Diffraction and Detector Considerations for High Resolution Imaging

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Abstract

High-energy electron beams with normalized emittances at or below 1 mm-mrad require camera and optics systems with micrometer level resolutions for accurate beam size measurements. We present here theory and measurements showing a light imaging system capable of $\leq 4$ micrometer resolution.

The FACET experimental program relies on a "butterfly" measurement[1] to quantify the emittance of electron beams used in Plasma Wakefield Acceleration (PWFA) experiments. The butterfly technique uses a set of quadrupole magnets to image the electron beam from one location in the beamline to a light emitting screen downstream along with a dipole magnet to disperse the beam in energy in one of the directions (e.g. the y-direction). This allows measurement of the beam size as a function of energy and, much like a quad scan, measurement of the emittance. Example butterfly images for FACET and FACET-II relevant parameters are shown in Figure 1. Since the imaged beam size decreases as the emittance decreases like $\sigma_x \sim \sqrt{\epsilon_x}$ at some point the beam will become a stripe on the camera and one will be unable to measure the beam size as a function of the energy and thus cannot measure the emittance.

The resolution of the light imaging system at FACET was approximately 13 $\mu$m. As will become clear later, this value was dominated by the size of the pixels used in the imaging system. This document focuses on improvements that can be made to the light imaging system to reduce its resolution to 4 $\mu$m which will allow normalized emittance measurements of 300 nm for 20 GeV beams with 0.5% energy spread. The majority of this document focuses on the light optics, so when discussing the properties of the magnetic beam transport we will specifically denote the parameters as "magnetic", i.e. magnetic magnification refers to the magnification of the electron beam transport.

The limits on resolution in a camera system can be understood *prima facie* from two limits. First, ignoring diffraction effects, an object imaged on to a CCD with a size smaller than the CCD pixel size will not be resolved. This sets the minimum size that a CCD can measure to the "effective pixel" size of the the CCD, which is the size of the pixel in the object plane. For example, a camera system with a pixel size of $p=6.5$ $\mu$m and a magnification of $M=2$ will have an effective pixel size of $p_e=p/M=3.25$ $\mu$m. In the second limit, if the contribution to the beam size at the CCD is dominated by the blurring due to diffraction effects, then the minimum object size measurable is related

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Figure 2: Diagram of a single lens diffraction system. R is the radius of the lens, s the distance from the object plane to the lens and s’ is the distance from the lens to the image plane.

to the diffraction of the lens in the camera system. That is to say that at some point diffraction ensures the object is always large compared to the effective pixel size, so the pixel size can be ignored.

This document is divided into three sections. In Section 1 we define what we mean by resolution, how it is measured and discuss the resolution and diffractive properties of single lens imaging systems. We show through measurements that multi-lens systems, such as the DSLR macro lenses, behave similarly. We also discuss the Depth of Field, which can be a limiting factor in high resolution imaging systems. Section 2 examines the role that discretization of the object by the CCD and noise plays in determining the minimum resolution of the camera system. In Section 3 we examine the relative performance of two cameras as an example of lens and camera selection.

1 Lens Point Spread Function

To define what we call resolution we must examine the diffraction properties of single lens system of focal length \( f \). The effect we intend to describe is the blurring of an image due to the "point spread function", PSF, of a lens. A diagram of the system in question is shown in Figure 2. The magnification of this system is \( M = s'/s \). In addition, we assume the imaging condition is satisfied,

\[
\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{s} \frac{(M+1)}{M} = \frac{1}{s'}(M + 1).
\]

We are interested in the intensity in the image plane due to a delta function intensity in the object plane. By using Fourier Optics\[2\] we find that for a single lens the intensity in the image plane is,

\[
I(r) \sim \left| \frac{J_1\left(\frac{kR}{s'} r\right)}{kR/s'} \right|^2,
\]

with \( k=2\pi/\lambda \) the wave number of the light and \( r \) is the radial coordinate. We assume green light in this work, \( \lambda=500 \) nm. This function, sometimes called a "jinc" function, represents the blurring of a small object, in this case a delta function, by the diffraction properties of an imaging single lens. This is the so called point spread function of the system. It can be shown that any given intensity distribution in the object plane can be transported to the image plane by the convolution of the object plane distribution with the point spread function \[2\]. This intensity function can be well approximated by a Gaussian, as described in Appendix A, when

\[
\frac{1.3}{\sigma_{psf}} = \frac{kR}{s'},
\]

where \( \sigma_{psf} \) is the r.m.s. size of the Gaussian. From this relation, and by Equation 1, the r.m.s. size of the Gaussian point spread function for this system can be written,

\[
\sigma_{psf} = \frac{1.3\lambda}{2\pi} \frac{(M + 1)}{f} R.
\]

An Air Force 1951 Target is used to measure \( \sigma_{psf} \). The convolution of the PSF of an imaging lens system with the lines on the target gives \( \sigma_{psf} \) as one half the line width when the contrast between light and dark is 50%, as described in Appendix C.
1.1 Resolution

To measure the point spread function of a real lens we must use a camera to generate images. This means we will have to incorporate our model for the smallest object measurable due to pixel size into the measurements we make. Due to the point spread function of the lens, the lines on the Air Force target will be transformed into Gaussian-like shapes. Since the blurring due to the PSF and the lines are Gaussian-like in shape, we assume the two effects add together in quadrature with a lower limit of the effective pixel size $p_e$, and define the resolution of a lens and camera system as,

$$r^2 = \left( \frac{p}{M} \right)^2 + \sigma_{psf}^2 = \left( \frac{p}{M} \right)^2 + \left( \frac{1.3\lambda}{\pi}(M+1)f# \right)^2,$$

(5)

where $f# = f/D$ is the f-number of the lens. For an $M=1$ system, assuming $\lambda = 500\,\text{nm}$, $\sigma_{psf} = 0.41f#$.

1.2 $f#$ and barrel $f#_b$

The calculations above require knowledge of the $f#$ of the lens in use. The number written on the barrel of the lens, $f#_b$, is found to be the “effective $f#_e = f#_b = (1 + M)f#$. All lenses used in this work are designed to be used at $M = 1$, so $f# = f#_b/2$. Typically, the effective pixel size dominates the smallest measured resolution of a camera and lens system, so the distinction between $f#$ and $f#_b$ results in errors of $\sim 10\%$ in resolution and thus may not be of practical importance.

1.3 Multi-element Lens Measurements

We compare our derived expression for the resolution of a lens and camera system with measurements made on a Tokina 105mm macro lens ($M=1$). We measured the resolution as a function of $f#_b$ for two different cameras: a PCO edge 5.0 with $6.5\,\mu m$ pixel size and an Allied Vision Manta 125 with $3.75\,\mu m$ pixel size. The data, shown in Figure 3, is fit to the function $r^2 = (ap)^2 + (bf#_b)^2$ with the results summarized in Table 1. The measurements show that the resolution of multi-element lens systems is well described by the single lens model and that the lower bound due to pixel size accurately describes the minimum resolution of said system.

1.4 Depth of Field

Depth of Field, $\Delta s$, is the distance the object plane must move from the design object plane for the resolution to degrade by some amount. As derived in Appendix B, this degradation is characterized by the parameter $\gamma$. From this
parameter, the Depth of Field $\Delta s$ can be estimated as,

$$
\Delta s = \frac{4\gamma \lambda}{\pi} \left( \frac{M + 1}{M} \right)^2 (f\#)^2.
$$

(6)

It is found that $\sigma_{psf}$ is worse by 5% when $\gamma \simeq 1.5$. For a lens with $M = 1$, $\lambda = 0.5\mu m$, $f\# = 4$ we calculate the resolution will be 5% worse when $\Delta s = 61\mu m$. A plot of the measured resolution of a lens as a function of position of the object relative to the true object plane is show in Figure 4. A fit of the form $r = a + b(x-c)^2$ is performed and it is found that the resolution is 5% worse when $\Delta s = 64\mu m$.

2 Measurement Limitations Due to Pixel Size

In this section we examine the effect of discretization on measurement of Gaussian distributions which ultimately limit the smallest Gaussian that can be measured. For this section we assume that the distribution is cutoff when the amplitude is below 1% of the peak value, which is 1 in this case. We assume a lens operating at $M = 1$.

For this measurement a distribution of RMS size $\sigma_d$ is generated at the object plane and transported through the lens by adding $\sigma_d$ and $\sigma_{psf}$ in quadrature. Noise is then added to the distribution to account for background light in the spectrometer system when the plasma is excited. The distribution is then discretized and a Gaussian fit is performed. From the fit, the known contribution to the beam size from the lens, $\sigma_{psf}$, is subtracted out in quadrature. The result of this process using a 6.5 $\mu m$ size pixel camera and $\sigma_{psf} = 4\mu m$ is shown in Figure 5.

The result of the simulated measurement is that the measured beam size and the input beam size begin to diverge at approximately the pixel size. This result is shown to be robust across translations of the distributions with respect to the pixels and noise levels up to 20%. It can also be shown that larger $\sigma_{psf}$ are better for measuring smaller beam sizes, as the beam is larger on the CCD, but generate systematic errors which increases uncertainty. Applying a cutoff of larger than 5% is found to add a systematic reduction in fit beam size $\sigma_f$.

3 Comparison of PCO Edge and Allied Vision Manta 125

Parameters for three potential cameras and lens configurations are shown in Table 2. The result is that the parameters of the PCO sCMOS camera are superior to the Manta camera even when the loss of photons due to increased effective $f\#$ for the $M = 2$ case is taken into account. The analysis assumes green light and draws from the published sensor data from the manufacturer.

3.1 PCO Edge and M = 2 Measurement

We now turn to the measurement of calculated values in Table 2 of a lens and camera system with $M = 2$. For this measurement we use a Nikon Nikkor f/4 200mm lens and a PCO Edge 5.0 camera. An extension of tube of 100 mm is added between the lens and the camera to reach a magnification of 2. The results are shown in Figure 6. One would expect from a simple lens perspective that we would need to add 200 mm of extension to a lens with a 200 mm focal length to move from $M = 1$ to $M = 2$, a discrepancy we address below. The "minimum working distance", or the distance from the front of the lens to the Air Force Target, is 205 mm.
Figure 5: A simulated measure of the fit beam size as a function of the input beam size.

Table 2: A comparison of the parameters of potential camera and lens configuration for FACET-II. This table is compiled assuming an $f\#$ of 4.

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<td>0.4</td>
<td>12</td>
<td>0.56</td>
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</table>

3.1.1 $M = 2$ Resolution

In Figure 6 a) we show that the measured minimum resolution decreases to $r = 3.7 \mu m$ for $f\#_b = 5.6$. From Equation 5 one would expect the resolution to be $4.76 \mu m$. A resolution of $r = 3.7 \mu m$ is recovered, using the measured magnification of 2, when the $f\#$ is 2.8. This indicates that the $f\#$ as inscribed on the lens is the effective $f\#_e = (1 + M) f\# = 2 * f\#$.

3.1.2 $M = 2$ Depth of Field

Figure 6 b) shows the Depth of Field of the $M = 2$ configuration. The point at which the resolution is 5% worse than the minimum is measured to be $\Delta s = \pm 61 \mu m$, which is consistent with a calculated value of $\Delta s = \pm 67 \mu m$ from Eq. 6 when using $f\#_b=5.6$.

3.1.3 $M = 2$ Extension and Effective Focal Length

The $f\#$ of a lens can be measured through Equation 5 by varying the magnification $M$. This measurement is shown in Figure 6 c). What is found is that the $f\#$ is smaller than that written on the lens by approximately a factor of 2. The errors can be improved by increasing the number of data points. This supports the conclusion that what is written on the lens is the effective $f\#_e$. Since the magnification ($M=1$) and the diameter of the lens (50 mm) are measured we can calculate an "effective" focal length of the lens from the $f\#_b$ written on the lens as,

$$f_e = \frac{f\#_b}{1 + M} D = 100 mm.$$ (7)

This effective focal length is consistent with the addition of 100 mm extension to move from $M = 1$ to $M = 2$.

Using a Nikon Nikkor 60mm f/2.8 lens, $M = 2$ is reached when the extender is 48 mm. For a Tokina 105mm f/2.8 lend, $M = 2$ is reached when the extender is 76 mm. For all three macro lenses measured here the minimum working distance remains virtually unchanged when changing magnification. As such, determination of the extenders needed to obtain a desired magnification is not consistent with a single lens system, and is found to be shorter than single lens theory would suggest. The lack of change of minimum working distance when increasing magnification is also inconsistent with a single lens theory but is advantageous.
The wave passes through a thin lens of focal length \( f \) with \( k = 2 \). We then propagate from the lens to the image plane, sometimes called the Transfer Function in frequency space, to propagate the beam from one \( z \) location to another, defined by position in the system \( z \). The object plane is defined to be \( z = 0 \). We use the convolution kernel \( h(x,y;z) \).

We assume there are no bends in the system so that \( x \) and \( y \) are the transverse coordinates in any plane, which is defined by position in the system \( z \). The object plane is defined to be \( z = 0 \). We use the convolution kernel \( h(x,y;z) \) (sometimes called the Transfer Function in frequency space) to propagate the beam from one \( z \) location to another,

\[
U'(x, y; z_1 + \Delta z) = U(x, y; z_1) \otimes h(x, y; \Delta z),
\]

where \( \otimes \) is the convolution operator. The convolution kernel for free space propagation is,

\[
h(x, y; z) = \frac{e^{ikz}}{i\lambda z} e^{\frac{i}{2}(x^2+y^2)},
\]

with \( k = 2\pi/\lambda \) the wavenumber of the radiation of interest. So to propagate from the objection plane to the lens we find,

\[
U_1(x, y; s) = \frac{A_0 e^{iks}}{i\lambda s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\zeta, \eta)e^{\frac{i}{\lambda s}((x-\zeta)^2 + (y-\eta)^2)} d\zeta d\eta,
\]

\[
U_1(x, y; s) = \frac{A_0 e^{iks}}{i\lambda s} e^{\frac{i}{2}(x^2+y^2)}.
\]

Next the wave passes through a thin lens of focal length \( f \) and pupil function \( p(x, y) = \theta(x^2 + y^2 - R^2) \) where \( R \) is the radius of the lens,

\[
U_2(x, y; s) = \frac{A_0 e^{iks}}{i\lambda s} e^{\frac{i}{2}(x^2+y^2)} e^{\frac{-iks}{f}}(x^2+y^2)p(x, y)
\]

\[
U_2(x, y; s) = \frac{A_0 e^{iks}}{i\lambda s} e^{\frac{i}{2}(x^2+y^2)}p(x, y).
\]

We then propagate from the lens to the image plane,

\[
U_3(x, y; s + s') = -\frac{A_0 e^{iks}}{\lambda^2 s s'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i}{\lambda s}((\zeta-x)^2 + (\eta-y)^2)} p(\zeta, \eta) e^{\frac{-iks}{f}}((\zeta-x)^2 + (\eta-y)^2) d\zeta d\eta,
\]

\[
U_3(x, y; s + s') = -\frac{A_0 e^{iks}}{\lambda^2 s s'} e^{\frac{i}{2}(x^2+y^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i}{\lambda s}((\zeta-x)^2 + (\eta-y)^2)} p(\zeta, \eta) e^{\frac{-iks}{f}}((\zeta-x)^2 + (\eta-y)^2) d\zeta d\eta.
\]

Appendices

A Point Spread Function of a Single Lens System

We refer again to Figure 2 for the system we wish to derive the Point Spread Function for. The Point Spread Function is defined to be the response of the system to an impulse input, e.g. a delta function. To find the PSF of the system of interest we use scalar diffraction theory.

The field in the object plane is,

\[
U_0(x, y) = A_0 \delta(x, y).
\]

(b) The Depth of Field which results in 5% worse resolution is \( \pm 61 \mu m \). For this measurement, \( f\# = 5.6 \).

Figure 6: The collection of measurements made using a Nikon Nikkor f/4 200mm lens and a PCO Edge 5.0 camera. a) The resolution minimum of 3.7 \( \mu m \) occurs at an f\# of 5.6. b) The Depth of Field which results in 5% worse resolution is \( \pm 60.98 \mu m \). c) When Equation 5 is used to measure the resolution as a function of magnification that the "true" f\# of the lens is closer to 2.8.
Note that the first exponential in the integral of the second part of Equation 13 is zero when the imaging condition is satisfied. We are interested in the intensity in the imaging plane when the imaging condition is satisfied,

$$|U_3(x, y; s + s')|^2 = \left| \frac{A^2_0}{(\lambda s s')^2} \int \int_{(\zeta^2 + \eta^2) \leq R^2} p(\zeta, \eta) e^{\frac{-ik}{s} (x \zeta + y \eta)} d\zeta d\eta \right|^2. \tag{14}$$

At this point we drop the constants out front as we are interested in the distribution of the intensity and not its absolute value. After integration we find,

$$|U_3(x, y; s + s')|^2 = \left| \frac{s'}{k R (x^2 + y^2)} J_1 \left( \frac{k s'}{R} R (x^2 + y^2) \right) \right|^2, \tag{15}$$

$|U_3|^2$ can be approximated by a Gaussian, when $\frac{1.3}{\sigma_{psf}} = \frac{k R}{s'}$, as shown in Figure 7.

### B Depth of Field of a Single Lens

To quantify the Depth of Field of a single lens, we start with Eq. 13 and are interested in how the distribution in the fixed image plane changes case when object moves out of the object plane by an amount $\Delta s$,

$$\frac{1}{s} + \frac{1}{s'} - \frac{1}{f} \rightarrow \frac{1}{s + \Delta s} + \frac{1}{s'} - \frac{1}{f}. \tag{16}$$

We assume the imaging condition is still satisfied, $1/f = 1/s + 1/s'$, so that we can write,

$$\frac{1}{s + \Delta s} + \frac{1}{s'} - \frac{1}{f} = \frac{1}{s + \Delta s} - \frac{1}{s} = \frac{\Delta s}{s^2} (1 - \frac{\Delta s}{s}), \tag{17}$$

which is substituted into Eq. 13. From this substitution the intensity in the image plane is found to be,

$$|U_3(x, y; s + s' + \Delta s)|^2 = \frac{A^2_0}{(\lambda (s + \Delta s) s')^2} \left| \int \int_{(\zeta^2 + \eta^2) \leq R^2} e^{\frac{-ik}{s'} (1 - \frac{\Delta s}{s}) (\zeta^2 + \eta^2)} p(\zeta, \eta) e^{\frac{-ik}{s} (x \zeta + y \eta)} d\zeta d\eta \right|^2. \tag{18}$$

This integral is rewritten in polar coordinates using $\zeta = r \cos \theta$ and $\eta = r \sin \theta$ and the theta integral is performed, giving,

$$|U_3(x, y; s + s' + \Delta s)|^2 = \frac{2\pi A^2_0}{(\lambda (s + \Delta s) s')^2} \left| \int_0^R r e^{-ikr^2} J_0(k \left( \frac{r}{R} \sqrt{x^2 + y^2} \right)) r dr \right|^2, \tag{19}$$

where,

$$\alpha = \frac{1}{2} \frac{\Delta s}{s^2} (1 - \frac{\Delta s}{s}). \tag{20}$$

Performing a $u$ substitution $u = r/R$ the integral is rewritten, one last time, as

$$|U_3(x, y; s + s' + \Delta s)|^2 = \frac{2\pi R^2 A^2_0}{(\lambda (s + \Delta s) s')^2} \left| \int_0^1 u e^{-i\alpha u^2} J_0(\beta u) du \right|^2, \tag{21}$$
Figure 8: Plot of $|U_3|^2$ for multiple values of $\gamma$, Equation 21.

with,

$$\gamma \equiv \frac{kR^2}{s^2} \Delta s \frac{1 - \Delta s}{s},$$

$$\beta \equiv \frac{kR}{s^2} \sqrt{x^2 + y^2}. \quad (22)$$

It turns out that Eq. 21 does not have a closed solution, but it can be shown to give the same result as Eq. 13 when $\Delta s \to 0$, as expected.

To quantify the changes in the Depth of field we examine how the width of the intensity distribution in the image plane changes when the object plane is moved by $\Delta s$. A plot illustrating several values of $\gamma$ is shown in Figure 8.

In Appendix A it was found that a reasonable approximation to $|U_3|^2$ occurs when $\sigma_{psf} = \frac{1.3}{M}$. Here we find that $\sigma_{psf}$ grows by 5% when $\gamma \simeq 1.5$. Equation 22 can be solved for the offset as a function of $\gamma$,

$$\Delta s = \frac{4\gamma \lambda}{\pi} \left( \frac{M + 1}{M} \right)^2 (f\#)^2, \quad (23)$$

where the $\Delta s^2$ has been dropped.

C  Determining $\sigma_{psf}$ from an Air Force Target

An Air Force 1951 target, shown in Figure 9, contains a series of lines of known width which are separated by that width. To find how a step function, such as the lines on this target, behave when passed through an imaging lens system we take the convolution of a line of width $W$ with a Gaussian of width $\sigma_{psf}$,

$$F(x) = \int_{-\infty}^{\infty} \Theta(\frac{W}{2} + y) \Theta(\frac{W}{2} - y) e^{-\frac{(x-y)^2}{2\sigma_{psf}^2}} dy = \int_{-\frac{W}{2}}^{\frac{W}{2}} e^{-\frac{(x-y)^2}{2\sigma_{psf}^2}} dy = \sqrt{\frac{\pi}{2}} \sigma_{psf} \left( Erf \left( \frac{x + W/2}{\sqrt{2} \sigma_{psf}} \right) - Erf \left( \frac{x - W/2}{\sqrt{2} \sigma_{psf}} \right) \right), \quad (24)$$

with $Erf[]$ the error function. It can be shown that the last function can be well approximated, up to a normalization, when $W \leq 2\sigma_{psf}$ by a Gaussian of width $\sigma_{t}^2 = \sigma_{psf}^2 + (W/\pi)^2$. We can then represent the system of two lines on an Air Force target separated, center-to-center, by a distance $2W$ as,

$$G(x) \simeq e^{-\frac{(x-W)^2}{2\sigma_t^2}} + e^{-\frac{(x+W)^2}{2\sigma_t^2}} \quad (25)$$

and want to solve $G(0) = 1/2$. This relation is satisfied when $W = 1.96 \sigma_{psf}$. An example calculation done without approximation is show in Figure 10.
Figure 9: Example of the line pattern on an Air Force Target. Copyright Wikimedia.

Figure 10: Example measure of $\sigma_{psf}$ using an Air Force 1951 target.

References


[2] Introduction to Fourier Optics by Goodman