdot v = 0$, as $\varepsilon_{i_1 \cdots i_n} dv^{i_1}[\boldsymbol{z}] \wedge dz^{i_2} \cdots \wedge dz^{i_n} = \varepsilon_{i_1 \cdots i_n} \partial_j v^{i_1} dz^j \wedge dz^{i_2} \cdots \wedge dz^{i_n} = \frac{1}{n}\left(\partial_j v^j\right) \varepsilon_{i_1 \cdots i_n} dz^{i_1} \wedge dz^{i_2} \cdots \wedge dz^{i_n}$.

As an illustration of dynamical time scales set through Nambu brackets, consider the two dimensional isotropic oscillator.

The $su\,(2)$ algebra of the oscillator (as in Schwinger) may be realized by

$$J_x = \tfrac{1}{2}\left(p_x p_y + xy\right)$$

$$J_y = \tfrac{1}{4}\left(p_y^2 + y^2 - p_x^2 - x^2\right)$$

$$J_z = \tfrac{1}{2}\left(x p_y - y p_x\right)$$

Each of these is an invariant under time-evolution as generated by (we have set $m = 1/\omega$)

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + x^2 + y^2\right)$$

and they satisfy the usual Poisson brackets

$$\{J_x, J_y\}_{\mathrm{PB}} = J_z\ , \quad \{J_y, J_z\}_{\mathrm{PB}} = J_x\ , \quad \{J_z, J_x\}_{\mathrm{PB}} = J_y$$

Moreover, the *square* of the 2d SHO Hamiltonian is given by the quadratic Casimir

$$J_x^2 + J_y^2 + J_z^2 = \tfrac{1}{16}\left(p_x^2 + p_y^2 + x^2 + y^2\right)^2 = \tfrac{1}{4} m^2 H^2$$

Now there is a simple $su\,(2)$ bracket identity relating Poisson brackets to Nambu 4-brackets. It follows from nothing but the PB algebra. For any function $A\,(x, y, p_x, p_y)$ on the phase-space of the oscillator

$$\left\{A, J_x^2 + J_y^2 + J_z^2\right\}_{\text{PB}} = 2 \times \{A, J_x, J_y, J_z\}_{\text{NB}}$$

Thus evolution through use of this particular 4-bracket corresponds to using the square of the 2d SHO Hamiltonian.

Evolution using the Hamiltonian to the first power apparently can *not* be expressed in terms of quantum 4-brackets by using linear combinations of the J's. The best we can do is

$$z_j \in \; x, p_x, y, p_y \qquad \text{so again} \qquad n = 4.$$

$$v_j = \frac{\partial\,(z_j, J_x, J_y, J_z)}{\partial\,(x, p_x, y, p_y)}$$

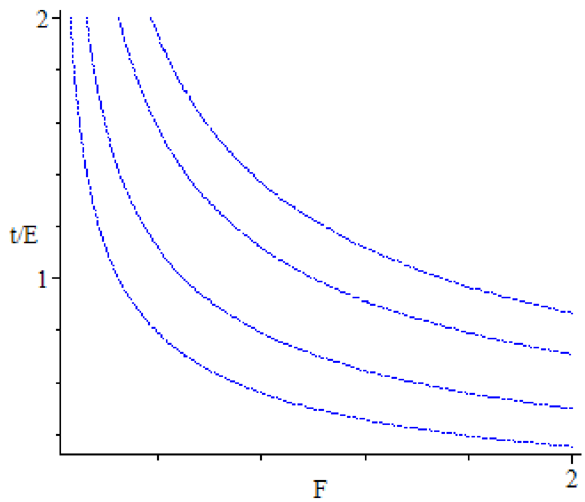
$$\frac{d}{d\tau} z_j = v_j = \tfrac{1}{8} m^2 \left\{z_j, H^2\right\}_{\text{PB}}$$

$$= \tfrac{1}{4} m^2 H \left\{z_j, H\right\}_{\text{PB}}$$

$$= \tfrac{1}{4} m^2 H \left(\frac{d}{dt} z_j\right)$$

Roughly speaking, for the 2d SHO it is not a simple time derivative that nicely fits into a Nambu multi-bracket formalism, but rather it is the derivative $\partial/\partial(t/E)$ for fixed energy levels. This can be written as a 4-bracket. That is to say, more precisely, we need to re-parameterize the time-energy plane by changing coordinates $(t, E) \rightarrow (\tau \equiv t/E, F \equiv E^2/2)$. This preserves areas on the time-energy plane (and therefore maintains the physics associated with time-energy distributions) since

$$\left\{t/E, E^2/2\right\}_{t,E} = 1$$



Constant time curves on the $\tau = t/E$, $F = E^2/2$ plane.

More importantly, this parameterization of the time-energy plane dictates that different energy levels evolve according to their own distinct energy-dependent times, as specified by τ. Since the relation between τ and t is invertible, for $E \neq 0$, this is not pathological.

Evolution of SHO extended data and dynamical time scales.

We set $m = 1/2$ and rescale $\tau \to 16\tau$ to clean up the numerics, to obtain

$$\frac{d}{d\tau} z = \frac{1}{2} \left\{ z, H^2 \right\}_{\text{PB}} = H \left\{ z, H \right\}_{\text{PB}} \ ,$$

for $z = (x, y, p_x, p_y)$. A representative solution is given by

$$x\left(E, \tau\right) = \sqrt{E} \left(\cos\left(E\tau\right) + \sin\left(E\tau\right)\right) \ ,$$

$$p_x\left(E, \tau\right) = \sqrt{E} \left(-\sin\left(E\tau\right) + \cos\left(E\tau\right)\right) \ ,$$

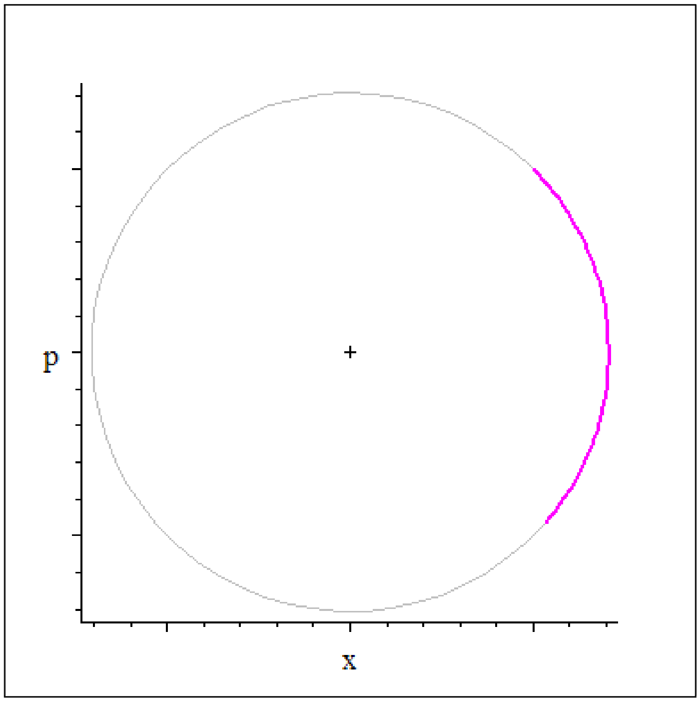
as opposed to the usual fixed frequency, standard time evolution:

$$x\left(E, t\right) = \sqrt{E} \left(\cos\left(t\right) + \sin\left(t\right)\right) \ ,$$

$$p_x\left(E, t\right) = \sqrt{E} \left(-\sin\left(t\right) + \cos\left(t\right)\right) \ .$$

The equivalence map between the single point particle solutions,
as given by the relation $\tau = t/E$, is evident.

The geometry of a single phase-space trajectory evolved under τ is indistinguishable from that of a single trajectory evolved under standard time t. Only the parameterization scale of the curve is different.

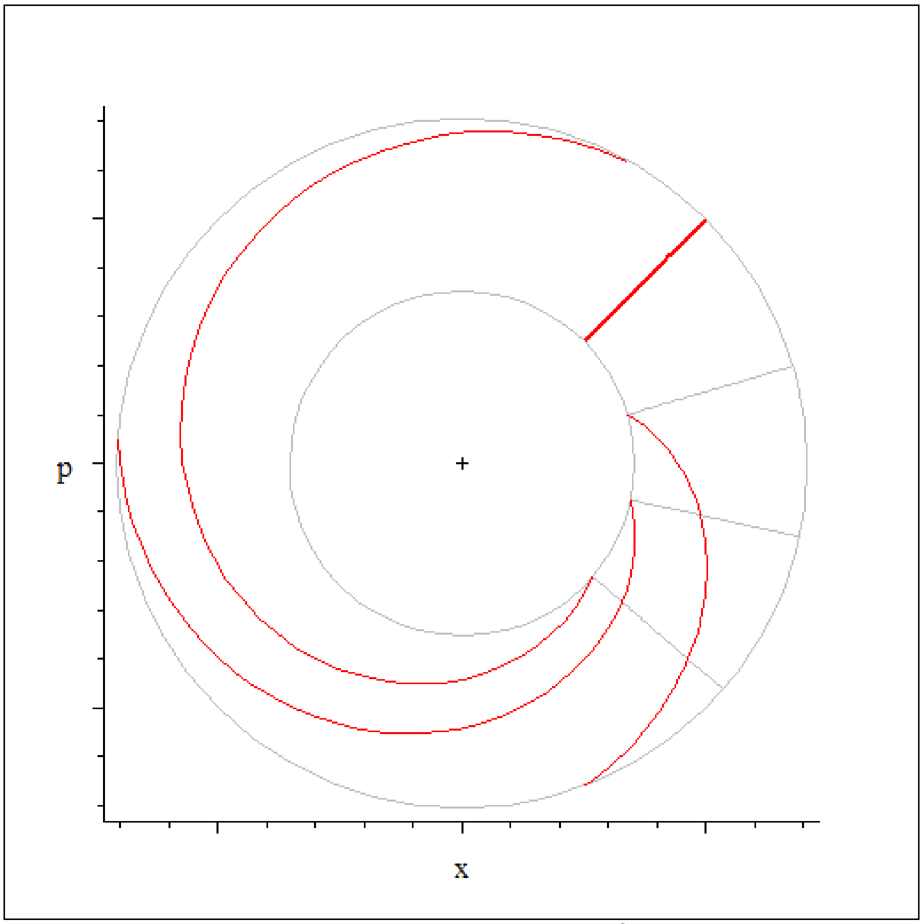


Evolution of a single phase-space point under the non-trivial action of any differentiable function of H.

To see the difference between the two forms of evolution, geometrically, we must compare two or more trajectories.

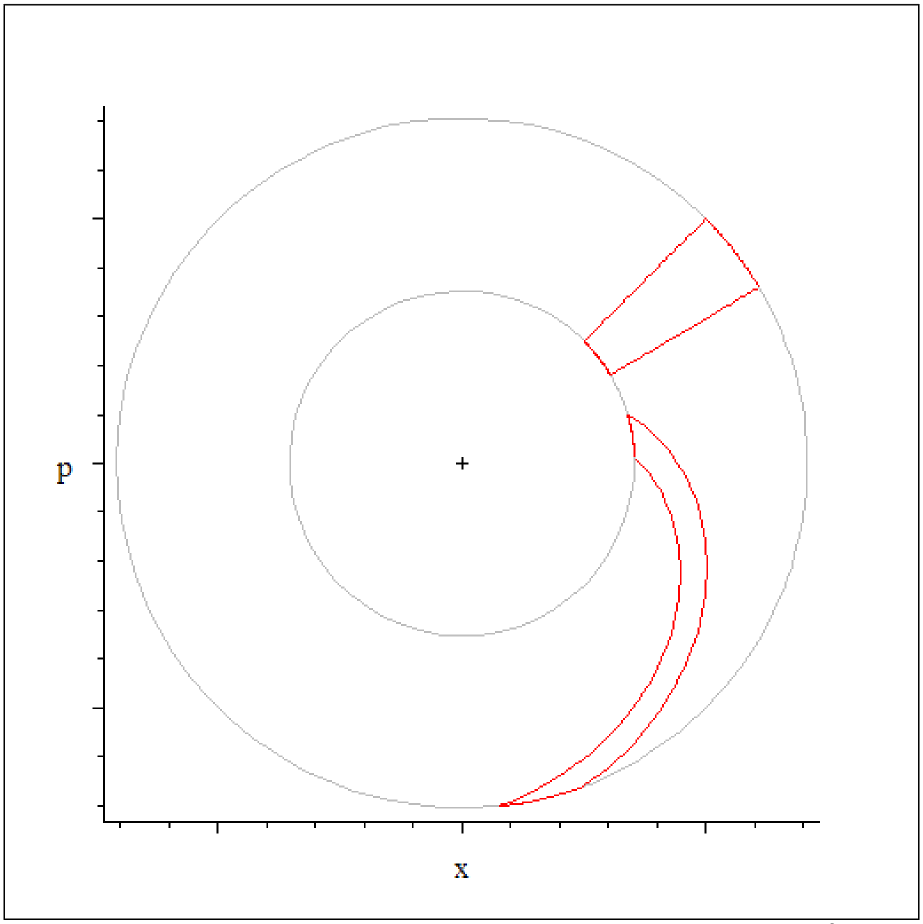
Extended data involving different energies evolve continuously under τ in an amusing way, as indicated in the pictures below.

That is to say, we need to consider the evolution of extended initial data in the phase-space, i.e. a *brane of data*. So consider a straight line segment of initial (x, p_x) points, and evolve the entire segment under the action of H^2. For three $\Delta\tau = 0.5$ steps, we have the following.



Evolution of extended data under action of H^2 ($\tau = 0.0, 0.5, 1.0, 1.5$).

The shearing of the data is evident. Nonetheless, the phase-space flow is still that of an incompressible fluid (Liouville's theorem still obtains).



Phase-space volume is preserved under the action of H^2.

2 Quantum brackets

We define quantum Nambu brackets (QNBs) as one of the possibilities originally suggested by Nambu.

$$[A_1, A_2, \cdots, A_k] \equiv \sum_{\substack{\text{all } k! \text{ perms } \{\sigma_1,\sigma_2,\cdots,\sigma_k\} \\ \text{of the indices } \{1,2,\cdots,k\}}} \operatorname{sgn}(\sigma) \quad A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_k} \ ,$$

Alternatively, recursively, there are left- and right-sided resolutions.

$$[A_1, \cdots, A_k] = A_1 [A_2, \cdots, A_k] + \text{ signed permutations}$$

$$= [A_1, \cdots, A_{k-1}] A_k + \text{ signed permutations}$$

Some of the most interesting features of QNBs center around their *not* being derivations, as easily follows from these resolutions. That is

$$[A_1, \cdots, A_k, \ BC\ ] - B \ [A_1, \cdots, A_k, C] - [A_1, \cdots, A_k, B] \ C$$

$$\neq 0$$

in general. The left-hand side (up to $\pm$) is the "derivator" $\Delta_{A_1,\cdots,A_k}(B, C)$.

The bracket for fixed $A_1, \cdots, A_k$ is a *derivation* iff $\Delta_{A_1,\cdots,A_k}(B, C) = 0$ for all B, C.

Even brackets are even better as they also admit *commutator resolutions.*

$$[A_1, \cdots, A_{2n}] = [A_1, A_2]\,[A_3, A_4]\ \cdots\ [A_{2n-1}, A_{2n}] \\ +\ \text{permutations}$$

So, even brackets have direct classical limits as a consequence of the fact that commutators $\Longrightarrow$ Poisson brackets. This we like.

$$\frac{1}{n!}\lim_{\hbar\to 0}\left(\frac{1}{i\hbar}\right)^n [A_1, A_2, \cdots, A_{2n}] = \{A_1, A_2\}_{\text{PB}}\,\{A_3, A_4\}_{\text{PB}}\ \cdots\ \{A_{2n-1}, A_{2n}\}_{\text{PB}} \\ +\ \text{permutations} \\ = \{A_1, A_2, \cdots, A_{2n}\}_{\text{NB}}\ .$$

Odd quantum brackets, on the other hand, do *not* become just classical odd brackets as $\hbar \to 0$, due to a mismatch in the number of derivatives: a $2n+1$ quantum bracket will only yield $2n$ derivatives in the classical limit, not $2n+1$ derivatives.

K Bering points out that the combination of Poisson brackets appearing in the classical Nambu bracket has another name:

$$\text{Pfaffian}\left(\{A_i, A_j\}_{\text{PB}}\right) = \sqrt{\det\left(\{A_i, A_j\}_{\text{PB}}\right)}\ .$$

So the above definition of classical Nambu brackets is actually just

$$\{A_1, A_2, \cdots, A_{2n}\}_{\text{NB}} = \text{Pfaffian}\left(\{A_i, A_j\}_{\text{PB}}\right)\ .$$

This may be a useful connection to make, say if it can help to establish all those cases in which the Nambu bracket reduces to a (sum of) single Poisson brackets, as in the $su(2)$ case mentioned earlier, or to establish other such relationships.

Moreover, in a well-defined sense, the commutator resolution above is a *quantum Pfaffian*, i.e. a precise definition when the entries in the matrix do not commute.

Brackets from fermionic Gaussians The quantum definition of the Nambu bracket that we have chosen, and hence the quantum Pfaffian, is naturally expressed using fermionic integrals for non-commuting Gaussians.

$$\int \exp\left(-\frac{1}{i\hbar}\sum_{i<j=1}^{2n}\theta_i\left[A_i,A_j\right]\theta_j\right)\ d\theta_1\cdots d\theta_{2n}=\frac{1}{n!\,(i\hbar)^n}\left[A_1,\cdots,A_{2n}\right]\ ,$$

where as usual $\{\theta_i,\theta_j\}=0$ (actually, a Grassmanian δ_{ij} does not seem to do any harm here), $\int d\theta_i=0$, and the normalization is $\int\theta_i d\theta_j=\delta_{ij}$. Alternatively, since $\int A(\theta)\,d\theta=\frac{\partial}{\partial\theta}A(\theta)$, we may write this as

$$\frac{\partial}{\partial\theta_1}\cdots\frac{\partial}{\partial\theta_{2n}}\exp\left(-\frac{1}{i\hbar}\sum_{i<j=1}^{2n}\theta_i\left[A_i,A_j\right]\theta_j\right)=\frac{1}{n!\,(i\hbar)^n}\left[A_1,\cdots,A_{2n}\right]\ .$$

Either of these fermionic representations have the correct classical limit, rather transparently, in which limit they become well-known fermionic expressions for $\sqrt{\det}$.

$$\int \exp\left(-\sum_{i<j=1}^{2n}\theta_i\left\{A_i,A_j\right\}_{\mathrm{PB}}\theta_j\right)\ d\theta_1\cdots d\theta_{2n}=\left\{A_1,\cdots,A_{2n}\right\}_{\mathrm{NB}}$$

Also, the integral representation is perhaps the easiest way to take the $n\to\infty$ limit, such as would be expected to appear in a field theory framework. The sums in the exponential become integrals, and the θ integrations meld into a fermionic functional integral.

$$\int\exp\left(-\frac{1}{2i\hbar}\int d\alpha\int d\beta\ \theta(\alpha)\left[A(\alpha),A(\beta)\right]\theta(\beta)\right)\ \mathcal{D}\theta$$

Also in this limit, $\int\theta(\alpha)\,d\theta(\beta)=\delta(\alpha-\beta)$. Evaluation and *application* of this fermionic functional integral remains to be carried out, however.

There is also the *red herring* known as the "fundamental identity" or FI. By straightforward combinatorics, we find that

$$
\begin{aligned}
&\left[\left[A_1,\cdots,A_n\right],B_1,B_2,\cdots,B_k\right]-\sum_{j=1}^{n}\left[A_1,\cdots,\left[A_j,B_1,B_2,\cdots,B_k\right],\cdots,A_n\right]\\
&=\sum_{n!\text{ perms }\sigma}\operatorname{sgn}(\sigma)\,\Big(\frac{1}{(n-1)!}\Delta_{B_1,B_2,\cdots,B_k}\left(A_{\sigma_1},\left[A_{\sigma_2},\cdots,A_{\sigma_n}\right]\right)\\
&+\frac{1}{(n-2)!}A_{\sigma_1}\Delta_{B_1,B_2,\cdots,B_k}\left(A_{\sigma_2},\left[A_{\sigma_3},\cdots,A_{\sigma_n}\right]\right)\\
&+\frac{1}{2!\,(n-3)!}\left[A_{\sigma_1},A_{\sigma_2}\right]\Delta_{B_1,B_2,\cdots,B_k}\left(A_{\sigma_3},\left[A_{\sigma_4},\cdots,A_{\sigma_n}\right]\right)+\cdots\\
&+\frac{1}{(n-2)!}\left[A_{\sigma_1},A_{\sigma_2},\cdots,A_{\sigma_{n-2}}\right]\Delta_{B_1,B_2,\cdots,B_k}\left(A_{\sigma_{n-1}},A_{\sigma_n}\right)\Big)\ ,
\end{aligned}
$$

where $A_1,\cdots,A_n$ are any n operators, and where $B_1,B_2,\cdots,B_k$ are any k operators.

The RHS vanishes when $\Delta_{B_1,B_2,\cdots,B_k}=0$, i.e. when the $(k+1)$-bracket with k fixed Bs is a derivation. Thus when $\Delta_{B_1,B_2,\cdots,B_k}=0$ we have the rather obvious subsidiary identity, the FI.

$$
\left[\left[A_1,\cdots,A_n\right],B_1,B_2,\cdots,B_k\right]=\sum_{j=1}^{n}\left[A_1,\cdots,\left[A_j,B_1,B_2,\cdots,B_k\right],\cdots,A_n\right]\ .
$$

From the point of view of the operator brackets as we have defined them, there does not seem to be any additional content in this identity beyond the fact that it holds for derivations.

When the $B_1,\cdots,B_k$ bracket is *not* a derivation, the FI is *not* guaranteed to hold.

For the appropriate identity that stems from the underlying associativity of the operator algebra, and which holds even when the action of a bracket is not a derivation, see J A de Azcárraga, et al., or P Hanlon and M Wachs, as cited in T Curtright and C Zachos, PRD**68** (2003) 085001.

Although not always the case, sometimes QNBs are derivations by virtue of algebra of the As.

For example, consider a free particle on the plane, again.

$$H = \left(p_x^2 + p_y^2\right)/2 \ , \quad L = xp_y - yp_x$$

$$\begin{aligned} [A, L, p_x, p_y] &= \{[A, p_y], [L, p_x]\} - \{[A, p_x], [L, p_y]\} \\ &= i\hbar \{[A, p_y], p_y\} + i\hbar \{[A, p_x], p_x\} \\ &= 2i\hbar [A, H] \end{aligned}$$

This particular example is therefore a derivation.

$$2 (i\hbar)^2 \frac{d}{dt} A (x, p_x, y, p_y) = [A, L, p_x, p_y]$$

and serves as the simplest example of consistently quantized Nambu mechanics. In this special case, the quantum brackets have the same algebraic structure as their classical limits.

3 Odd QNBs are odd

Consider quantum 3-brackets.

$$[A, B, C] = [A, B] \, C + [C, A] \, B + [B, C] \, A$$

There is an "identity crisis" for the 3-bracket.

$$[\mathbf{1}, B, C] = [B, C] \neq 0 \quad !!!$$

Smells like trouble. But things are not so bad for commuting B and C.

However, even without an identity crisis, odd QNBs can be really odd. This stems from the fact that conventional, unconstrained phase-spaces are always even-dimensional, and not odd.

For example, consider once more a free particle on the plane but now using the 3-bracket $[A, p_x, p_y]$.

Does this give something proportional to the time-derivative?
No. Not exactly ...

$$[x, p_x, p_y] = [x, p_x] \, p_y = i\hbar p_y$$

$$[y, p_x, p_y] = - [y, p_y] \, p_x = -i\hbar p_x$$

so

$$i\hbar \frac{d}{d\tau} \boldsymbol{x} = [\boldsymbol{x}, p_x, p_y] = i\hbar \, \widetilde{\boldsymbol{p}} = i\hbar \frac{d}{dt} \, \widetilde{\boldsymbol{x}}$$

This is time evolution accompanied by a $\pi/2$ rotation in the phase-space, as designated by the tilde.

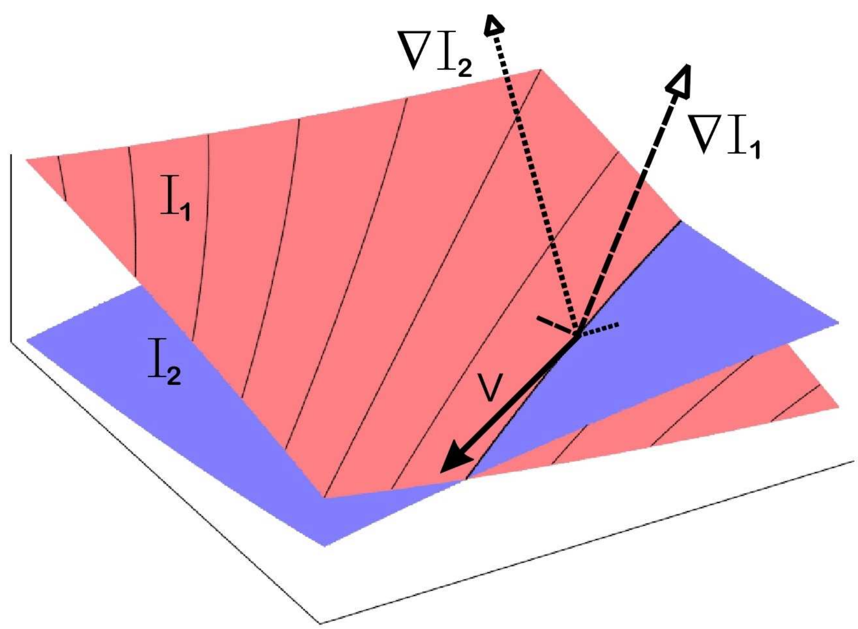
$$\widetilde{\boldsymbol{p}} = (p_y, -p_x)$$

So that

$$\boldsymbol{x}\,(\tau) = \boldsymbol{x}\,(0) + \widetilde{\boldsymbol{p}} \; \tau$$

This sort of behavior can occur whenever the number of entries is less than the dimension of the system phase-space, as always happens for odd brackets when applied to conventional $\boldsymbol{x}, \boldsymbol{p}$ phase-space.

Aside: This is also true classically.



For a system with N degrees of freedom, hence $2N$-dimensional phase-space,

$$\mathsf{v} \propto dI_1 \wedge \cdots \wedge dI_{n-1}$$

is guaranteed only when

$$n = 2N \ .$$

As further, very general illustration of quantum 3-brackets without an identity crisis, consider the quantum mechanics of any axially symmetric system, $[H, L] = 0$ with Hamiltonian H, and with L generating rotations about the axis of symmetry, for example $L = L_z$.

For such H, L, and any other operator A, the quantum 3-bracket simplifies to

$$[A, H, L] = LAH - HAL \ .$$

Another way to write this is as the sum of two commutators.

$$[A, H, L] = [\{A, L\}, H] + [HL, A] \ .$$

Nevertheless, this 3-bracket is still *not* a derivation. For general A and B,

$$\Delta_{H,L}(A, B) \equiv [AB, H, L] - A [B, H, L] - [A, H, L] B$$

$$= [A, H] [B, L] - [A, L] [B, H] \ .$$

The second line is valid only because $[H, L] = 0$.

Note that derivators are neither symmetric nor antisymmetric under $A \leftrightarrow B$. For example, if $A = B$, then $\Delta_{H,L}(A, A) = [[A, H], [A, L]] \neq 0$ for generic A.

The structure and effects of this quantum 3-bracket can be understood using *projections*. Since the operators L and H commute, we may resolve the identity in terms of projection operators which are simultaneous eigenoperators of both L and H.

$$\mathbf{1} = \sum_{\lambda,\omega} \mathbb{P}_{\lambda\omega} \ , \quad \mathbb{P}_{\lambda\omega}\delta_{\lambda\lambda'}\delta_{\omega\omega'} = \mathbb{P}_{\lambda\omega}\mathbb{P}_{\lambda'\omega'} \ ,$$

$$L\mathbb{P}_{\lambda\omega} = \hbar\lambda \ \mathbb{P}_{\lambda\omega} = \mathbb{P}_{\lambda\omega}L \ , \quad H\mathbb{P}_{\lambda\omega} = \hbar\omega \ \mathbb{P}_{\lambda\omega} = \mathbb{P}_{\lambda\omega}H \ .$$

The last line implies $[L, \mathbb{P}_{\lambda\omega}] = 0 = [H, \mathbb{P}_{\lambda\omega}]$, of course.

Therefore, any operator A may be written as a sum of simultaneous left- and right-eigenoperators of L and H.

$$A = \sum_{\lambda_l,\omega_l,\lambda_r,\omega_r} \mathbb{P}_{\lambda_l\omega_l} \, A \, \mathbb{P}_{\lambda_r\omega_r} = \sum_{\lambda_l,\omega_l,\lambda_r,\omega_r} A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \ ,$$

$$A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \equiv \mathbb{P}_{\lambda_l\omega_l} \, A \, \mathbb{P}_{\lambda_r\omega_r} \ ,$$

$$LA_{\lambda_l,\omega_l,\lambda_r,\omega_r} = \hbar\lambda_l \ A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \ , \quad A_{\lambda_l,\omega_l,\lambda_r,\omega_r}L = \hbar\lambda_r \ A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \ ,$$

$$HA_{\lambda_l,\omega_l,\lambda_r,\omega_r} = \hbar\omega_l \ A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \ , \quad A_{\lambda_l,\omega_l,\lambda_r,\omega_r}H = \hbar\omega_r \ A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \ .$$

On such individual eigenoperators, the action of the 3-bracket $[A, H, L] = LAH - HAL$ reduces to an interesting eigenvalue equation, where the eigenvalue is an element of "area" on the quantum LH-plane.

$$[A_{\lambda_l,\omega_l,\lambda_r,\omega_r}, H, L] = \hbar^2 \left(\lambda_l\omega_r - \omega_l\lambda_r\right) \ A_{\lambda_l,\omega_l,\lambda_r,\omega_r} \ .$$

This (signed) area is given by the planar cross-product of the pairs of left- and right- L and H eigenvalues.

$$\left(\lambda_l\omega_r - \omega_l\lambda_r\right) = \left(\lambda_l, \omega_l\right) \wedge \left(\lambda_r, \omega_r\right) \ .$$

But obviously this is *not* conventional time evolution. Once again, the bracket combines time evolution with some rotation effects.

In particular, when exponentiated, this bracket does not yield the usual time-dependent phase. In fact, finite evolution generated by the 3-bracket $[A, H, L]$ will *not* be a unitary nor even a similarity transformation. More on this later.

4 Bracket reduction - classical versus quantum

Perhaps an alternate way of obtaining odd brackets would be through reduction from higher even brackets. That is a logical possibility, but there are some subtleties in the quantum case.

Classically, on the x, p_x, y, p_y phase-space,

$$\{A, B, C, p_y\}_{\text{NB}} = \frac{\partial\,(A, B, C)}{\partial\,(x, p_x, y)}$$

is a bona fide classical 3-bracket. The RHS involves partial derivatives only with respect to the three variables x, p_x, y. Any p_y dependence in A, B, C goes along for the ride.

But QMly things are not so simple. From the commutator resolution, in general,

$$[A, B, C, p_y] = i\hbar\ \{[A, B], \partial_y C\} + i\hbar\ \{[C, A], \partial_y B\} + i\hbar\ \{[B, C], \partial_y A\}$$

$$\neq \text{constant}\ \times [A, B, C]$$

(Incidentally, the first line of this equation fully exhibits the relevant structure in the case that the underlying phase-space is higher than 4-dimensional.)

Now, the first line in this last equation could be used to just define a *new* quantum three-bracket which in general will differ from the previous operator definition of $[A, B, C]$. And, indeed, the RHS of the first line *does* have the correct classical limit. However, if the classical limit is not taken, then there are vestigial quantum effects of p_y that may be understood as resulting from higher p_y derivatives implicit in the quantum commutators and anticommutators. We next look at these vestigial effects in more detail.

Since any reasonable distribution on the phase-space can be obtained from linear exponentials,

$$\exp\left(iax + ibp_x + icy + idp_y\right) \ ,$$

either through partial derivatives with respect to the parameters, a, b, c, and d, or else through Fourier/Laplace transforms, it suffices to replace one of A, B, or C, with such an exponential, say A. Then compute

$$\left[\exp\left(iax + ibp_x + icy + idp_y\right), B, C, p_y\right]$$

$$= i\hbar \left\{\left[\exp\left(iax + ibp_x + icy + idp_y\right), B\right], \partial_y C\right\}$$

$$- \hbar c \left\{\left[B, C\right], \exp\left(iax + ibp_x + icy + idp_y\right)\right\}$$

$$+ i\hbar \left\{\left[C, \exp\left(iax + ibp_x + icy + idp_y\right)\right], \partial_y B\right\} \ .$$

Let's take an explicit example so that things don't get out of hand. Let

$$H = \frac{1}{2}\left(p_x^2 + x^2\right) + p_y \ .$$

This sets up y to act merely as a linear measure of time, since $[y, H] = i\hbar$ gives just $y\,(t) = y\,(0) + t$. Another Hamiltonian invariant is then easy to find from the oscillator behavior of the remaining variables. Take

$$I = (p_x + ix)\exp\left(-iy\right) \ .$$

Then we have

$$[H, I] = 0 = [H, p_y] \ , \quad [I, p_y] = i\hbar\ \partial_y I = \hbar\ I$$

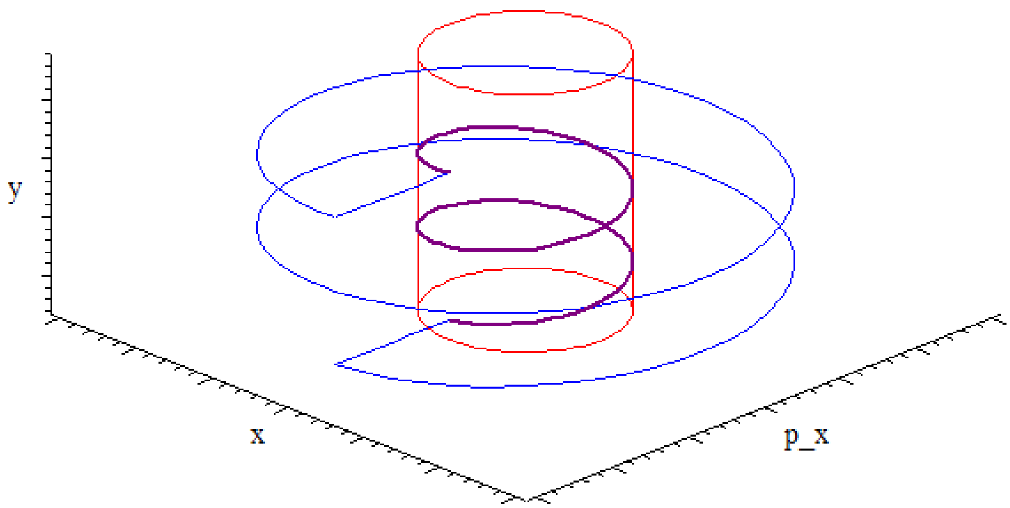
and

$$[A, H, I, p_y] = \hbar\ \{[A, H]\,, I\} = \hbar\ [\{A, I\}\,, H]$$

$$\neq \text{constant} \ \times [A, H, I]$$

This quantum 4-bracket is *not* proportional to the originally defined quantum 3-bracket, in general.

The classical phase-space picture for this example is given by



A right circular cylinder of constant H intersects a helical inclined plane of constant $\mathrm{Re}\, I$ to define a trajectory.

Operator products are equivalent to star products of appropriate classical functions on the phase-space, through use of the Weyl correspondence. (See our forthcoming book. Or better yet, read our papers on hep-th.)

$$\star \ = \exp \frac{i\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p_x} - \frac{\overleftarrow{\partial}}{\partial p_x} \frac{\overrightarrow{\partial}}{\partial x} + \frac{\overleftarrow{\partial}}{\partial y} \frac{\overrightarrow{\partial}}{\partial p_y} - \frac{\overleftarrow{\partial}}{\partial p_y} \frac{\overrightarrow{\partial}}{\partial y} \right)$$

These star products allow immediate comparison to classical expressions, since they can be easily reduced to ordinary products, for the case at hand.

Expressing things in terms of ordinary products, the calculation of interest then ultimately reduces to

$$\begin{aligned}
&[\exp\left(iax + ibp_x + icy + idp_y\right), H, I, p_y]_\star \\
&\qquad = \left(2\hbar I \cos\left(\frac{\hbar d}{2}\right) + i\hbar^2 \left(a - ib\right) e^{-iy} \sin\left(\frac{\hbar d}{2}\right) \right) \times \\
&\qquad \times \hbar \left(-ap_x + bx - c\right) \exp\left(iax + ibp_x + icy + idp_y\right) \\
&\qquad\qquad + i\hbar^3 \left(a - ib\right) e^{-iy} \sin\left(\frac{\hbar d}{2}\right) \exp\left(iax + ibp_x + icy + idp_y\right)
\end{aligned}$$

where $H = \frac{1}{2}\left(p_x^2 + x^2\right) + p_y = p_y + \frac{1}{2}\left(p_x + ix\right) \star \left(p_x - ix\right) + \frac{1}{2}\hbar$ and

$$[\exp\left(iax + ibp_x + icy + idp_y\right), H]_\star = \hbar\left(-ap_x + bx - c\right) \exp\left(iax + ibp_x + icy + idp_y\right)$$

The previously mentioned vestigial p_y effects are clearly evident in the above.

It actually suffices to illustrate the point just to consider $\exp(icy + idp_y)$, i.e. to set $a = 0 = b$ in the previous exponential. Then we have

$$\frac{1}{2(i\hbar)^2}\left[\exp(icy + idp_y), H, I, p_y\right]_\star = c\ I\ \cos\left(\frac{\hbar d}{2}\right)\ \exp(icy + idp_y)$$

to be compared with the classical result

$$\{\exp(icy + idp_y), H, I, p_y\}_{\rm NB} = \frac{\partial(\exp(icy + idp_y), H, I)}{\partial(y, x, p_x)}$$

$$= c\ I\ \exp(icy + idp_y)\ .$$

That is to say,

$$\frac{1}{2(i\hbar)^2}\left[\exp(icy + idp_y), H, I, p_y\right]_\star = \cos\left(\frac{\hbar d}{2}\right)\{\exp(icy + idp_y), H, I, p_y\}_{\rm NB}\ .$$

The prefactor on the RHS exhibits the result of higher order p_y partials, with a manifest classical limit.

On the other hand, we also have

$$\left[\exp(icy + idp_y), H, I\right]_\star = -\hbar\cos\left(\frac{\hbar d}{2}\right)\{\exp(icy + idp_y), H, I, p_y\}_{\rm NB}$$

$$+ 2i\sin\left(\frac{\hbar d}{2}\right)\ H\ I\ \exp(icy + idp_y)$$

Were there only the first line on the RHS of this last result, we would conclude that $\left[\exp(icy + idp_y), H, I, p_y\right]_\star \propto \left[\exp(icy + idp_y), H, I\right]_\star$. But, alas, there is also the second line and therefore $\left[\exp(icy + idp_y), H, I, p_y\right]_\star$ is *not* proportional to $\left[\exp(icy + idp_y), H, I\right]_\star$, except when $d = 0 \bmod (2\pi/\hbar)$.

5 Bracket equivalence classes

Even though

$$[A, B, C, p_y] \neq \text{constant} \ \times \ [A, B, C]$$

as an operator statement, nevertheless the effects of the left- and right-hand sides can be equivalent.

Lemma ("4 = 3 + 2") If

$$[H, J] = 0 = [H, K] \ , \quad [J, K] = L$$

then for any A

$$[A, H, J, K]_{\text{4-bracket}} = [A, H, L]_{\text{3-bracket}} + [A, HL]_{\text{2-bracket}}$$

and therefore

$$\Delta_{H,J,K} = \Delta_{H,L}$$

We shall say that these two QNBs are in the *same equivalence class* since they differ only by a derivation, with the latter expressed as a commutator. As we shall see later, this equivalence can be expressed precisely through a (unitary) similarity transformation.

More generally, QNBs can be equivalent in their effects to linear combinations of lower-order QNBs. This can lead to quite elaborate structure for partitioning into equivalence classes.

Applying this Lemma to the previous explicit example leads to the following.

$$[A, H, I, p_y]_{\text{4-bracket}} = \hbar \ [A, H, I]_{\text{3-bracket}} + \hbar \ [A, HI]_{\text{2-bracket}}$$

So to compare to the result for classical brackets, the quantum 3-bracket here is not equal to the quantum 4-bracket. Nevertheless, the 3- and 4-brackets in question are in the same equivalence class.

Models possessing $su(2)$ invariance provide another simple illustration of bracket equivalence (see the Appendix). The concept is actually quite generally applicable.

6 Quantum Solenoidal Flow

Quantum symplectic traces Classical symplectic traces have quantum corrections, in general. Recall in particular (sum repeated $i = 1, \cdots, N$ is understood)

$$\{A, B, x_i, p_i\}_{\text{NB}} = (N-1)\{A, B\}_{\text{PB}}$$

Corrections to the corresponding quantum 4-bracket follow from the commutator resolution.

$$\begin{aligned}[A, B, x_i, p_i] &= \{[x_i, p_i], [A, B]\} + \{[p_i, A], [x_i, B]\} - \{[x_i, A], [p_i, B]\} \\ &= 2i\hbar N \times [A, B] + \hbar^2 \left\{\frac{\partial}{\partial x_i}A, \frac{\partial}{\partial p_i}B\right\} - \hbar^2 \left\{\frac{\partial}{\partial p_i}A, \frac{\partial}{\partial x_i}B\right\} \qquad (1)\end{aligned}$$

where the RHS terms include *anticommutators* of partial derivatives of the operators, as well as the sought for $[A, B]$. In the classical limit, this gives

$$\lim_{\hbar\to 0} \frac{1}{2(i\hbar)^2} \times [A, B, x_i, p_i] = (N-1)\{A, B\}_{\text{PB}}$$

To make explicit the corrections, while retaining generality, take exponentials of arbitrary linear combinations of xs and ps.

$$A = e^{\alpha_x \cdot x + \alpha_p \cdot p}, \quad B = e^{\beta_x \cdot x + \beta_p \cdot p}$$

The αs and βs are parameters. We find the

Quantum Trace Lemma:

$$\left[e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}, x_i, p_i\right]_\star$$

$$= 2i\hbar \left(N - \frac{\frac{\hbar}{2}\left(\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x\right)}{\tan \frac{\hbar}{2}\left(\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x\right)} \right) \; \left[e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}\right]_\star$$

The prefactor on the RHS involves (sums of) areas on the parameter phase-space planes.

$$\begin{aligned}
\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x &= \sum_i \left(\alpha_x\right)_i \left(\beta_p\right)_i - \sum_i \left(\alpha_p\right)_i \left(\beta_x\right)_i \\
&= \sum_i \left(\alpha_x, \alpha_p\right)_i \wedge \left(\beta_x, \beta_p\right)_i
\end{aligned}$$

In the classical limit, obviously $\lim_{\hbar\to 0} \frac{\hbar}{2}\left(\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x\right) \cot \frac{\hbar}{2}\left(\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x\right) = 1$ gives

$$\begin{aligned}
\left\{e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}, x_i, p_i\right\}_{\rm NB} &= \lim_{\hbar\to 0} \frac{1}{2\left(i\hbar\right)^2} \left[e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}, x_i, p_i\right]_\star \\
&= \left(N-1\right) \left\{e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}\right\}_{\rm PB}
\end{aligned}$$

as expected. **Proof of Lemma:** Just reduce the star products to ordinary products.

$$\begin{aligned}
\left[e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}\right]_\star &= 2i \sin \frac{\hbar}{2}\left(\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x\right) \times e^{\alpha_x \cdot x+\alpha_p \cdot p+\beta_x \cdot x+\beta_p \cdot p} \\
\left\{e^{\alpha_x \cdot x+\alpha_p \cdot p}, e^{\beta_x \cdot x+\beta_p \cdot p}\right\}_\star &= 2 \cos \frac{\hbar}{2}\left(\alpha_x \cdot \beta_p - \alpha_p \cdot \beta_x\right) \times e^{\alpha_x \cdot x+\alpha_p \cdot p+\beta_x \cdot x+\beta_p \cdot p}
\end{aligned}$$

Now a couple more derivatives, as required in (1), and QED.

Examples of quantum solenoidal flow Define the quantum phase-space divergence as

$$\frac{\partial}{\partial z_j}[z_j, I_1, \cdots] = \frac{\partial}{\partial p_i}[p_i, I_1, \cdots] + \frac{\partial}{\partial x_i}[x_i, I_1, \cdots]$$

(sum $i = 1, \cdots, N$ is understood on RHS). That is

$$i\hbar\frac{\partial}{\partial z_j}[z_j, I_1, \cdots] = [x_i, [p_i, I_1, \cdots]] - [p_i, [x_i, I_1, \cdots]]$$

For definiteness, consider a 3-bracket. That is nicely behaved when the two generating entries in the bracket commute. This leads to a quick
Lemma: If $[H, L] = 0$ then

$$[x_i, [p_i, H, L]] - [p_i, [x_i, H, L]] = [H, L, x_i, p_i]$$

The quantum divergence of this 3-bracket is a symplectic *trace* of a quantum 4-bracket.

As discussed above, this trace is not necessarily proportional to the commutator $[H, L]$ as might be expected based on the classical limit. However, there are situations where this does hold.

For example, suppose L generates rotations in a fixed plane, say the m, n plane: $L = x_m p_n - x_n p_m$. Then with *no* assumptions about H we have another
Lemma: If L is a rotation in any fixed plane, then

$$[H, L, x_i, p_i] = 2\,(N-1)\,i\hbar\,[H, L]$$

for any H. Putting these two lemmata together gives a little
Proposition: If L is a Hamiltonian-preserving rotation in any plane, i.e. $[H, L] = 0$, then H, L generated 3-bracket quantum flow is solenoidal.

$$i\hbar\frac{\partial}{\partial z_j}[z_j, H, L] = 0\ .$$

The same conclusion holds for 4-bracket flows that are equivalent to such 3-bracket flows.

Finally, we note that the peculiar anisotropic oscillator, used previously to illustrate the intricacies of quantum bracket reduction, is another particular example of quantum solenoidal flow. In that case, both $[H, I] = 0$ and $[H, I, x_i, p_i] = 0$ by direct calculation. Thus from the first Lemma above, $\frac{\partial}{\partial z_j}[z_j, H, I] = 0$.

7 Parameterization and interpretation

The previous 3- and 4-bracket equivalence can be explicitly parameterized by "exponentiating" the effects of the bracket. For example, consider the 3-bracket in the Lemma, and define parameter-dependent operators $A(\tau)$ by

$$i\hbar^2 \frac{d}{d\tau} A(\tau) = [A(\tau), H, L]$$

$$= LA(\tau) H - HA(\tau) L$$

where $[H, L] = 0$ was used in the last step. The formal solution, with $A(\tau = 0) = A$, is given by iteration. The result for the 3-bracket flow is

$$A(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\tau}{\hbar^2} \right)^n [\cdots [[A, H, L], H, L], \cdots, H, L]$$

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \left(\frac{i\tau}{\hbar^2} \right)^m \left(\frac{-i\tau}{\hbar^2} \right)^n H^m L^n A \ H^n L^m \ ,$$

where the nth term on the first RHS line involves n nested 3-brackets, and where the second RHS line obtains when $[H, L] = 0$.

In terms of the previously discussed eigenoperators, this is

$$A(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\tau}{\hbar^2} \right)^n \sum_{\lambda_l, \omega_l, \lambda_r, \omega_r} \mathbb{P}_{\lambda_l \omega_l} \left[\cdots \left[\left[A, H, L \right], H, L \right], \cdots, H, L \right] \mathbb{P}_{\lambda_r \omega_r}$$

$$= \sum_{\lambda_l, \omega_l, \lambda_r, \omega_r} e^{i(\omega_l \lambda_r - \lambda_l \omega_r)\tau} A_{\lambda_l, \omega_l, \lambda_r, \omega_r} = \sum_{\lambda_l, \omega_l, \lambda_r, \omega_r} \mathbb{P}_{\lambda_l \omega_l}\, e^{i(\omega_l \lambda_r - \lambda_l \omega_r)\tau} A\, \mathbb{P}_{\lambda_r \omega_r} \ .$$

Each eigenoperator has its particular areal phase develop linearly in τ over the course of the flow.

The variable τ has dimensions of time, but obviously it is *not* the conventional time parameter t. The relation between τ and t is angular momentum dependent.

When $[H, L] = 0$, the result can be written in deceptively simple short-hand through the use of an operator ordering prescription.

$$A(\tau) = e^{i\tau \underset{\rightarrow}{H} \ \underset{\leftarrow}{L}/\hbar^2} A \ e^{-i\tau \underset{\rightarrow}{L} \ \underset{\leftarrow}{H}/\hbar^2}$$

The underlining arrows, *not* the left-right order on the page, indicate from which side the generators are to act upon the A: $HAL \equiv \underset{\rightarrow}{H} \ \underset{\leftarrow}{L} \ A = A \ \underset{\rightarrow}{H} \ \underset{\leftarrow}{L}$. (When $[H, L] \neq 0$ this notation is not the best! Nevertheless, the formal solution for $A(\tau)$ can always be written in terms of exponentials with a suitable ordering prescription, even when $[H, L] \neq 0$.)

Caveat emptor! This is *not* a unitary transformation.

$$A(\tau)\, B(\tau) \neq (AB)(\tau)$$

The ordering is important.

$$\left(e^{i\tau \underset{\rightarrow}{H} \ \underset{\leftarrow}{L}/\hbar^2} \right)^{-1} = e^{-i\tau \underset{\rightarrow}{H} \ \underset{\leftarrow}{L}/\hbar^2} \neq e^{-i\tau \underset{\rightarrow}{L} \ \underset{\leftarrow}{H}/\hbar^2}$$

Any QNB can be similarly "exponentiated" to obtain τ dependent $A(\tau)$.

For example, consider the 4-bracket of the Lemma. Then

$$i\hbar^2 \frac{d}{d\tau} A(\tau) = [A(\tau), H, J, K]$$

$$A(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\tau}{\hbar^2} \right)^n [\cdots [[A, H, J, K], H, J, K], \cdots, H, J, K]$$

and the previous "4=3+2" relation between brackets becomes a unitary equivalence between the 3- and 4-bracket flows

$$A_{H,J,K}(\tau) = e^{-i\tau HL/\hbar^2} A_{H,L}(\tau) \; e^{i\tau HL/\hbar^2}$$

To distinguish the flows generated by 3- and 4-brackets, we have used subscripts on the evolved operators. The ordering of the additional phase on the RHS is traditional, as written on the page.

Applied to the explicit example above, this becomes

$$A_{H,I,p_y}(\tau) = e^{-i\tau HI/\hbar} A_{H,\hbar I}(\tau) \; e^{i\tau HI/\hbar}$$

But once again, no matter which way you look at it, $(AB)(\tau) \neq A(\tau) B(\tau)$.

When the flow of expectation values is considered, it becomes clear that this strange product behavior is *not* a problem. For consistency, all operator products are to flow in unison, and not individually. That is, the correct procedure is to compute $(AB \cdots)(\tau)$, and then compute the expectation value. With this procedure, the flow can always be swept into the density operator, flowing backward in τ, and thence described by a τ propagator and/or Green's function.

8 Propagators and Green functions

When operator products are τ evolved in unison, $AB\cdots C \to (AB\cdots C)(\tau)$, their expectation values may be computed at later τ values by shifting the evolution over to the density operator.

$$\langle (AB\cdots C)(\tau)\rangle = Tr\left(\rho \times (AB\cdots C)(\tau)\right) = Tr\left(\rho(\tau) \times AB\cdots C\right) \ ,$$

where in the last step, the density operator flows backwards in τ.

For example, consider flow generated by the H, L 3-bracket.

$$(AB\cdots C)(\tau) = e^{i\tau \underset{\rightarrow}{H}\ \underset{\leftarrow}{L}/\hbar^2}\,(AB\cdots C)\ e^{-i\tau \underset{\rightarrow}{L}\ \underset{\leftarrow}{H}/\hbar^2}\ .$$

Then, with standard hermiticity of H and L combined with the cyclic properties of the trace, and with due care to the ordering, we have

$$Tr\left(\rho \times (AB\cdots C)(\tau)\right) = Tr\left(\rho \times \left(e^{i\tau \underset{\rightarrow}{H}\ \underset{\leftarrow}{L}/\hbar^2}\,(AB\cdots C)\ e^{-i\tau \underset{\rightarrow}{L}\ \underset{\leftarrow}{H}/\hbar^2}\right)\right)$$

$$= Tr\left(\left(e^{-i\tau \underset{\leftarrow}{L}\ \underset{\rightarrow}{H}/\hbar^2}\rho\ e^{i\tau \underset{\leftarrow}{H}\ \underset{\rightarrow}{L}/\hbar^2}\right) \times (AB\cdots C)\right)\ .$$

where $[H, L] = 0$. Thus the trace relation holds with

$$\rho(\tau) = e^{-i\tau \underset{\rightarrow}{H}\ \underset{\leftarrow}{L}/\hbar^2}\rho\ e^{i\tau \underset{\rightarrow}{L}\ \underset{\leftarrow}{H}/\hbar^2}\ .$$

The flow of the density operator is therefore backward in τ compared to that of the operators whose expectation values are of interest. Infinitesimally

$$i\hbar^2 \frac{d}{d\tau}\rho(\tau) = \underset{\rightarrow}{H}\ \underset{\leftarrow}{L}\ \rho(\tau) - \rho(\tau)\ \underset{\rightarrow}{L}\ \underset{\leftarrow}{H} \equiv H\rho(\tau)L - L\rho(\tau)H\ ,$$

as opposed to the operator evolution given above. This feature of backwards evolution is well-known to be true in standard Hamiltonian dynamics.

Also note that the flow of the operators must be in unison, as above, for this procedure to carry through. It would not be true that all the τ evolution of the expectation value could be swept into the density operator alone if we were to evolve the operators separately, since $(AB\cdots C)(\tau) \neq A(\tau)B(\tau)\cdots C(\tau)$ and therefore $\langle A(\tau)B(\tau)\cdots C(\tau)\rangle = Tr\left(\rho \times A(\tau)B(\tau)\cdots C(\tau)\right) \neq Tr\left(\rho(\tau)\times(AB\cdots C)\right)$.

Evolution of the density operator can be realized in the phase-space formulation of quantum mechanics. In that formalism, the density operator is represented by the Wigner function, $f(x,p)$. General Wigner functions (WFs), including the non-diagonal cases, are defined for pure state systems described by wave functions $\psi_a(x)$ like so:

$$f_{ab}(x,p) = \frac{1}{2\pi\hbar}\int dy\, \psi_a\left(x - \tfrac{1}{2}y\right)\ e^{ipy/\hbar}\, \psi_b^*\left(x + \tfrac{1}{2}y\right) = \psi_a(x) \star \delta(p) \star \psi_b^*(x)\ \ .$$

Here we have ordered the indices to correspond with the bra-ket notation for the underlying operator, $|a\rangle\langle b|$, and in the last step we have obtained the compact expression of Braunss through the use of Groenewold's non-commutative but associative (NBA) "star" product operation.

$$\star\ \equiv\ e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)}\ .$$

Such pure state Wigner functions obey the orthonormality conditions

$$\iint_{-\infty}^{+\infty} dxdp\, f_{ab}(x,p) = \delta_{ab}\ ,\quad (2\pi\hbar)\ f_{ab} \star f_{cd} = f_{ad}\ \delta_{bc}\ ,$$

when the underlying wave functions are orthonormal.

Propagation of WFs is effected through the use of a phase-space propagator in the usual sort of integral relation.

$$f\left(x,p;\tau\right)=\int dX\,dP\,G(x,p;X,P;\tau)\,f\left(X,P\right)\ .$$

For conventional time evolution generated by the Hamiltonian, H,

$$G_H(x,p;X,P;t)=2\pi\hbar\sum_a\sum_b\ f_{ab}(x,p)\,e^{-i(E_b-E_a)t/\hbar}\ f_{ba}(X,P)\ ,$$

while for τ evolution, the phase is modified appropriately. For example, again considering the H,L generated 3-bracket evolution, we have

$$G_{H,L}(x,p;X,P;\tau)=2\pi\hbar\sum_a\sum_b\ f_{ab}(x,p)\,e^{-i(l_aE_b-l_bE_a)\tau/\hbar^2}\ f_{ba}(X,P)\ .$$

We have assumed the labels on the Wigner functions designate particular energy and angular momentum left- and right-eigenvalues, as indicated. The propagator obeys the same infinitesimal τ evolution as the density operator, but with operator products supplanted by $\star$ products.

$$i\hbar^2\frac{d}{d\tau}G_{H,L}(x,p;X,P;\tau)=H\star G_{H,L}(x,p;X,P;\tau)\star L-L\star G_{H,L}(x,p;X,P;\tau)\star H\ .$$

The corresponding Green function is given by

$$g_{H,L}\left(x,p;X,P;\omega\right)\equiv\int_0^\infty d\tau\,e^{-i(\omega-i\varepsilon)\tau}G_{H,L}(x,p;X,P;\tau)$$

$$=2\pi\hbar\sum_a\sum_b\ f_{ab}(x,p)\ \int_0^\infty d\tau\,e^{-i(\omega-i\varepsilon)\tau}e^{-i(l_aE_b-l_bE_a)\tau/\hbar^2}\ f_{ba}(X,P)$$

$$=2\pi\hbar\sum_a\sum_b\ f_{ab}(x,p)\,\frac{i}{\omega-i\varepsilon-\left(l_bE_a-l_aE_b\right)/\hbar^2}\,f_{ba}(X,P)$$

The poles of this Green's function have residues consisting of the factorized Wigner function bilinears. Thus the Wigner functions can be recovered, in principle, from these residues. There is a "degeneracy" issue, however, since it is possible for different L and H eigenvalues to give the same value for $l_bE_a-l_aE_b$, hence the same pole location, and thus a residue which is a sum of the corresponding Wigner function bilinears. But this degeneracy issue is no more problematic than the usual situation corresponding to degenerate energies.

As a simple example that can be explicitly worked out, reconsider the free particle on the plane with generators p_x and p_y in a 3-bracket. Then

$$G_{p_x,p_y}(x, p_x, y, p_y; X, P_X, Y, P_Y; \tau)$$

$$= \delta\left(x - X + p_y \tau\right)\ \delta\left(y - Y - p_x \tau\right) \times$$

$$\times\, \delta\left(p_x - P_X\right)\ \delta\left(p_y - P_Y\right)$$

which is different than the usual free Hamiltonian $H = \frac{1}{2}\left(p_x^2 + p_y^2\right)$ generated flow

$$G_H(x, p_x, y, p_y; X, P_X, Y, P_Y; t)$$

$$= \delta\left(x - X + p_x t\right)\ \delta\left(y - Y + p_y t\right) \times$$

$$\times\, \delta\left(p_x - P_X\right)\ \delta\left(p_y - P_Y\right)$$

but obviously mathematically equivalent to it.

Thanks for your attention, and thanks to ANL for enthusiastically hosting this conference.

Appendix: Particle on a 2-sphere and bracket equivalence The commutator algebra of the charges ($L_0 \equiv L_z$, $L_\pm \equiv L_x \pm iL_y$) is

$$[L_+, L_-] = 2\hbar L_0 \ , \quad [L_0, L_-] = -\hbar L_- \ , \quad [L_0, L_+] = \hbar L_+ \ ,$$

giving rise to $[L_-, L_0^2] = \{[L_-, L_0], L_0\} = \hbar \{L_-, L_0\}$, etc. The invariant quadratic Casimir is

$$H = L_+ L_- + L_0 (L_0 - \hbar) = L_- L_+ + (L_0 + \hbar) L_0 \ .$$

This is also the Hamiltonian for the free particle on the sphere.

We use the algebra and the commutator resolution of the 4-bracket

$$[A, B, C, D] = \{[A, B], [C, D]\} - \{[A, C], [B, D]\} - \{[A, D], [C, B]\} \ ,$$

to obtain

$$[A, L_0, L_+, L_-] = 2\hbar [A, H] \ ,$$

as well as the more elaborate

$$[A, H, L_+, L_-] = 2\hbar \{[A, H], L_0\} = 2\hbar [\{A, L_0\}, H] \ .$$

In the latter case, the time derivation is entwined with the effects of a rotation.

So for $SU(2)$ invariant systems with $H = I/2$, we obtain the complete analog of classical time development as a derivation, namely

$$i\hbar^2 \frac{dA}{dt} = \hbar [A, H] = \frac{1}{4} [A, L_0, L_+, L_-] \ ,$$

where the QNB in question happens to be a derivation too. By contrast, the entwined form gives rise to

$$i\hbar^2 \left\{ \frac{dA}{dt}, L_0 \right\} = \hbar \{[A, H], L_0\} = \frac{1}{4} [A, H, L_+, L_-] \ .$$

Since the latter of these is manifestly not a derivation, one should not expect Leibniz rule and classical-like fundamental identities to hold. Of course, since a derivation is entwined in the structure, substitution $A \to A\mathcal{A}$ and application of Leibniz's rule to just the time derivation alone will necessarily yield correct but complicated expressions.

The "4=3+2" Lemma applies to the present situation to yield.

$$[A, H, L_0] = [\{A, L_0\}, H] + [HL_0, A] \ ,$$
$$[A, H, L_+, L_-] = 4\hbar [A, H, L_0] + 4\hbar [A, HL_0] \ .$$

The first line here is a special case, valid for $HL_0 = L_0 H$, of one of several general non-manifestly-antisymmetric ways of writing 3-brackets.